Function Theory on Symplectic Manifolds - problem session 3

**Spoiler alert:** Hints are provided in the end. Try to think about the exercises before looking at them.

1. Define an action of $\mathbb{T}^n$ on $\mathbb{C}^n$ by
   $$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (e^{2\pi it_1}z_1, \ldots, e^{2\pi it_n}z_n).$$

   Find the associated Poisson commutative subspace $\mathcal{A} \subset C^\infty(\mathbb{C}^n)$.

2. Define an action of $\mathbb{T}^n$ on $\mathbb{C}P^n$ by
   $$(t_1, \ldots, t_n) \cdot [z_0 : z_1 : \ldots : z_n] = [z_0 : e^{2\pi it_1}z_1 : \ldots : e^{2\pi it_n}z_n].$$

   (i) Find the associated Poisson-commutative subspace $\mathcal{A} \subset C^\infty(\mathbb{C}P^n)$. You may want to use (or prove) the fact that the map
   $$\{[z_0 : z_1 : \ldots : z_n] \in \mathbb{C}P^n : z_0 \neq 0\} \to D^{2n}(\frac{1}{\pi}), \quad [z_0 : z_1 : \ldots : z_n] \mapsto (z_1, \ldots, z_n)/||z||^2,$$
   is a symplectomorphism. Here $D^{2n}(\frac{1}{\pi}) \subset \mathbb{C}^n$ is the open $\frac{1}{\pi}$-disc in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, with the standard symplectic form (compare with question 1.iv from the first problem session), and $||z||^2 = |z_0|^2 + \cdots |z_n|^2$.

   (ii) Prove that the image of the moment map $\Phi : \mathbb{C}P^n \to \mathbb{R}^n$ is the standard $n$-simplex $\Delta \subset \mathbb{R}^n$.

   (iii) Prove that the preimage of the barycentre of $\Delta$ is a stem.

3. Let $\zeta : C^\infty(M) \to \mathbb{R}$ be a symplectic quasi-state, and let $\tau$ be the associated quasi-measure. Let $X \subset M$ such that $\tau(X) = 1$. Prove that if $F \in C^\infty(M)$ has $F|_X \equiv c$ then $\zeta(F) = c$.

**Hints**

1. Start with the case $n = 1$; recall question 1.iv from the first problem session.
2. (i) Use the previous question.

(iii) Prove that permutations of homogeneous coordinates on $\mathbb{C}P^n$ are Hamiltonian
diffeomorphisms (note that $H^1(\mathbb{C}P^n; \mathbb{R}) = 0$). Consider their effect on $\Delta$, and
hence on the fibers of $\Phi$.

3. Reduce to the case $c = 0$ and $F \geq 0$. 