**Temporally Disordered Bond Percolation on the Directed Square Lattice**

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Simple models of directed bond percolation with temporal disorder are introduced and studied via series expansions and Monte Carlo simulations. Series have been derived for the percolation probability on the directed square lattice. Analysis of the series revealed that the critical exponent $\beta$ and critical point $p_c$ change continuously with the strength of the disorder. Monte Carlo simulation confirmed the continuous change of critical exponents. Estimates for the temporal correlation-length exponent $\nu_\parallel$ for weak disorder showed that $\nu_\parallel < 2$ in apparent violation of the Harris criterion.

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Models exhibiting critical behavior similar to directed percolation (DP) are encountered in a wide variety of problems such as fluid flow in porous media, chemical reactions, population dynamics, catalysis, epidemics, forest fires, biological and even galactic evolution. Bond percolation on the directed square lattice can be seen as a one-dimensional stochastic cellular automaton in which the preferred direction $t$ is time. DP is thus a model for a branching process in which a site $x$ occupied at time $t$ may give rise to zero or one offspring on each of the sites $x \pm 1$ at time $t + 1$. The evolution of the model is determined by the conditional probabilities $P(\sigma_x | \sigma_i, \sigma_r)$ of finding the site $(x,t)$ in state $\sigma_x$ given that the sites $(x-1,t-1)$ and $(x+1,t-1)$ were in states $\sigma_i$ and $\sigma_r$, respectively, with $\sigma_i = 1$ if site $i$ is occupied and 0 otherwise. One has a very free hand in choosing these probabilities as long as one respects conservation of probability, $P(1|\sigma_i, \sigma_r) = 1 - P(0|\sigma_i, \sigma_r)$, and the condition $P(1|0,0) = 0$ which ensures that the process has an absorbing state. Bond percolation corresponds to the choice

$$P(0|\sigma_i, \sigma_r) = (1 - p)^{\sigma_i + \sigma_r}. \quad (1)$$

The behavior of DP is controlled by the branching probability $p$. When $p$ is smaller than a critical value $p_c$ the branching process always dies out, while for $p > p_c$ there is a nonzero probability $P(p)$ it will survive indefinitely. At $p_c$ the survival probability vanishes as a power law,

$$P(p) \propto (p - p_c)^\beta, \quad p \rightarrow p_c^+. \quad (2)$$

Studies have generally been limited to cases in which the transition probabilities are independent of both $x$ and $t$. However, in many real physical situations this idealization is unrealistic. Often some degree of randomness is present; e.g., in a real catalyst one might expect that not all sites on the surface have the same reaction rates, leading one to study models with spatial disorder. Apart from the inherent theoretical interest, models with temporal disorder, i.e., where the transition probabilities change randomly with $t$ (but are independent of $x$), could be of interest in the study of percolation through media with layers of different permeability or the growth of a population in which the reproductive probability changes randomly from generation to generation, e.g., due to changes in the environment.

The effect of quenched spatial disorder on the critical behavior of directed percolation was first considered by Kinzel [1] and studied numerically by Noest [2]. Recently Moreira and Dickman [3] studied the contact process (which essentially is a continuous time version of DP) on site diluted square lattices. The main conclusion from the numerical work is that quenched spatial disorder results in a marked change in the critical behavior. However, the behavior is independent of the strength of the disorder (concentration of diluted sites), and thus there seems to be two universality classes, one for the pure undiluted model and another for disordered (diluted) models. This is fully consistent with the expectations following the Harris criterion [4], which states that disorder should change the critical exponents if $dv_\perp \leq 2$, where $d$ is the spatial dimensionality and $v_\perp$ the spatial correlation-length exponent of the pure model.

