Critical behavior of the three-dimensional contact process

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I determine the critical behavior of a nonequilibrium three-dimensional lattice model exhibiting a phase transition to an absorbing state. I study the model in the vicinity of the critical point, and in the subcritical region, via time-dependent Monte Carlo simulations. The method used in the subcritical region is very efficient. The results for the directly measured critical exponents, $\nu=1.11 \pm 0.01$, $\eta=0.114 \pm 0.004$, and $z=1.052 \pm 0.003$, are consistent with those of directed percolation. $\delta=0.732 \pm 0.004$ is obtained from the hyperscaling relation $4\delta+2\eta=\tilde{d}$, and $\beta=0.813 \pm 0.011$ from $\beta=\nu\delta$. These results are the most precise so far for a three-dimensional model with directed percolation critical behavior.

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Several models proposed in the study of various physical, chemical, and biological systems [1,2] have been shown to exhibit nonequilibrium phase transitions and critical points. A common feature of many models is that they contain a single component and evolve according to a Markov process governed by local, intrinsically irreversible transition rules; such models are collectively known as interacting particle systems [3]. Examples are the contact process [4], a model for the spread of an epidemic, Schlögl's first and second model [5] for autocatalytic chemical reactions, surface reaction models [6,7], directed percolation (DP) in $d+1$ dimensions [8,9], and Reggeon field theory [10]. These models exhibit a continuous phase transition from an active steady state to a unique absorbing state. Results from computer simulations [6, 11, 12], series expansions [6, 10, 13, 14], and field-theoretic renormalization-group (RG) calculations [15–18] show that these and other models [19–22] exhibit the same critical behavior and thus belong to the same universality class. So presently there is substantial evidence in favor of the conjecture [15,17,18] that the critical behavior of directed percolation or Reggeon field theory is generic to models with a scalar order parameter exhibiting a continuous phase transition to a unique absorbing state.

Precise estimates for the critical exponents characterizing the DP universality class have been obtained for one- [10,13,23] and two-dimensional [9,10] systems. The exponents for three-dimensional (3D) systems are not known very accurately, although some results have been obtained from renormalization-group calculations [15], series expansions [24], and computer simulations [19]. I study the 3D contact process, using time-dependent Monte Carlo simulations and obtain the most precise estimates so far for the critical exponents.

The contact process, originally proposed by Harris [4] as a model for the growth of an epidemic, is a nonequilibrium stochastic lattice model with a single component. I study the contact process on a simple-cubic lattice, where each site can be either vacant or occupied by a single particle. The evolution is a Markov process consisting of a sequence of elementary transitions, each involving a single process at a randomly chosen site (sequential dynamics). Particles are annihilated spontaneously with probability $p$ independent of the state of other sites. With probability $1-p$ a particle can create another particle at a randomly chosen nearest neighbor (provided it is vacant). As there is no spontaneous creation of particles, the system can become trapped in the vacuum state, which is an absorbing state for the Markov process. When the annihilation probability is too high the system always enters the absorbing state, but for sufficiently small values the system has an active state with a nonzero average particle concentration $\rho$. The system exhibits a continuous phase transition from the active state to the absorbing state at a critical value $p_c$. The steady-state concentration of particles (which is the appropriate order parameter) decays asymptotically as $\rho \propto (p_c - p)^{\delta}$ when $p \to p_c$.

There are two primary ways of studying the critical behavior of systems exhibiting a phase transition to an absorbing state via Monte Carlo simulations. The "traditional" approach is to study the steady-state behavior, e.g., the concentration of particles and fluctuations. Determining critical behavior from steady-state simulations is often very hard due to large fluctuations, slow relaxation, finite-size effects, and problems in locating the critical point. Some of these difficulties can be at least partially circumvented by a finite-size scaling analysis [19]. But even in this case the typical precision, in the estimates for the critical exponents, are 5%–10%. The method used in this work is known as time-dependent Monte Carlo simulations [11]. Numerous earlier studies [11,12,17,21,22] have revealed that this is a very efficient method for locating critical points and determining critical exponents for systems with an absorbing state; often the estimates are an order of magnitude or so better than steady-state results.

