Lecture 1. Elementary ideas of 3D integrability

What is the quantum tetrahedron equation and what are its benefits. Classical limit.

The tetrahedron equation is the three-dimensional generalization of the Yang-Baxter (triangle) equation. To link together two- and three-dimensional methods, we commence with a short reminder of 2D quantum inverse scattering method.

Sketch of OUISM and YBE. The most useful object in 2D is the $L$-operator

\[ L_{\alpha,i} \]

an operator acting in the tensor product of two spaces $V_\alpha \otimes V_i$

All spaces may be equipped by independent $\mathbb{C}$-valued (spectral) parameters. In what follows, we imply that an “auxiliary” space $V_\alpha$ is simple (e.g. dim $V_\alpha = 2$) while “quantum” space $V_i$ may be more complicated.

The alphabetic indices $\alpha, \beta, ...$ and numerical indices $i = 1, 2, 3, ...$ are just useful labels for the components of tensor product of a number of spaces.

The auxiliary YBE in $V_\alpha \otimes V_\beta \otimes V_1$, sometimes called the zero curvature representation, reads

\[ R_{\alpha\beta} L_{\alpha,1} L_{\beta,1} = L_{\beta,1} L_{\alpha,1} R_{\alpha\beta} . \]

The associativity of (2) is the standard YBE for $R$-matrices (in geometry this is a sort of Bianchi identity, in algebra this is a sort of Jacobi identity)

\[ R_{\alpha\beta} R_{\alpha\gamma} R_{\beta\gamma} = R_{\beta\gamma} R_{\alpha\gamma} R_{\alpha\beta} \]

Repeated use of (2) for ordered product of $L$-operators provides

\[ R_{\alpha\beta} \left( \prod_{i=1..n} L_{\alpha,i} L_{\beta,i} \right) = \left( \prod_{i=1..n} L_{\beta,i} L_{\alpha,i} \right) R_{\alpha\beta} \]

where

\[ \prod_{i=1..n} L_{\alpha,i} L_{\beta,i} = L_{\alpha,1} L_{\beta,1} L_{\alpha,2} L_{\beta,2} \cdots L_{\alpha,n} L_{\beta,n} = \left( \prod_{i=1..n} L_{\alpha,i} \right) \left( \prod_{i=1..n} L_{\beta,i} \right), \text{ etc.} \]
Multiplying (4) by $R^{-1}_{ab}$, taking then trace over $V_\alpha \otimes V_\beta$ and using the cyclic property of the trace to cancel $R_{ab}R^{-1}_{ab} = 1$, we come to commutativity of transfer matrices

$$
(\text{Trace}_{V_\alpha} \prod L_{\alpha,i} ), \left(\text{Trace}_{V_\beta} \prod L_{\beta,i} \right) = 0
$$

what means the existence of complete(?) set of integrals of motion which may be diagonalized simultaneously with the help of a proper separation of variables method (Bethe Ansatz).

Pictorial representation of R & L-matrices, YBE and transfer matrices is very helpful:

what means that each edge carries some index – a label of basis vector in corresponding space $V$; a bounded edge implies the summation over basis vector index – the matrix multiplication in corresponding space $V$.

Applications of YBE to quantum algebras, link invariants atc. are well known.

Note, hunting for new models people often considered and studied an alternative to (2) relation

$$
L_{\alpha,1}L_{\alpha,2}R_{12} = R_{12}L_{\alpha,2}L_{\alpha,1}
$$

for more general matrix $R_{12}$ acting in quantum spaces $V_1 \otimes V_2$.

**3D generalization of Equation (7):** a hypothetical auxiliary tetrahedron equation

$$
L_{\alpha\beta\gamma,1}L_{\alpha\gamma,2}L_{\beta\gamma,3}R_{123} = R_{123}L_{\beta\gamma,3}L_{\alpha\gamma,2}L_{\alpha\beta,1}
$$

where

- $(\alpha\beta\gamma)$ indices copy the structure of YBE
- associativity of (8) is the “standard” tetrahedron equation

$$
R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123}
$$

The name “tetrahedron equation” follows from the pictorial representation of (8):
Similarly to 2d case, the repeated use of (8) provides the exchange relation

\[
R_{123} = R_{123} \left( \prod_{i=1}^{m} L_{\alpha_i, \beta_i, 1} L_{\alpha_i, \gamma_i, 1} L_{\beta_i, \gamma_i, 1} \right).
\]

A fragment of this ordered product graphically looks like

Multiplying (10) by \( R_{123}^{-1} \) and taking the trace, we get

\[
R_{\alpha \beta} R_{\alpha \gamma} R_{\beta \gamma} = R_{\beta \gamma} R_{\alpha \gamma} R_{\alpha \beta},
\]

where

\[
R_{\alpha \beta} = \text{Trace}_{V_i} \left( \prod_{i=1}^{m} L_{\alpha_i, \beta_i, 1} \right)
\]

is a Yang-Baxter \( R \)-matrix acting in \( V_\alpha \otimes V_\beta \).