Here I study a simple generalized model of directed bond percolation which incorporates temporal disorder. This is achieved by allowing spreading from some rows to be deterministic, which corresponds to setting $p = 1$ in the transition probabilities. Each row has a probability $\alpha$ of being "deterministic" independent of other rows. Otherwise spreading from a row takes place with the probabilities of Eq. (1). Kinzel [1] argues that the Harris criterion remains valid, one just has to replace $dv_\perp$ with $\nu_\parallel$. Since $\nu_\parallel = 1.7338$ for two-dimensional DP this would indicate that temporal disorder should change the critical exponents. I have studied this model via series expansions for $P(q)$ ($q = 1 - p$ is the appropriate expansion variable) and Monte Carlo simulations. The major result is that the series analysis and simulations show that the critical exponents change continuously with $\alpha$. This is in sharp contrast to the situation with spatial disorder. Furthermore, estimates for the temporal correlation-length exponent $\nu_\parallel$ for weak disorder show that $\nu_\parallel < 2$ in apparent violation of the Harris criterion.
Consider the calculation of the series expansion of the percolation probability for the pure model \([5,6]\) on a square lattice oriented as in Fig. 1. Bond percolation may also be interpreted as a static model in which each bond is independently present with probability \(p\). Two sites are connected if one can find a path through occupied bonds, while following the allowed directions. For an infinite system, when \(p \geq p_c\), there is an infinite cluster spanning the lattice. The order parameter \(P(q)\) is the probability that a given site belongs to the infinite cluster. The generalization to the temporally disordered model is straightforward. One has to calculate the polynomials \(P_{n,m}(q)\), which is the sum of all finite-lattice configurations of \(n\) rows with \(m\) deterministic rows. The contribution of each finite lattice is calculated using the algorithm of \([5,6]\) with the obvious generalizations necessary to accommodate deterministic rows. \(P_n(q)\) is obtained as

\[
P_n(q) = \sum_{m=0}^{n} \alpha^m(1 - \alpha)^{n-m} P_{n,m}(q),
\]

where the prefactor to \(P_{n,m}\) accounts for the probability of finding a configuration with \(m\) deterministic rows. As was the case for the pure system the coefficients of \(q^k\) for \(k \leq n + 1\) are constant, i.e., they do not change as more rows are included. One can therefore define

\[
P(q) = \lim_{n \to \infty} P_n(q),
\]

and obtain the first \(n + 2\) terms from \(P_n(q)\).

For the pure system Baxter and Guttmann \([5]\) demonstrated that the series for \(P(q)\) can be extended considerably by determining correction terms to \(P_n(q)\),

\[
P_n - P_{n+1} = q^{n+2} \sum_{r \geq 0} q^r d_{n,r}.
\]

We shall call \(d_{n,r}\) the \(r\)th correction term. Obviously if one can find formulas for \(d_{n,r}\) for all \(r \leq k\) then one can use the series coefficients of \(P_n(q)\) to extend the series for \(P(q)\) to order \(n + k + 2\) since

\[
a_{n+2+j} = a_{n,n+2+j} - \sum_{i=0}^{j} d_{n+k-i,j}
\]

for all \(j \leq k\), where \(a_i\) and \(a_{n,i}\) are the coefficients in \(P(q)\) and \(P_n(q)\), respectively. For the pure system the first correction term was conjectured to be the Catalan numbers \([5]\), \(d_{n,0} = (2n)!/[n!(n + 1)!]\). This conjecture has since been proved \([7,8]\). Baxter and Guttmann also found that the higher order correction terms can be expressed as linear functions of \(d_{n,0}\).

For the generalized model I find that the first correction term is given by a quite simple recurrence relation

\[
d_{n+2,0} = (1 - \alpha)[4 \alpha d_{n,0} + (2 + 4n)d_{n+1,0}]/(n + 3),
\]

with \(d_{0,0} = 1\) and \(d_{1,0} = 1 - \alpha\). Notice that for \(\alpha = 0\) this recurrence relation reduces to the one for the Catalan numbers. The higher order correction terms are obviously quite complicated, and I have not attempted to find general formulas. However, for rational values of \(\alpha\) the higher order correction terms can be expressed as

\[
d_{n,r} = \sum_{j=1}^{2r} (n^r a_{r,j} + b_{r,j}) d_{n+r-k,0},
\]

for \(n \geq r\). Since the coefficients in \(P_n\) are rational numbers (when \(\alpha\) is rational) the equations determining the coefficients \(a_{r,j}\) and \(b_{r,j}\) are easily solved exactly using a symbolic mathematical package. The rational series coefficients of \(P(q)\) can therefore also be calculated exactly. I have calculated \(P_n\) for \(n \leq 20\) and can find the formulas for the correction terms for \(r \leq 4\), which yields a series correct to order \(q^{26}\) for any given rational value of \(\alpha\).

