Function Theory on Symplectic Manifolds - problem session 3

**Spoiler alert:** Hints are provided in the end. Try to think about the exercises before looking at them.

The Calabi homomorphism

Let \((M,\omega)\) be a symplectic manifold with exact symplectic form, \(\omega = d\lambda\).

1. Prove that the Calabi homomorphism is well-defined, i.e., that \(\text{Cal}(f)\) depends only on \(f \in \text{Ham}(M,\omega)\) and not on the Hamiltonian function generating the flow. Proceed in the following steps:

   (i) First prove that there is a compactly supported function \(S \in C^\infty_c(M)\) such that

   \[
   f^\ast \lambda - \lambda = dS.
   \]

   Show that this \(S\) is unique.

   (ii) Let \(f \in \text{Ham}(M,\omega)\) be the time-one map of the Hamiltonian flow generated by the normalized Hamiltonian \(F_t\). Prove that

   \[
   \text{Cal}(f) = \int_0^1 dt \int_M F_t \omega^n = -\frac{1}{n+1} \int_M S \omega^n.
   \]

   You will need to use the fact that the map

   \[
   i_\xi : \Omega^*(M) \to \Omega^{*-1}(M)
   \]

   is a (graded) derivation, that is

   \[
   i_\xi (\alpha \wedge \beta) = (i_\xi \alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge i_\xi \beta.
   \]

   (iii) Deduce that \(\text{Cal}(f)\) is well-defined.

2. Prove that \(\text{Cal} : \text{Ham}(M,\omega) \to (\mathbb{R},+)\) is a surjective group homomorphism.
The median quasi-state

Let \( \zeta \) denote the median quasi-state on \( S^2 \).

3. Verify the monotonicity and quasi-linearity axioms on Morse functions.

4. Check that the monotonicity and normalization axioms, together with
\[ \zeta(F + c) = \zeta(F) + c \text{ for } F \text{ Morse and } c \in \mathbb{R}, \]
imply that \( \zeta \) is 1-Lipschitz in the uniform norm. Therefore it extends to \( \zeta : C(M) \to \mathbb{R} \). Check that this extension is still monotone.

5. Prove that, for a Morse function \( F \in C^\infty(M) \) and continuous function \( u : \mathbb{R} \to \mathbb{R} \),
\[ \zeta(u(F)) = u(\zeta(F)). \]
Deduce that if \( H \in C(M) \), and \( F = u(H) \), \( G = v(H) \) for continuous \( u, v : \mathbb{R} \to \mathbb{R} \), then \( \zeta(F + G) = \zeta(F) + \zeta(G) \). In other words, \( \zeta \) is linear on the closed subalgebra of \( C(M) \) generated by \( H \). This property is called topological quasi-linearity. A functional \( C(M) \to \mathbb{R} \) satisfying the normalization, monotonicity\(^1\) and topological quasi-linearity axioms is called a topological quasi-state (introduced by Aarnes). On closed surfaces, any topological quasi-state is in fact a symplectic quasi-state (Entov-Polterovich), and hence we’ve shown that \( \zeta \) is a symplectic quasi-state.

Hints

1. (i) Suppose that \( \{f_t\} \) is a path in \( \text{Ham}(M, \omega) \) with \( f_0 = 1 \) and \( f_1 = f \). Then
\[ f^* \lambda - \lambda = \int_0^1 \frac{d}{dt} f_t^* \lambda \ dt. \]
Use Cartan’s formula for the Lie derivative: if \( X \) is a vector field and \( \alpha \) a differential form,
\[ \mathcal{L}_X \alpha = di_X \alpha + i_X d\alpha. \]
Additionally, use that if \( \varphi_t \) is the flow of the vector field \( X_t \), then
\[ \frac{d}{dt} \varphi_t^* \alpha = \varphi_t^* \mathcal{L}_X \alpha. \]
Convince yourself you can interchange \( d \) with \( \int dt \).
(ii) \( \omega^n \) is a top form, so \( \alpha \wedge \omega^n = 0 \) for any differential form \( \alpha \). Apply \( i_\xi \) to both sides, for the right \( \xi \) and \( \alpha \).

2. Use the cocycle (product Hamiltonian) formula.

3. For quasi-linearity recall that if \( \{F, G\} = 0 \) then \( G \) is constant along the Hamiltonian flow of \( F \). Use this to show that \( G \) is constant on level sets of \( F \) (this is true only in dimension 2!), and hence descends to the Reeb graph of \( F \).

5. For the first part, \( u(F) \) descends to the Reeb graph of \( F \). For the second, since \( \zeta \) is Lipschitz, you may assume that \( H \) is a smooth Morse function.

\(^1\)In fact, the weaker positivity axiom is required, but monotonicity follows by a theorem of Aarnes.