Thus, as the Yang-Baxter equation provides the commutativity of transfer matrices of any length, the tetrahedron equation provides the Yang-Baxter equation for the objects (12) with any hidden depths \( m \) (in terms of quantum groups, the construction (12) corresponds to \( U_q(\hat{sl}_m) \) and various, reducible as well as irreducible, representations. See Lecture 3).
**Auxiliary tetrahedron equation as automorphism.** Let us allow us a bit of imagination: suppose, in equation (8) \( V_\alpha = \mathbb{C}^2 \) with the basis \(|0\rangle\) and \(|1\rangle\) while \( V_i \) is more complicated. Thus, \( L_{\alpha\beta,1} \) has the structure of \( 4 \times 4 \) matrix in the basis of \( V_\alpha \otimes V_\beta \).

\[
V_\alpha \otimes V_\beta : |0\rangle \otimes |0\rangle , |1\rangle \otimes |0\rangle , |1\rangle \otimes |1\rangle
\]

with matrix elements – operators acting in \( V_1 \), for instance

\[
L_{\alpha\beta,1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & a_1 & b_1 & 0 \\
0 & c_1 & d_1 & 0 \\
0 & 0 & 0 & z_1
\end{pmatrix}
\]

Here

\[
A_j \overset{\text{def}}{=} \{1, a_j, b_j, c_j, d_j, z_j\}
\]

stands for shortness for the list of operators acting in the space \( V_j \). Since all elements of \( A_j \) are operators, we may consider them as elements of some algebra; this algebra we also will denote by the letter \( A_j \) so that \( j \) would stand for a component in tensor product of several copies of an algebra \( A \).

The auxiliary TE (8) may be then rewritten as

\[
L_{\alpha\beta}[A_1]L_{\alpha\gamma}[A_2]L_{\beta\gamma}[A_3] = L_{\beta\gamma}[A'_1]L_{\beta\gamma}[A'_2]L_{\alpha\beta}[A'_3]
\]

where

\[
A'_j \overset{\text{def}}{=} R_{123}A_j R_{123}^{-1} , \quad j = 1, 2, 3 ,
\]

i.e. whenever algebras \( A_j \) are taken, the auxiliary tetrahedron equation means the automorphism of tensor cube of algebras \( A_1 \otimes A_2 \otimes A_3 \).

**Classical limit.** *Any decent quantum theory must have classical limit*

Classical limit appears when ubiquitous quantum group’s parameter tends to unity, \( q \to 1 \). Allowing a bit of imagination again, we anticipate in this limit

\[
A_j = \{a_j, b_j, c_j, d_j\} \to \mathbb{C} \text{ – valued fields, up to possible constraints,}
\]

as well as

\[
A'_j = R_{123}\{a_j, b_j, c_j, d_j\} R_{123}^{-1} \overset{\text{def}}{=} \{a'_j, b'_j, c'_j, d'_j\} \to \text{also } \mathbb{C}-\text{valued stuff.}
\]

In this anticipated limit the auxiliary tetrahedron equation (17) becomes the local Yang-Baxter equation defining the primed multi-component fields \( A'_j \) of its RHS in terms of unprimed multi-component fields \( A_j \) of LHS,

\[
A'_j = F_j(A_1, A_2, A_3) , \quad j = 1, 2, 3 .
\]

(Or vice versa).

- Graphically, classical fields \( A_j \) and \( A'_j \) are associated with the edges of three-dimensional vertex. Such a vertex is the local element of some 3D lattice. Equations (21) are just local Hamiltonian equations of motion for three-dimensional classical integrable system with evidently well posed Cauchy problem.
If one regards (21) as the functional transformation (synonym map\(^1\), change of variables, ...)

\[ R_{123} \overset{\text{def}}{=} (R_{123} \cdot R_{123}^{-1})_{q \rightarrow 1} : A_j \rightarrow A'_j = F_j(A_1, A_2, A_3), \ j = 1, 2, 3 \]

then this map satisfies the functional tetrahedron equation – equivalence of two sequences of functional transformations.