These series were analyzed using Dlog-Padé approximants (see \([9]\) for a review), which yields accurate estimates for \(\beta\) and \(p_c\) as long as \(\alpha\) is not too close to 1. The main result of the analysis of \(P(q)\) is that both \(\beta\) and \(p_c\) change continuously with \(\alpha\). Figure 2 shows the estimates of \(p_c\), while Fig. 3 shows the estimates of \(\beta\), for different values of \(\alpha\). The continuous change of \(\beta\) is in sharp contrast to the situation for spatial disorder, while the change of \(p_c\) is to be expected.

In order to confirm the influence of temporal disorder on the critical behavior and obtain estimates for additional exponents I performed extensive Monte Carlo simulations for a few values of \(\alpha\). I followed the epidemic analysis approach of Grassberger and de la Torre \([10,11]\) and studied a large ensemble of independent realizations all starting at \(t = 0\) with a single occupied site at the origin. I measured the survival probability \(P(t)\), the mean number of occupied sites \(n(t)\), and \(R^2(t)\), the mean-square distance of spreading from the origin. The average was performed over 1000 configurations of

![FIG. 1. The directed square lattice with orientation given by the arrows.](Image)
deterministic/probabilistic rows, and for each configuration 1000 independent runs were simulated for a total of \(10^6\) samples. Each run had a maximal duration of up to 10,000 time steps. In a little more detail the Monte Carlo simulations proceeded as follows: (1) Each row was assigned a random number \(r_t\) chosen uniformly between 0 and 1. (2) The evolution from a single seed was simulated until the population died out or 10,000 time steps had elapsed. In each time step the occupied sites in row \(t\) were visited, and the neighboring sites in row \(t+1\) became occupied with probability 1 if \(r_t < \alpha\) (deterministic spreading) or else with probability \(p\) (probabilistic spreading). Once occupied, a site is marked to avoid double occupancy in the case that both its neighbors in deterministic spreading). Once occupied, a site is marked to avoid double occupancy in the case that both its neighbors in spreading). Once occupied, a site is marked to avoid double occupancy in the case that both its neighbors in deterministic spreading. The branching probability \(p\) is an external control parameter given initially as input [in this case I used the estimate \(p_c\) obtained from the series analysis]. Step 1 was performed 1000 times with step 2 performed 1000 times for each occurrence of step 1.

At the critical point \(p_c(\alpha)\) one expects \(P(t)\) to decay asymptotically as a power law with exponent \(\delta\), while \(n(t)\) and \(R^2(t)\) should increase as power laws with exponents \(\eta\) and \(z\), respectively [10,11]. The exponents are expected to satisfy the hyperscaling relation

\[
4\delta + 2\eta = dz.
\]

where \(d\) is the spatial dimension of the lattice perpendicular to the preferred direction; in this case \(d = 1\).

In Table I are listed exponent estimates for various values of \(\alpha\). The estimates for the pure system, \(\alpha = 0\), are from recent low-density series expansions [12]. The estimates of the critical point and the exponent \(\beta\) were obtained from the Dlog Padé analysis of the percolation probability series, while the estimates of \(\delta\), \(\eta\), and \(z\) were obtained from the asymptotic behavior as measured from the Monte Carlo simulations. The exponents \(\nu_{||}\) and \(\nu_{\perp}\) were obtained from the scaling relations \(\nu_{||} = \beta/\delta\) and \(z\nu_{||}/2\). For the purely deterministic system, \(\alpha = 1\), it is clear that the percolation probability (and the survival probability) equals 1 and therefore \(\beta = \delta = 0\). Similarly, it is easy to see from the definitions of \(n(t)\) and \(R^2(t)\) that \(\eta = 1\) and \(z = 2\). It then follows from the second of the scaling relations mentioned above that \(\nu_{||} = \nu_{\perp}\), and 2 is the obvious value.