The general idea underlying time-dependent simulations is to study the evolution of a state close to the absorbing state. In the simulations presented here I always started, at $t=0$, with a single particle at the origin. As the number of particles is very low, an efficient algorithm can be devised by keeping a list of particles.
mentary step a particle from the list is chosen randomly. The particle is removed with probability \( p \), otherwise a nearest neighbor is chosen randomly and if it is vacant a new particle is placed there. After each attempted change the time is incremented by \( 1/n(t) \), where \( n(t) \) is the number of particles at time \( t \). Thus one time step equals one attempted update per lattice site. I made a number, \( N_S \), of independent runs for various values of \( p \). Each run had a maximal duration of \( t_M \) time steps, but most of the runs stopped earlier because all particles had disappeared. I measured the survival probability \( P(t) \), the average number of particles \( \bar{n}(t) \), and the average mean-square distance of spreading \( \bar{R}^2(t) \).

Following Grassberger and de la Torre [11], I review the scaling behavior of models with a continuous transition to an absorbing state. According to the scaling hypothesis, one expects that functions of \( x \), \( t \), and \( \Delta \) (where \( \Delta = \delta t \)) depends on these variables only through \( x^2/t^\delta \) and \( \Delta t^{1/\tau} \) at some time power of \( x^2 \), \( t \), or \( \Delta \).

For the particle density one expects

\[
\rho(x,t) \propto t^{-\delta} F(x^2/t^\delta, \Delta t^{1/\tau}),
\]

and for the probability of survival, i.e., the probability that the system has not entered the absorbing state at time \( t \), one expects

\[
P(t) \propto t^{-\delta} \phi(\Delta t^{1/\tau}).
\]

From Eq. (1) we find for the average number of particles \( \bar{n}(t) \) and the mean-square distance of spreading \( \bar{R}^2(t) \):

\[
\bar{n}(t) = \int d^d x \rho(x,t) \propto t^\eta \phi(\Delta t^{1/\tau})
\]

and

\[
\bar{R}^2(t) = \frac{1}{\bar{n}(t)} \int d^d x x^2 \rho(x,t) \propto t^{\gamma} \phi(\Delta t^{1/\tau}).
\]

In the subcritical region \( (p > p_c) \) the correlations are short ranged and one therefore expects \( P(t) \) and \( N(t) \) to decay exponentially. From Eq. (3) one sees that this can only be the case if \( f(y) \propto (\Delta t^{1/\tau})^\gamma \), for \( y \to -\infty \), thus yielding

\[
\bar{n}(t) \propto (\Delta t^{1/\tau})^{-\eta \gamma} e^{-b(\Delta t^{1/\tau})}, \text{ for } t \to -\infty.
\]

From Eqs. (2)-(4) we see that if \( \phi(y) \), \( f(y) \), and \( g(y) \) are nonsingular at \( y = 0 \) the behavior of \( P(t) \), \( \bar{n}(t) \), and \( \bar{R}^2(t) \) as \( t \to \infty \) at \( p_c \) determines the critical exponents \( \delta \), \( \eta \), and \( \tau \):

\[
P(t) \propto t^{-\delta},
\]

\[
\bar{n}(t) \propto t^\eta,
\]

\[
\bar{R}^2(t) \propto t^\gamma.
\]

Away from the critical point, the behavior departs from a power law. In plots of \( \ln P(t) \), \( \ln \bar{n}(t) \), and \( \ln R^2(t) \) vs \( \ln t \) we should see asymptotically a straight line at \( p = p_c \). The curves for \( P(t) \) and \( \bar{n}(t) \) often have distinct positive (negative) curvature in the supercritical (subcritical) regime. This permits one to obtain a precise estimate for \( p_c \). The asymptotic slopes of the curves at the critical point define the dynamic critical exponents \( \delta \), \( \eta \), and \( \gamma \). The power-law behavior at criticality is often modified by corrections to scaling, so that, e.g., \( P(t) \) is more accurately given by

\[
P(t) \propto t^{-\delta} (1 + at^{-1} + bt^{-\delta} + \cdots).
\]

Similar expressions are expected to hold for \( \bar{n}(t) \) and \( \bar{R}^2(t) \). In order to estimate the critical exponents it is very useful to look at the local slopes:

\[
-\delta = \frac{\ln[P(t)/P(t/m)]}{\ln(m)}
\]

and similarly for \( \eta \) and \( \tau \). In the present work \( m = 5 \). In plots of the local slopes versus \( 1/t \) the critical exponents are given by the interception with the \( y \) axis.