The reminder of quantum world in the classical limit is the Hamiltonian structure and Poisson brackets. For instance, in the limit

\[ q = e^{-\hbar}, \ h \rightarrow 0 \]

the Heiseberg algebra of standard quantum mechanics \([x, p] = i\hbar\) is replaced by the Poisson algebra of standard classical mechanics \(\{x, p\} = 1\). Thus,

\begin{itemize}
  \item \(A_j = \{a_j, b_j, c_j, d_j\}\) in classical limit should be a local\(^2\) Poisson algebra
  \item The map \(R_{123} : A_j \rightarrow A'_j = F_j(A_1, A_2, A_3)\) should be a canonical map.
  \item The canonicity, i.e. the existence of generating function, guarantees the Hamiltonian dynamics and the variational principle: generating function is the Hamiltonian/Lagrangian density of the action, equations of motion in Hamiltonian or Lagrangian form follow from the least action principle.
\end{itemize}

**Forthcoming lectures.** In the next lectures we will develop reverse way. We’ll start from classical integrable system of discrete differential geometry, observe maps satisfying FTE, give geometrical meaning of local YBE, find the Poisson structure and action for discrete differential geometry and finally construct quantum algebras \(A_j\) and quantum 3D \(R\)-matrices.

**Lecture 2. Quadrilateral and circular nets**


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\(^1\)In quantum case, operator \(R_{123}\) is the adjoint action of \(R_{123}\), \((R_{123} \cdot \varphi) = R_{123} \varphi R_{123}^{-1}\) for any operator \(\varphi\). In classical limit, \(R_{123}\) is an operator acting in the space of functions of \(A_1, A_2, A_3\): for any function \(\varphi = \varphi(A_1, A_2, A_3)\)

\[ (R_{123} \cdot \varphi)(A_1, A_2, A_3) \overset{\text{def}}{=} \varphi(A'_1, A'_2, A'_3). \]

\(^2\)“local” means that all fields with different \(j\) are in involution – reminder of tensor product.
**Geometric integrability.** A quadrilateral net (Q-net) is a $\mathbb{Z}^3$ lattice imbedded in sufficiently many-dimensional (Euclidean) space $\mathbb{R}^M$:

\begin{equation}
 n = n_1 e_1 + n_2 e_2 + n_3 e_3 \in \mathbb{Z}^3 \rightarrow x = x(n) \in \mathbb{R}^M, \; M \geq 3,
\end{equation}

such that all quadrilaterals

\begin{equation}
 Q_{n,(ij)} = \left( x(n), x(n + e_i), x(n + e_j), x(n + e_i + e_j) \right)
\end{equation}

are planar ones. For the sake of shortness it is convenient to use the following notations:

\begin{equation}
 x(n) = x \Rightarrow x(n + e_i) = x_i, \; x(n + e_i + e_j) = x_{ij} \text{ etc.}
\end{equation}

The subject of interest is a single isolated hexahedron of such net:

The principle of geometric integrability (i.e. the integrability by means of bow-compass and ruler) is based on the following elementary theorem (A. Doliwa et al):

**Theorem 1.** Given the points $x, x_1, x_2, x_3, x_{12}, x_{13}, x_{23}$, the point $x_{123}$ of quadrilateral net can be restored uniquely.

**Proof:** According to the quadrilateral principle, the hexahedron is essentially three-dimensional object: for instance the vector $x_1 - x, x_2 - x$ and $x_3 - x$ are by definition the basis for all its edges. Therefore, the initial data defines the plane $(x_1, x_{12}, x_{13})$, the plane $(x_2, x_{23}, x_{12})$ and the plane $(x_3, x_{13}, x_{23})$ in three-dimensional Euclidean space. Three arbitrary planes in 3D intersect at one and only one point – the point $x_{123}$. □

The statement of the theorem is the elementary local step of evolution. The Cauchy problem can be formulated as follows. Initial data may be chosen on the boundary of $\mathbb{Z}_0^3$ octant – the points $x(0), x(n_1 e_1 + n_2 e_2), x(n_1 e_1 + x_3 e_3)$ and $x(n_2 e_2 + n_3 e_3)$. Using the statement of the theorem, one can restore step-by-step all other points of the net inside $\mathbb{Z}_0^3$. The following picture gives you a visualization of a few steps of such “staircase” evolution:
Note, on each step of evolution we have a surface of quadrilaterals – a sort of space-like surface.

