Series Analysis of Kosterlitz-Thouless Transitions

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Summary

- Potts models: standard and planar
- Planar Potts
- Kosterlitz-Thouless Transitions
- 6-state model: Monte Carlo and high-field
- 6-state model: Low-T series by finite lattice method
- Two-parameter 5-state model: series analysis and phase-space

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Potts Models

Potts model variables, $x_j$, at each lattice site take one of $q$ values: 1, 2, ... $q$, generalising $q = 2$ (Ising) case. For $q > 3$ two natural generalisations (with scalar product interactions):

**Standard**: The $x_j$ are vectors in $(q - 1)$-D space. Only two distinct pair-energies: $x_j, x_k$ are the same or different. Widely-studied, generalised to non-integer $q$. First-order transition for $q > 4$ on 2-D lattices (Baxter).

**Planar**: The $x_j$ are treated as vectors in 2-D space. Pair energy $\propto -\cos(2\pi(x_j - x_k))$.

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Planar Potts models

The $q$-state Planar Potts model has low-T ordered state. Ordered state disappears as $q \to \infty$ (planar rotator).

Planar rotator has low-T spin-wave states (as does larger-$q$ planar Potts, above ordered phase).

Disordered phase at high-temperatures.

Expect 3 phases for $q$ sufficiently large, in practice for $q \geq 5$. For 2005 Stat Mech meeting
Spin-wave states

Primary excitation of planar rotator (and planar Potts analogues) is spin-wave. Spin-waves preclude long-range order in planar rotator (Mermin-Wagner theorem). Spin-wave spectrum exhibits algebraic decay of correlations. Bound vortex-pairs are additional excitations, ultimately leading to breakdown of spin-wave phase and a transition to a disordered state.
K-T transitions

Phase between ordered and disordered phases has long-range algebraic decay of correlations (with varying exponent $\eta$). Termed ‘massless’ phase, based on particle physics identification of correlation length with inverse mass. Onset of massless phase has expected critical behaviour with

$$M \sim \exp\left(-\frac{c}{\sqrt{T_c - T}}\right)$$

known as a Kosterlitz-Thouless transition.

In geology, K-T transition means Cretaceous-Tertiary, time of extinction of dinosaurs, with iridium-rich layer from asteroid impact.
Monte Carlo: 6-state planar

Simulations for 0.45, 0.47, 0.50, 0.58, 0.62, 0.70.
c.f. $x_1 \approx 0.49, x_2 \approx 0.60$ series: Barber and Enting (1981).

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Barber and Enting (1981) analysed massless phase \( T_1 < T < T_2 \) using high-field series for varying \( \delta \) where \( M \sim H^{1/\delta} \) as \( H \to 0 \), expecting \( \delta = 1 \) for \( T > T_2 \) and \( \delta = \infty \) for \( T < T_1 \) (actually diagnosed by poor convergence of estimates). Hyperscaling \( (\delta - 1)/(\delta + 1) = (2 - \eta)/d \).

Series to \( \mu^9 \) (\( \mu = \exp(-H/k_BT) \)) from code method (partial generating functions: from sublattice summations).

Low temperature series for \( M \) to \( x^{16} \) too short to analyse (and wrong). With extension (2005) to \( x^{31} \), analysis is still problematic.

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Finite Lattice Method

Expansions start (formally or actually) from partition function $Z$, or limit of $Z^{1/N}$, often as multi-variate series so derivatives can be taken. Approximations from inclusion-exclusion relations extract $Z^{1/N}$ behaviour from finite rectangular lattices with $Z_{m,n}$ calculated by transfer matrix techniques: e.g.

$$Z^{1/N} \approx \frac{[Z_{m+1,m+1}Z_{m,m}]}{[Z_{m+1,m}Z_{m,m+1}]}$$

More efficient cutoff uses $Z_{m,n}$ with $m + n \leq k$. 

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Order parameter series extended to $x^{41}$. Series inconsistent with power-law. 

$$M \sim \exp(-c/\sqrt{x_c - x})$$ implies 

$$X = \ln M \sim (x_c - x)^{-0.5}$$

Fit differential approximants to series $x^{-4}X$

No indication of confluent singularity.

<table>
<thead>
<tr>
<th></th>
<th>$x_c$</th>
<th>exponent</th>
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<tbody>
<tr>
<td>1st order DA</td>
<td>0.4881 ± 0.0006</td>
<td>0.55 ± 0.05</td>
</tr>
<tr>
<td>2nd order DA</td>
<td>0.4888 ± 0.0008</td>
<td>0.50 ± 0.05</td>
</tr>
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Padé approximants estimate $c = 0.02750$

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• Low-T series for $\chi$ to order 35 — more field terms, so fewer temperature terms.

• Use known $x_c$ from $M$.

• Not a conventional singularity

• $\chi \sim \exp\left(\frac{A}{(x_c - x)^\alpha}\right)$

• estimate from DAs to $\ln \chi$ suggest $\alpha \approx 0.9$

• probably $\alpha = 1$ ?????
Phase space of 5-state models

Rotation/reflection symmetries imply 2 possible pair energies (relative to $x_j = x_k$). For $|x_j - x_k| = 1$ or 2, weights are $z_1$ and $z_2$. If energies sufficiently different, expect 2 phase transitions (Wu, Cardy). High-T vs low-T (and associated transition lines) related by duality (Wu).

Solution by Fateev and Zamolodchikov may mark bifurcation point.

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Analysis: 5-state models

Low-$T$ series for order parameter using FLM expanded in $x$ with $z_1 = x^a$, $z_2 = x^b$.

$M$ to order $x^{41}$ for $a/b = 1/2, 1/3, 1/4$.

$$M \sim \exp\left(-c/\sqrt{x_c - x}\right) \text{ implies } X = \frac{d}{dx} \ln M \sim (x_c - x)^{-1.5}$$

Fit differential approximants (DAs) to series $X$

$$P_2(x)\frac{d^2}{dx^2}X + P_2(x)\frac{d}{dx}X + P_0(x)X = 0$$

Roots of $P_2$ give $x_c$ and indicial equation gives exponent (-1.5 expected) (Guttmann 1989).

Poor convergence, especially if $a/b$ small.
Results: 5-state models

Exponent vs critical point

Exponent vs 1/number of terms fitted

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Series: 5-state models

Low-$T$ expansions for order parameter using FLM. Expanded in $x$ with $z_1 = x^a$, $z_2 = x^b$ ($a$, $b$ small integers).

Estimated critical points for $a/b = 1/2, 1/3, 1/4$ suggest massless phase is narrow for $q = 5$.

Paths of fixed $a/b$ dotted.

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Implications

Kosterlitz-Thouless Transitions can be successfully studied by series analysis of favourable cases. Prediction of $-\frac{1}{2}$ exponent in exponential confirmed. It is likely to be hard to confirm the role of the Fateev-Zamolodchikov point.
Possible future directions

• Longer low-$T$ series;
• Revisit high-field series analysis (as per Barber and Enting)
• More insight into structure of singularity:
  $\exp(-c/\sqrt{T_c - T})$ vs $\beta (T_c - T)^\beta \exp(-c/\sqrt{T_c - T})$
• Cases that are easier (for series): $Z6$ to $Z5$ (to Ashkin-Teller?).