Positive and real-positive solutions to the equation $axa^* = c$ in C^*-algebras

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Abstract

We obtain new representations for the general positive and real-positive solutions of the equation $axa^* = c$ in a C^*-algebra using the characterization of positivity based on a matrix representation of an element and the generalized Schur complement. Applications to the equation $AXA^* = C$ for operators between Hilbert spaces and for finite matrices are given.

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1 Introduction

In this paper we propose a new approach to the finding of the solutions of the equation $axa^* = c$ in a C^*-algebra which yields both the positive and real-positive solutions. In this way we obtain a new representation of the positive solutions for the corresponding matrix equation, and solve the problem for the real-positive solutions. To interpret our results for rectangular matrices we embed the matrices as blocks into the C^*-algebra of matrices of the same order. The same approach works also for operators between different Hilbert spaces.

The main tool used in this paper is the characterization of positivity of an element $x \in \mathcal{A}$ based on its matrix representation, originally given by Albert [1] for complex matrices, and extended to C^*-algebras by Cvetković-Ilić et al. in [3] (see Lemma 1.1).

By $\mathcal{A}$ we denote a complex unital C^*-algebra. An element $a \in \mathcal{A}$ is *hermitian* if $a^* = a$, and *positive*, written $a \geq 0$, if it is hermitian and its spectrum $\sigma(a)$ lies in

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$[0,\infty)$. A *projection* $p \in \mathcal{A}$ is a hermitian idempotent: $p^2 = p = p^*$. An element $a \in \mathcal{A}$ is *regular* (in the sense of von Neumann) if there exists $b \in \mathcal{A}$ such that $aba = a$; any such b is called an *inner inverse* of a. A *Moore-Penrose inverse* of a is an element $a^\dagger \in \mathcal{A}$ satisfying the four Penrose equations [8],

$$aa^\dagger a = a, \quad a^\dagger a a^\dagger = a^\dagger, \quad (a^\dagger a)^* = a^\dagger a, \quad (aa^\dagger)^* = aa^\dagger; \tag{1.1}$$

if a Moore-Penrose inverse exists, it is unique. It is well known (see for instance Harte and Mbekhta [6]) that

$$a \text{ is regular} \iff a\mathcal{A} \text{ is closed} \iff a^\dagger \text{ exists.}$$

We recall the well known fact that

$$aa^\dagger c = c \iff c\mathcal{A} \subset a\mathcal{A}.$$

An *orthogonal system* in $\mathcal{A}$ is an ordered pair (p_1, p_2) of projections such that $p_1 + p_2 = 1$ and $p_1 p_2 = 0 = p_2 p_1$. Every element $x \in \mathcal{A}$ can be represented by a 2×2 matrix over $\mathcal{A}$ relative to an orthogonal system (p_1, p_2),

$$\varphi(x) = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix},$$

where $x_1 = p_1 x p_1$, $x_2 = p_1 x p_2$, $x_3 = p_2 x p_1$, $x_4 = p_2 x p_2$. The mapping $\varphi\colon \mathcal{A} \to \mathcal{A}^{2\times 2}$ is an algebra $*$-monomorphism when $\mathcal{A}^{2\times 2}$ is equipped with the usual matrix operations. Note that $x = x_1 + x_2 + x_3 + x_4$. We shall write

$$x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \tag{1.2}$$

if the orthogonal system (p_1, p_2) is understood.

The following crucial lemma was proved by Albert [1] for matrices, and generalized to C^*-algebras in [3, Theorem 2.2] by Cvetković-Ilić et al. Let $x \in \mathcal{A}$ be represented by a matrix (1.2) relative to an orthogonal system (p_1, p_2), and let x_1 be regular. The expression $s(x) = x_4 - x_3 x_1^\dagger x_2$ is the so called *generalized Schur complement* of x relative to (p_1, p_2).

Lemma 1.1. ([3, Theorem 2.2].) *Let x be a hermitian element in a C^*-algebra $\mathcal{A}$, let (1.2) be the representation of x relative to an orthogonal system (p_1, p_2), and let x_1 be regular. Then x is positive if and only if the following conditions are satisfied:*

(i) $x_1 \geq 0$,

(ii) $x_1 x_1^\dagger x_2 = x_2$,

(iii) $s(x) = x_4 - x_3 x_1^\dagger x_2 \geq 0$.