**Angles and the map.** Turn now to arithmetic behind this geometrical evolution and consider the hexahedron and quadrilaterals in more details. A quadrilateral as the two-dimensional object is described by five parameters – for instance by three angles and by two edge lengths. A hexahedron as the three-dimensional object is described by 12 parameters – for instance by 9 proper angles and by three edge lengths. Let us consider all planar angles of the hexahedron:

In this picture the previous hexahedron is just dissected into two (non-planar!) hexagons. On each quadrilateral we put four angles

\[ A_j = \{\alpha_j, \beta_j, \gamma_j, \delta_j\} : \alpha_j + \beta_j + \gamma_j + \delta_j = 2\pi. \]

Thus, there are exactly nine independent angles of \(A_1, A_2, A_3\) on the left hand side hexagon and all the other angles may be recalculated with the help of spherical triangle cosine theorems, for instance the right hand side angle \(\beta_2'\) is defined by

\[ \cos \theta = \cos \beta_2 - \cos \alpha_1 \cos \delta_3 \frac{\sin \alpha_1 \sin \delta_3}{\sin \beta_1 \sin \beta_3} = \cos \beta_2' - \cos \alpha_1 \cos \beta_3 \]

where \(\theta\) is the dihedral angle of the edge \((x, x_2)\). All the other angles for the right hand side hexagon can be calculated similarly. Thus, the geometry provides the unique map

\[ R_{123} : A_1, A_2, A_3 \to A'_1, A'_2, A'_3 \]

Familiar, isn’t it? Three-dimensional vertex from the first lecture and hexahedron of Q-net are graphically just dual objects (they form dual lattices).

One more important remark. We denote the angles of hexahedron in very specific way. In the following picture we draw auxiliary lines across the hexagons from above:
The auxiliary lines are oriented in the same way as in graphical representation of the Yang-Baxter or of the auxiliary tetrahedron equation (up to occasional replacement \((\alpha, \beta, \gamma) \rightarrow (p, q, r)\) since the Greek letters are in use). One can easily see, notations for all angles of hexahedron correspond to the same pattern:

![Diagram](image)

**Functional tetrahedron equation.**

**Theorem 2.** The map \(R_{123}\) (30) for the angles of quadrilaterals satisfy the functional tetrahedron equation,

\[
R_{123} R_{145} R_{246} R_{356} = R_{356} R_{246} R_{145} R_{123}
\]

understood as sequence of maps.

**Proof:** The dual graph to a vertex tetrahedron is a quadrilateral (“rhombic”) dodecahedron. The same 4D quadrilateral dodecahedron can be dissected into four hexahedrons in two different ways what corresponds to left and right hand side of (31):

![Diagram](image)
The dodecahedron has twelve boundary quadrilaterals corresponding to six “in” quadrilaterals \(A_1, \ldots, A_6\) of equation (31) and six “out” quadrilaterals, say \(A_1^*, \ldots, A_6^*\). The initial angles \(A_1, \ldots, A_6\) completely define 4D cube – similarly to the proof of Theorem 1 this follows from the uniqueness of intersection point of four 3-planes in 4D space. Left and right hand side of the tetrahedron equation correspond to two dissection of 4D cube, similarly to dissection of a hexahedron into two hexagons. Uniqueness of 4D cube guarantees the uniqueness of expression of \(A_1^*, \ldots, A_6^*\) in terms of \(A_1, \ldots, A_6\), i.e. the coincidence of transformations given by the left and the right hand sided of (31). Note, smart choice of angle notations corresponding to the Yang-Baxter/tetrahedral orientation guarantees that all the maps in (31) are similar. ■

**Auxiliary problem.** For given angles \(A = \{\alpha, \beta, \gamma, \delta\}\) corresponding to auxiliary lines \(p, q\), see the quadrilateral picture above, define the following matrix acting in tensor product of two-dimensional vector spaces \(V_p \otimes V_q\)

\[
L_{pq}[A] = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \sin \gamma & \sin(\delta + \beta) & \sin \delta \\
0 & \sin \gamma & \sin \delta & 0 \\
0 & 0 & 0 & -\sin \alpha \sin \delta
\end{pmatrix},
\]

confer with equations (14) and (15).