The results for \( P(t) \), \( \bar{n}(t) \), and \( \bar{R}^2(t) \) are shown in Fig. 1. In this case \( N_S = 1 \times 10^6 \) and \( t_M = 1000 \). The corresponding local slopes \( -\delta(t) \), \( \eta(t) \), and \( \tau(t) \) are shown in Fig. 2. From these results, especially the curves for \( \bar{n}(t) \) and \( \eta(t) \), I estimate that \( p_c = 0.43162(2) \), where the number in parentheses is the estimated uncertainty. Esti-

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**FIG. 1.** Plots of the survival probability (upper panel), the average number of particles (middle panel), and the average mean-square distance of spreading (lower panel). Each panel contain five curves with, from top to bottom, \( p = 0.43158 \), \( 0.43160 \), \( 0.43162 \), \( 0.43164 \), and \( 0.43166 \).
mates for the critical exponents are listed in Table I. The curves for $-\delta(t)$ exhibit pronounced curvature, making a direct estimation somewhat difficult. I have therefore used the hyperscaling relation \[ zd = 4\delta + 2\eta, \] where \( d \) is the dimension of the system, in estimating \( \delta \).

From Eq. (5) one sees that the decay constant \( \xi \), governing the long-time behavior of \( \bar{n}(t) \) in the subcritical regime, is proportional to \( (p - p_c)^{\gamma} \). I have simulated the model in the subcritical region over a wide range of values for the annihilation probability \( p \); the values of \( p - p_c \) span almost 3 decades. The number of independent runs \( N_S \) and the maximal duration \( t_M \) of each run varied from \( N_S = 5 \times 10^5 \), \( t_M = 3000 \) closest to \( p_c \), to \( N_S = 1 \times 10^7 \), \( t_M = 100 \) furthest away from \( p_c \). In plots of \( \ln \bar{n}(t) \) vs \( t \) one should see asymptotically a straight line with slope \( \xi \). Again one can obtain better results by studying the local slope of the curves. In this case I define the local slope, \( \xi(t) \), at time \( t \) as the predicted slope obtained by a least-squares fit to a straight line of \( \ln \bar{n}(t) \) with \( t \) ranging from \( t - t_i \) to \( t + t_i \). In a plot of \( \xi(t) \) vs \( t \) one should see a constant plateau, with possible deviations for short times (the asymptotic behavior is not reached yet) and long times (poor statistics). Figure 3 shows the local slope \( \xi(t) \) for various values of \( p \). Figure 4 shows a log-log plot of the estimated decay constants \( \xi \), as obtained from the data for \( \bar{n}(t) \) versus the distance, \( p - p_c \), from the critical point. The value used for \( p_c \) was 0.43162. Only for values of \( p - p_c > 0.02 \) does \( \xi \) begin to deviate significantly from the scaling behavior. The slope of the curve yields the value \( \nu = 1.11(1) \). From the scaling relations [11], \( \beta = \nu \delta \), I find that \( \beta = 0.813(11) \).

![FIG. 3. Local slopes \( \xi(t) \) as obtained from the data for \( \bar{n}(t) \) for various values of \( p \) in the subcritical regime. \( \xi(t) \) has been scaled by a factor \( 10^3 \).](image)

![FIG. 4. The decay constant \( \xi \) governing the exponent decay of the number of particles in the subcritical region as a function of the distance from the critical point. The value of \( p_c \) is 0.43162 and the slope of the line is \( \nu = 1.11 \).](image)

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<tr>
<td>( \eta )</td>
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<td>( z )</td>
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<tr>
<td>( \nu )</td>
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<td>1.10(10)</td>
<td>1.102</td>
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<tr>
<td>( \beta )</td>
<td>0.813(11)</td>
<td>0.79(7)</td>
<td>0.822</td>
<td>0.82(3)</td>
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The results for the critical exponents of the contact process are in good agreement with other simulation results [19], results from series expansions [24], and field-theoretic renormalization-group calculations [15], as can be seen from Table I. The results are also consistent with the general trend of the RG predictions [15], namely, that RG calculations overestimate $\beta$, $\delta$, and $z$ and underestimate $\nu$ and $\eta$.

I have studied a simple 3D nonequilibrium lattice model, the contact process, belonging to the universality class of directed percolation. From time-dependent Monte Carlo simulations in the vicinity of the critical point I have obtained accurate estimates for the location of the critical point and the critical exponents $\eta$ and $z$. The critical exponent $\delta$ is then determined via the hyperscaling relation $zd = 4\delta + 2\eta$. Simulations of the model in the subcritical region yielded an accurate estimate for the critical exponent $\nu$ and, thus, through the scaling relation $\beta = \nu \delta$, a precise estimate for the order-parameter exponent $\beta$. The method applied in this work to estimate $\nu$ has proven to be very efficient. The estimates for the critical exponents are the most precise so far for any 3D model belonging to the universality class of directed percolation.

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