The *real part* (or *hermitian part*) of $a \in \mathcal{A}$ is defined by $\operatorname{Re} a = \frac{1}{2}(a + a^*)$. We mention some properties that will be useful later. First, $\operatorname{Re} a$ is always hermitian, $\operatorname{Re} a = a$ if a is hermitian, and $\operatorname{Re} a = 0$ if a is skew-hermitian. Further,

$$\operatorname{Re}(a \pm b) = \operatorname{Re} a \pm \operatorname{Re} b, \quad \operatorname{Re} x^* a x = x^* (\operatorname{Re} a) x.$$

An element $a \in \mathcal{A}$ is called *real-positive* or Re-*positive* if $\operatorname{Re} a \geq 0$.

2 Positive solutions

We will study the general positive solutions to the equation

$$axa^* = c \tag{2.1}$$

where $a, c \in \mathcal{A}$ are given and a is regular. A necessary condition for the solvability of the equation (2.1) by a hermitian x is that c itself is hermitian. This will be assumed throughout.

Lemma 2.1. *Let $a, c \in \mathcal{A}$ with a regular and c hermitian. Then* (2.1) *has a hermitian solution if and only if $aa^\dagger c = c$, and a positive solution if and only if $aa^\dagger c = c$ and c is positive.*

Proof. If $axa^* = c$, then $c\mathcal{A} \subset a\mathcal{A}$, that is, $aa^\dagger c = c$. If x is positive, then so is c. Conversely, if $aa^\dagger c = c$, then also $c(a^-)^* a^* = c$, and $x_0 = a^\dagger c (a^-)^*$ is a hermitian solution to (2.1). If c is positive, then so is x_0. □

In order to derive the general form of the positive solutions to (2.1) we assume, as in Lemma 2.1, that a is regular, c positive, and that (2.1) is consistent. In addition we need to assume that c is regular to ensure the regularity of a particular positive solution x_1. It is interesting that the general solution depends on two parameters $e, f \in \mathcal{A}$, one arbitrary and one positive.

Theorem 2.2. *Let* $a \in \mathcal{A}$ *be regular,* $c \in \mathcal{A}$ *positive and regular, and let* $aa^\dagger c = c$. *Then the general positive solution to* (2.1) *can be expressed as*

$$\begin{aligned} x = x_1 &+ x_1 x_1^\dagger e(1-a^\dagger a) + (1-a^\dagger a)e^* x_1 x_1^\dagger \\ &+ (1-a^\dagger a)e^* x_1^\dagger e(1-a^\dagger a) + (1-a^\dagger a)f(1-a^\dagger a), \end{aligned} \tag{2.2}$$

where $x_1 = a^\dagger c(a^\dagger)^*$ *is a particular positive solution,* $e \in \mathcal{A}$ *is arbitrary and* $f \in \mathcal{A}$ *is positive.*

Proof. A positive solution to (2.1) exists by Lemma 2.1.

Assume that c is regular and that x is a positive solution to $axa^* = c$. We write

$$p_1 = aa^\dagger, \quad p_2 = 1 - aa^\dagger, \quad q_1 = a^\dagger a, \quad q_2 = 1 - a^\dagger a. \tag{2.3}$$

Then p_i, q_i, $i = 1, 2$, are projections in $\mathcal{A}$, that is, hermitian idempotents. From the definition,

$$p_1 a = aq_1 = p_1 a q_1 = a, \quad q_1 a^\dagger = a^\dagger p_1 = q_1 a^\dagger p_1 = a^\dagger. \tag{2.4}$$

Since the equation $axa^* = c$ is solvable, we have $p_1 c p_1 = aa^\dagger caa^\dagger = aa^\dagger c(a^*)^\dagger a^* = c$. Further, $q_1 x q_1 = a^\dagger a x a^\dagger a = a^\dagger a x a^*(a^\dagger)^* = a^\dagger c(a^\dagger)^* = x_1$. Hence

$$c = p_1 c = c p_1 = p_1 c p_1, \quad q_1 x q_1 = a^\dagger c(a^\dagger)^* = x_1, \tag{2.5}$$

which shows that x_1 is a positive solution of (2.1). Let us express x as a matrix (1.2) relative to the orthogonal system (q_1, q_2). We express the condition $x \geq 0$ in terms of this matrix using Lemma 1.1:

(i) x_1 is positive and regular, and $x_3 = x_2^*$,

(ii) $x_1 x_1^\dagger x_2 = x_2$,

(iii) $x_4 - x_3 x_1^\dagger x_2 \geq 0$.