**Theorem 3.** The set of all cosine theorems defining uniquely the map (30) \(R_{123} : A_1, A_2, A_3 \rightarrow A_1', A_2', A_3'\) can be written as the local Yang-Baxter equation

\[
L_{pq}[A_1]L_{pr}[A_2]L_{qr}[A_3] = L_{qr}[A_3']L_{pr}[A_2']L_{pq}[A_1']
\]

**Proof:** a direct verification for instance. ■

In fact, the matrix (32) arises when one considers the edge lengths. Let \(\ell_p, \ell_q, \ell_p', \ell_q'\) be the edge length of a single quadrilateral shown above. Three angles \(A = \{\alpha, \beta, \gamma, \delta\} = 2\pi - \alpha - \beta - \gamma\) and two edges \(\ell_p, \ell_q\) are complete set of geometrical data; two opposite edges \(\ell_p'\) and \(\ell_q'\) can be calculated easily:

\[
\begin{pmatrix}
\ell_p' \\
\ell_q'
\end{pmatrix} = X_{pq}[A]\begin{pmatrix}
\ell_p \\
\ell_q
\end{pmatrix}
\]

Matrix \(L_{pq}\) is a “fermionic exponent” of \(X_{pq}\) (a combinatorial trick comes from free-fermion models),

\[
X_{pq} = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \Leftrightarrow L_{pq} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & a & b & 0 \\
0 & c & d & 0 \\
0 & 0 & 0 & z
\end{pmatrix}, \quad z = -\det X.
\]

Both left and right hand sides of the local Yang-Baxter equation (33) actually relate three “outgoing” edge length of auxiliary hexagons to “incoming” edge length (see the directions of auxiliary \(pqr\)-lines).
Circular net. The map (30) involving three independent angles in each $A_j$ is extremely complicated, perhaps it needs a long separate study. But hopefully, there is a significant simplification – the case of circular quadrilaterals:

\begin{equation}
A = \{\alpha, \beta : \gamma = \pi - \beta, \delta = \pi - \alpha\},
\end{equation}

what means that each quadrilateral has thus a circumcircle. There is a geometric theorem (Miquel theorem) stating that if three quadrilaterals of a hexahedron are circular then all the other quadrilaterals are also circular, i.e. the whole hexahedron has a circumsphere.

Matrix $L_{pq}$ for the circular net\(^3\) simplifies:

\begin{equation}
L_{pq}[A] = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & k & a^* & 0 \\
0 & -a & k & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\end{equation}

where

\begin{equation}
a^* = (\sin \alpha)^{-1} \sin(\alpha - \beta), \quad a = (\sin \alpha)^{-1} \sin(\alpha + \beta) \quad \text{and} \quad k = (\sin \alpha)^{-1} \sin \beta
\end{equation}

Elements of $L$-matrix now are related by

\begin{equation}
k^2 = 1 - aa^*
\end{equation}

The further considerations, including a Poisson structure and quantization, are related to the reduction of the map (30) to the circular case (36,38,39).

**LECTURE 3. QUANTIZATION**

**Poisson structure and quantization, geometrical action and quantum geometry**

Poisson structure. The local Yang-Baxter equation (33) may be written in components and explicitly solved with respect to primed fields $a'_j, a''_j$:

\begin{equation}
(k_2a'_1)' = k_3a'_1 - k_1a'_2a_3, \quad (k_2a_1)' = k_3a_1 - k_1a_2a'_3,
\end{equation}

\begin{equation}
(k_2a'_2)' = a''_1a'_3 + k_1k_3a'_2, \quad (a_2)' = a_1a_3 + k_1k_3a_2,
\end{equation}

\begin{equation}
(k_2a''_3)' = k_1a''_1 - k_3a_1a''_2, \quad (k_2a_3)' = k_1a_3 - k_3a''_1a_2,
\end{equation}

where $k_2^2 = 1 - a_3a''_1$ is taken into account, two additional relations compatible with (41) are

\begin{equation}
k_1k_2 = k'_1k'_2, \quad k_2k_3 = k'_2k'_3.
\end{equation}

\(^3\)If one considers the Minkowski metric in a prescription that all edges of a Q-net are time-like, then imposes a Minkowski analogue of Miquel circular conditions, the result will be

\begin{equation}
L_{pq}[A] = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & k & a^* & 0 \\
0 & a & -k & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad k^2 = 1 - aa^*.
\end{equation}
Using (39) and reverse formulas

\[(43) \quad \cos \alpha = \frac{a - a^*}{2k}, \quad \cos \beta = \frac{a + a^*}{2}\]

one can recalculate the map in terms of angles.

