Function Theory on Symplectic Manifolds - problem session 2

**Spoiler alert:** The hints provided are sometimes very close to solutions, and have therefore been relegated to the end of the sheet. Try to think about the exercises before looking at them.

The following lemma, which was a key step in proving the Eliashberg-Gromov $C^0$-rigidity theorem, was quoted without proof:

**Lemma 0.1.** Let $f$ be a compactly supported diffeomorphism of $M$ such that $f^*\omega \neq c\omega$ for any $c \in \mathbb{R}$. Then there exist $F, G \in C^\infty(M)$ such that $||\{F, G\}|| = \frac{1}{2}$ but $||\{f^*F, f^*G\}|| \geq 1$.

This sheet is aimed at proving this lemma in a sequence of exercises.

1. Let $(E, \omega)$ be a symplectic vector space (i.e., $\omega$ is a non-degenerate exterior 2-form on $E$). Recall that we define a 2-form $\omega^*$ on $E^*$ by

$$\omega^*(\alpha, \beta) = \omega(I_\omega^{-1}\alpha, I_\omega^{-1}\beta) = \alpha(I_\omega^{-1}\beta).$$

(recall that $I_\omega: E \to E^*$ is defined by $I_\omega(\xi) = i_\xi\omega = \omega(\xi, \cdot)$.)

(i) Check that $\omega^*$ is a symplectic form (i.e., non-degenerate) on $E^*$.

(ii) Suppose that $\omega_1, \omega_2$ are two symplectic forms on $E$. Prove that

$$\omega_1 = c \cdot \omega_2 \iff \omega_1^* = \frac{1}{c} \cdot \omega_2^*.$$

(iii) Show that on a symplectic manifold,

$$\omega^*(dF, dG) = \{G, F\}.$$

The following (easy) fact may be useful in the following exercises: any symplectic vector space $(E, \omega)$ has a *symplectic basis*, i.e., a basis $\{e_i, f_i\}_{i=1}^n$ such that

$$\omega(e_i, f_j) = \delta_{ij}, \quad \omega(e_i, e_j) = \omega(f_i, f_j) = 0.$$

Thus, any symplectic vector space is isomorphic to $(\mathbb{R}^{2n}, \sum dp_j \wedge dq_j)$. 

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2. Let \( \omega_1, \omega_2 \) be two symplectic forms on \( E^{2n} \) such that
\[
\omega_1(\xi, \eta) = 0 \Rightarrow \omega_2(\xi, \eta) = 0.
\]
Prove that \( \omega_2 = c\omega_1 \) for some \( c \neq 0. \)

If \((E, \omega)\) is a symplectic vector space, we denote by \( \text{Sp}(E) = \text{Sp}(E, \omega) \) the group of linear isomorphisms of \( E \) which preserve \( \omega \). For the standard symplectic vector space \( \mathbb{R}^{2n} \) we use the notation \( \text{Sp}(2n) = \text{Sp}(\mathbb{R}^{2n}) \).

3. Let \((E, \omega)\) be a symplectic vector space and let \( \alpha, \beta \in E^* \setminus 0 \) such that \( \omega^*(\alpha, \beta) = 0 \).

Prove that for any \( c \neq 0 \) there exists \( A \in \text{Sp}(E) \) such that \( A^*\alpha = c\alpha \) and \( A^*\beta = c\beta. \)

4. Let \( A \in \text{Sp}(2n) \) and \( r > 0 \). Prove that there exists a Hamiltonian diffeomorphism \( \varphi \in \text{Ham}(\mathbb{R}^{2n}) \), supported in the ball of radius \( r \), such that \( \varphi(0) = 0 \) and \( D_0\varphi = A \). You will need the fact that the group \( \text{Sp}(2n) \) is connected (for example, since it deformation retracts onto \( U(n) \), which can be seen using polar decomposition and the fact that \( \text{Sp}(2n) \cap O(2n) = U(n) \)).

5. Complete the proof of the lemma: let \( x \in M \) and set \( y = f(x) \). Show (using exercise 2) that we can find \( \alpha, \beta \in T^*_y M \) such that
\[
\omega^*_y(\alpha, \beta) = 0 \quad \text{and} \quad \omega^*_x(f^*\alpha, f^*\beta) > 0.
\]
Now rescale \( \alpha, \beta \) so that
\[
\omega^*_y(\alpha, \beta) = 0 \quad \text{and} \quad \omega^*_x(f^*\alpha, f^*\beta) = 1.
\]
Now choose compactly supported smooth \( F_0, G_0 \) such that \( d_yF_0 = \alpha \) and \( d_yG_0 = \beta \). Set \( K = ||\{F_0, G_0\}|| \). Use exercises 3 and 4 to construct \( \varphi \in \text{Ham}(M, \omega) \) such that
\[
\varphi(y) = y, \quad \varphi^*\alpha = \sqrt{2K}\alpha, \quad \varphi^*\beta = \sqrt{2K}\beta.
\]
Finally, define
\[
F = \frac{1}{\sqrt{2K}}\varphi^*F_0, \quad G = \frac{1}{\sqrt{2K}}\varphi^*G_0,
\]
and check that \( F \) and \( G \) satisfy the requirements of the lemma.

Hints
1. For part (ii): Prove that \( I_{\omega^*} = -I_{\omega}^{-1} \), hence \( (\omega^*)^* = \omega \), so suffices to prove the \( \Rightarrow \) direction. For that direction, prove first that the condition implies that \( I_{\omega_2}^{-1} = \frac{1}{c} \cdot I_{\omega_1}^{-1}. \)

2. Consider a symplectic basis for \( \omega_1. \)

3. Prove that \( \alpha, \beta \) can be completed to a symplectic basis of \( E^* \). Define \( A \) on its dual basis of \( E. \)

4. Consider a path in \( \text{Sp}(2n) \) connecting \( I \) and \( A \). Show that this path is in fact the Hamiltonian flow of some (non compactly supported) time-dependent function, and use an appropriate cut-off function.

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