The results from the simulations clearly confirm the continuous change of exponents from the purely probabilistic values at \(\alpha = 0\) to the purely deterministic values at \(\alpha = 1\). Note that the hyperscaling relation (9) holds among the measured exponents for all values of \(\alpha\). The exponents generally seem to approach the values of the fully deterministic model continuously and the change appears to be monotonic. However, the central estimates of the exponent \(\nu_{||}\) apparently exceed those of the fully deterministic model for large values of \(\alpha\). Because of the large error bars on the estimates of this exponent the results are compatible with a “true” exponent value below the deterministic limit. It should, however, be noted that if the application and arguments leading to the Harris criterion, as advocated by Kinzel [1], are correct, one would expect that \(\nu_{||} \geq 2\) for any value of \(\alpha > 0\). In particular, for small values of \(\alpha\), where the model clearly should represent a temporally disordered system, this should be the case. From Table I it is evident that for \(\alpha = 0.05\) and 0.25 the estimates for \(\nu_{||}\) are clearly smaller than 2. There are several possible explanations of this discrepancy:

1. The Harris criterion does not apply to this model.
2. The scaling relation used to derive the estimate for \(\nu_{||}\) is not valid for this model.
3. The exponents are estimated from a single source initial configuration (often called “dynamic” exponents). It is possible that “steady-state” exponents (typically measured from a finite size lattice initially fully occupied) are different. In the pure probabilistic model there is no such difference, but it has been found that for certain models (e.g., models with infinitely many absorbing states [13]) the dynamic exponents depend on the initial configuration and only under special circumstances do they agree with the steady-state exponents.

In order to further investigate temporally disordered models, and possibly confirm that the continuous change of exponents is not a mere artifact of the particular way in which I introduced disorder, I have used Monte Carlo
simulations to study a second model of temporally disordered directed percolation. In this second model spreading from a given row takes place with the transition probabilities of Eq. (1), but each row has its own branching probability \( p_t \) which is a random number chosen uniformly in the interval \( [p - \alpha, p + \alpha] \). The central value \( p \) of this interval is an externally supplied control parameter. For large \( \alpha \) the upper limit of the interval may exceed 1, and thus if \( p_t \geq 1 \) spreading from that particular row is deterministic. Again I used epidemic analysis with 1000 realizations of the disorder and 1000 samples simulated for each disorder configuration. So the algorithm is as described above, except that spreading is always probabilistic and performed with branching probability \( p_t \) from row \( t \). For a given value of \( \alpha \) Monte Carlo simulations were performed for various values of \( p \). As usual in the epidemic approach [10,11] the hallmark of a second-order phase transition is the onset of power-law behavior of \( P(t) \), \( n(t) \), and \( R^2(t) \) at the critical point \( p_c \). For \( p < p_c \) ( \( p > p_c \) ) log-log plots of these quantities exhibit a marked downwards (upwards) deviation from a straight line for large values of \( t \). This makes an accurate determination of \( p_c \) possible with the exponents estimated from the asymptotic slope of the “critical” curves. In Table II I have listed estimates of \( p_c \) and exponents \( \delta, \eta \), and \( z \) for a few values of \( \alpha \) obtained from Monte Carlo simulations. It is obvious that the exponents again change continuously as \( \alpha \) is increased. Again the hyperscaling relation (9) holds among the measured exponents for all values of \( \alpha \).

In summary, I find that the introduction of temporal disorder in directed percolation leads to critical exponents which change continuously with the strength of the disorder. This effect is independent of the specific way in which disorder is introduced. For small values of the disorder parameter \( \alpha \) the estimates of the critical exponent \( v || \) were clearly less than 2, in an apparent violation of the Harris criterion. I shall investigate this question in greater detail in future work.

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