**Theorem 4.** The circular map (41) is the symplectic map:

\[(44) \quad \sum_{i=1}^{3} d\alpha_i \wedge d\beta_i = \sum_{i=1}^{3} d\alpha_i' \wedge d\beta_i'\]

**Proof** is just direct verification. In calculations, it is more convenient to use Poisson brackets for the fields \(a_j, a_j^*\): locally\(^4\) \(\{\alpha, \beta\} = 1\) provides

\[(45) \quad \{a, a^*\} = 2k^2, \quad \{k, a\} = -ka, \quad \{k, a^*\} = ka^*\]

Local conservation of symplectic form (44) provides the conservation of global symplectic form

\[(46) \quad \omega = \sum_{\text{space-like } v} d\alpha_v \wedge d\beta_v\]

where the sum is taken over all quadrilaterals of space-like discrete surface. Thus we have Hamiltonian evolution in discrete space-time. The local conservation of symplectic form (44) allows to define a generation function, for instance by

\[(47) \quad dL(\alpha', \alpha) = \sum_{j=1}^{3} \beta_j' d\alpha_j' - \beta_j d\alpha_j \quad \Leftrightarrow \quad \beta_j' = \frac{\partial L}{\partial \alpha_j'}, \quad \beta_j = -\frac{\partial L}{\partial \alpha_j}\]

where \(\beta_j', \beta_j\) are functions of \(\alpha_j', \alpha_j\) calculated from (41). The generating function is a local Lagrangian density, the geometrical action is the sum of Lagrangian density over the whole \(\mathbb{Z}^3\) circular net,

\[(48) \quad A = \sum_{\mathbb{Z}^3} L(\alpha_j', \alpha_j) .\]

Geometrical equations of motion (41) [in their more complicated Lagrangian form] follows from the variational least action principle.

Upon some reparameterization the Lagrangian density \(L\) may be expressed in terms of Lorentzian function, details are beyond the scope of this introductory lectures.

**Quantization.** In general it is a replacement of \(\mathbb{C}\)-valued fields by operator-valued fields.

Equations (41) follow from the local Yang-Baxter equation without any assumptions about an algebraic nature of matrix elements \(a_j, a_j^*, k_j\) of (38) (of course LYBE contains a lot of extra equations). In fact, all the system (41) and additional relations (42) are simply some matrix elements of (33) in two-dimensional auxiliary spaces \(V_p \otimes V_q \otimes V_r\).

The primary principle is the locality. Whichever algebras

\[(49) \quad A_j = \{1, a_j, a_j^*, k_j\}\]

\(^4\)Repeat again, the locality means that elements of \(A_j\) and \(A_{j'}\) with \(j \neq j'\) are in involution, the non-trivial brackets exist only for the fields with the same \(j\).
are taken, the tensor product structure $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3$ implies the commutativity of operators with different indices (operators in different components of tensor product). In particular, the primed fields with different indices must commute, for instance

\[(50) \quad [(k_1^{-1}a_1')', a_2'] = [(k_1^{-1}a_1')', a_2'] = \ldots = 0\]

It is not difficult to solve these equations and conclude that the most general exchange relations for (49) providing the locality are

\[(51) \quad a^*a = 1 - q^{-1}k^2, \quad aa^* = 1 - qk^2, \quad ka^* = qa^*k, \quad ka = q^{-1}ak\]

for some (arbitrary) $q$ common for all components $\mathcal{A}_j$. Verifying exchange relations (51) for primed fields with the same index in (41) and checking all the rest of the local Yang-Baxter equation (33), we come to

**Theorem 5.** 1). The map (41) is the automorphism of tensor cube of $q$-oscillator algebra (51). 2). The map (41) for $q$-oscillators is the unique solution of the local Yang-Baxter equation (33) [with operator-valued entries] for $L$-operator (38).

Note, the $q$-oscillator algebra (51) corresponds to the straightforward Dirac’s quantization of the angles $\{\alpha, \beta\} = 1 - [\alpha, [\alpha, \beta]] = \hbar$ in expressions (39) with $q = e^{i\hbar}$.

**Representations.** For the construction of quantum intertwiner $R_{123}$, eq. (18), for our automorphism (41) we need to choose a representation of $q$-oscillator algebra (51). There are three different representations:

- A Fock space representation, spectrum of $k = q^{n+1/2}, \ n \geq 0$ and $|q| < 1$. After dimension-rank transmutation of the first lecture it will correspond to highest weight representations of $U_q(\hat{sl}_m)$.
- A cyclic representation for $q^N = 1, \ N$-odd. This corresponds to cyclic representations of quantum groups.
- A modular representation, spectrum of $k = i e^{i\pi b z}, \ z \in \mathbb{R}, q = e^{i\pi k^2}$. This corresponds to modular double of quantum groups.

In this lectures we choose the Fock space representation. The Fock vacuum $|0\rangle$ is annihilated by $a$,

\[(52) \quad |n\rangle = \frac{a^n}{(q^2;q^2)_n}|0\rangle, \quad |k\rangle n\rangle = |n\rangle q^{n+1/2}, \quad n \geq 0,
\]

where

\[(53) \quad (x; q^2)_n = (1 - x)(1 - q^2x) \cdots (1 - q^{2(n-1)}x).
\]

Equations (41,42), being written in the form

\[(54) \quad \langle n_1, n_2, n_3| (A'_1 R_{123} = R_{123} A_j) |n'_1, n'_2, n'_3\rangle,
\]

provide a set of recursion relations for the matrix elements $\langle n_1, n_2, n_3| R_{123}|n'_1, n'_2, n'_3\rangle$. For instance,

\[(55) \quad \langle n| (a_{143} + k_1 k_3 a_2) R_{123}|n'\rangle = \langle n| R_{123} a_2 |n'\rangle
\]

gives

\[(56) \quad \langle n_1, n_2 n_3| R|n'_1, n'_2 - 1, n'_3\rangle = \langle n_1 + 1, n_2, n_3 + 1| R|n'_1, n'_2, n'_3\rangle + q^{n_1 + n_2 + 1} \langle n_1, n_2 + 1, n_3| R|n'_1, n'_2, n'_3\rangle
\]
Existence of solution of all these recursion relations is guaranteed by the automorphism property. Moreover, the solution of recursion relations is unique and given by

$$\langle n | R | n' \rangle = \delta_{n_1+n_2,n'_1+n'_2} \delta_{n_2+n'_3,n'_2+n'_3} \frac{q^{(n'_1-n_2)(n'_2-n_3)}}{(q^2;q^2)_{n'_2}} \frac{q^{2(1-n'_2+n_3)}}{(q^2;q^2)_{n'_2}}$$

(57)

$$\times \ _2\Phi_1(q^{-2n'_2}, q^{2(1+n'_4)}; q^{2(1-n'_2+n_3)}; q^{2(1+n_1)})$$

where

$$\ _2\Phi_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a; q^2)_n (b; q^2)_n (c; q^2)_n}{(q^2; q^2)_n} z^n$$

(58)

is the q-deformed Gauss hypergeometric function. Note, the hypergeometric function in (57) is truncated. Important property of (57) is

$$R_{123}^2 = 1.$$

The limit $$q = e^{-\hbar} \to 1$$ with big occupation numbers $$k = q^{n+1/2} < 1$$ corresponds to classical geometry of circular lattice. In this limit

$$\langle n | R | n' \rangle \to \exp \left( -\frac{\mathcal{L}(k,k')}{\hbar} \right),$$

(60)

where $$\mathcal{L}$$ is the classical Lagrangian density (47) in variables $$k, k'$$. Function $$\mathcal{L}$$ may be obtained from (57) by the saddle point method. Similarly, the partition function of $$\mathbb{Z}^3$$ lattice has the limit

$$Z \overset{\text{def}}{=} \sum_{\text{inner spins cubic lattice}} \prod \langle n | R | n' \rangle \to \exp \left( -\frac{\mathcal{A}}{\hbar} \right)$$

(61)

where $$\mathcal{A}$$ is the classical action (48) calculated on the solution of equations of motion with fixed boundary variables $$k$$ (the Dirichlet problem, not the Cauchy one).

**Application to quantum groups.** Consider in conclusion the applications of derived solutions of tetrahedron equations to the related series of solutions of Yang-Baxter equation, see eq. (12) in the first lecture.

You maybe noted already, we have no spectral parameters in our solution (57) of the tetrahedron equation. Appearance of spectral parameters is the topological effect, spectral parameter can be introduced as boundary field. For instance, formula (12) for $$R$$-matrix in $$V_\alpha^m \otimes V_\beta^m$$ must be improved as follows:

$$R_{\alpha\beta}(u) = \sum V_1 \left( u^{N_1} \prod_{i=1}^{\infty} L_{\alpha_i, \beta_i, 1} \right),$$

(62)

where $$N_1$$ is the occupation number in Fock space $$V_1$$, $$k_1 = q^{N_1+1/2}$$. Repeating the considerations (10) with fields and using the delta-function structure of (57),

$$u^{N_1+N_2+N_3} R_{123} = R_{123} u^{N_1+N_2+N_3}$$

(63)

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5For all other representations the form of matrix elements/kernel of $$R$$-matrix is almost the same, just replace hypergeometric function by its cyclic or integral extensions.

6while quantum $$a$$ and $$a^*$$ is the conjugated pair, at the saddle point they become real.
we come to the Yang-Baxter equation
\[ R_{\alpha\beta}(u)R_{\alpha\gamma}(uv)R_{\beta\gamma}(v) = R_{\beta\gamma}(v)R_{\alpha\gamma}(uv)R_{\alpha\beta}(u). \]

The two-dimensional spaces \( V_\alpha, V_\beta, \text{etc.} \) are in fact the Fock spaces for Fermi oscillator\(^7\). Define the “occupation number” operator in \( V_\alpha = \mathbb{C}^2 \) as follows:

\[ N_\alpha |0\rangle = 0, \quad N_\alpha |1\rangle = |1\rangle \text{ or equivalently } N_\alpha = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]

One can verify immediately,
\[ u^{N_\alpha - N_\beta} L_{\alpha\beta,1} = L_{\alpha\beta,1} u^{N_\alpha - N_\beta}, \quad u^{N_\alpha + N_\beta} L_{\alpha\beta,1} = L_{\alpha\beta,1} u^{N_\alpha + N_\beta}. \]

Then, from (62) it follows that
\[ \left[ \sum_i N_\alpha, R_{\alpha\beta} \right] = \left[ \sum_i N_\beta, R_{\alpha\beta} \right] = 0 \]

Thus the space \( V_\alpha^{\otimes m} \) (as well as \( V_\beta^{\otimes m} \)) is reducible. Its irreducible subspace is fixed by the integer eigenvalue \( \sum_i N_\alpha \). One may easily see,
\[ \dim V_\alpha^{\otimes m} \left| \sum_i N_\alpha = m' \right| = \frac{m!}{m'(m-m')!}, \quad 0 \leq m' \leq m \]

what is the dimension of \( m' \)-th antisymmetric tensor representation of \( U_q(\mathfrak{sl}_m) \). Direct test of matrix elements shows that \( R_{\alpha\beta} \) indeed provides the direct sum of all fundamental \( R \)-matrices (and \( L \)-operators) for affine \( U_q(\widehat{\mathfrak{sl}}_m) \).

Similarly we can consider more complicated \( R \)-matrix in tensor power of Fock spaces \( V_1^{\otimes m} \otimes V_2^{\otimes m} \),

\[ R_{12}(u) = \text{Trace}_{V_3} \left( u^{N_3} \prod_{i=1..m} R_{V_i,V_3} \right), \]

and in the tensor power \( V_\alpha^{\otimes m} \otimes V_1^{\otimes m} \),

\[ R_{\alpha 1}(u) = \text{Trace}_{V_3} \left( u^{N_3} \prod_{i=1..m} L_{\alpha i, V_3} \right), \]

The tensor power of Fock spaces \( V_1^{\otimes m} \) has the center \( \sum_i N_{1,i} \),

\[ \dim V_1^{\otimes m} \left| \sum_i N_{1,i} = m' \right| = \frac{(m+m'-1)!}{(m-1)!m'!}, \quad m' = 0, 1, 2, \ldots \]

what corresponds to the infinite series of all symmetric tensor representations of \( U_q(\mathfrak{sl}_m) \). All possible Yang-Baxter equations are satisfied due to the tetrahedron equations.

Note, for the case \( U_q(\mathfrak{sl}_2) \) the formula (69) gives the direct sum of all highest weight representations, it gives the universal \( R \)-matrix.

Note also, consideration of Fermi oscillators allows to alternate bosonic Fock spaces and fermionic spaces in \( V^{\otimes m} \). This reproduces the quantum super-algebras and their representation theory.

\(^7\)Operator \( L_{\alpha\beta,1} \) can be rewritten in terms of Fermi creation, annihilation and occupation number operators. In general, there are eight different tetrahedron equations corresponding to different choices of even/odd parity for six tetrahedral spaces.
Discussion. Plank scale $\lambda = 10^{-35}m$, Plank energy is one pound of TNT-equivalent (armored detectors?). Strings or discrete space-time? Discrete space-time $\rightarrow$ quantum discrete space-time, quantum gravity.

Literature

I give here only few relevant references.

The role of the tetrahedron equation for the effective Yang-Baxter equations and commutativity of transfer matrices was established in [1] while the first solution of TE was suggested in [2] in the framework of scattering theory. The method of the local Yang-Baxter equation was proposed in [3]. A lot of exercises with the functional tetrahedron equation may be found in [4, 5, 6, 7], however the $q$-oscillator quantization was obtained in [8]. Quadrilateral nets were proposed in [9], but a bit more general classical equations were studied algebraically earlier in [10]. I would like to acknowledge an extensive literature concerning the discrete differential geometry; a review of related topics may be found in the book [11]. The main subject of these lectures, the relation between discrete differential integrability and quantum tetrahedron equation, appeared recently [12].

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References