We show that the positivity and regularity of x_1 follow from the hypotheses. Since $x_1 = a^\dagger c(a^\dagger)^*$ and $c \geq 0$, also $x_1 \geq 0$. Let $w = a^* c^\dagger a$. Then

$$x_1 w x_1 = a^\dagger c(a^\dagger)^* a^* c^\dagger a a^\dagger c(a^\dagger)^* = a^\dagger c p_1 c^\dagger p_1 c(a^\dagger)^* = a^\dagger c(a^\dagger)^* = x_1,$$

that is, w is an inner inverse for x_1, and x_1 is regular.

Condition (ii) is equivalent to $x_2 = x_1 x_1^\dagger \tilde{e}$, where $\tilde{e} \in \mathcal{A}q_2$.

Condition (iii) implies

$$x_4 - x_3x_1^\dagger x_2 = x_4 - x_2^*x_1^\dagger x_2 = x_4 - \tilde{e}^*x_1^\dagger x_1x_1^\dagger x_1x_1^\dagger \tilde{e} = x_4 - \tilde{e}^*x_1^\dagger \tilde{e} \geq 0,$$

that is, $x_4 = \tilde{e}^*x_1^\dagger \tilde{e} + \tilde{f}$, where $\tilde{f} \in q_2\mathcal{A}q_2$ is positive.

Summarizing, we obtain

$$x = x_1 + x_2 + x_3 + x_4 = x_1 + x_1x_1^\dagger eq_2 + q_2e^*x_1x_1^\dagger + q_2e^*x_1^\dagger eq_2 + q_2fq_2,$$

with $e, f \in \mathcal{A}$, $f \geq 0$. Thus x is of the form (2.2).

Conversely, assume that x is of the form (2.2). A direct verification shows that $axa^* = c$. Expressing x as a $\mathcal{A}^{2\times 2}$ matrix (1.2) relative to the orthogonal system (q_1, q_2), we have

$$x_1 = q_1xq_1 = a^\dagger c(a^\dagger)^*, \quad x_2 = q_1x_1x_1^\dagger eq_2, \quad x_3 = x_2^*, \quad x_4 = q_2(e^*x_1^\dagger e + f)q_2.$$

The conditions of Lemma 1.1 are satisfied, which shows that $x \geq 0$. $\square$

We apply the preceding theorem to Hilbert space operators. If H, K are Hilbert spaces, $\mathcal{B}(H, K)$ denotes the set of all bounded linear operators from H to K, and $R(A)$ stands for the range of $A \in \mathcal{B}(H, K)$. If $H = K$, we write $\mathcal{B}(H)$ instead of $\mathcal{B}(H, H)$. The set $\mathcal{B}(H)$ is a C^*-algebra. An operator $A \in \mathcal{B}(H, K)$ is *regular* if there exists $B \in \mathcal{B}(K, H)$ such that $ABA = A$; it is well known that A is regular if and only if the range of A is closed. The Moore-Penrose inverse of $A \in \mathcal{B}(H, K)$ is defined as the (unique) operator $A^\dagger \in \mathcal{B}(K, H)$ satisfying the Penrose equations (1.1). It is a standard result that A possesses the Moore-Penrose inverse if and only if it has closed range. The positivity and real-positivity is defined only for operators acting on the same Hilbert space H, that is, in the C^*-algebra $\mathcal{B}(H)$.

Corollary 2.3. *Let H, K be Hilbert spaces, and let $A \in \mathcal{B}(H, K)$, $C \in \mathcal{B}(K)$ be such that A has closed range, C is hermitian and $R(C) \subset R(A)$. Then the equation $AXA^* = C$ has a positive solution $X \in \mathcal{B}(H)$ if and only if C is positive. If, in addition, C has closed range, then the general positive solution has the form*

$$\begin{aligned} X = {} & X_0 + X_0X_0^\dagger E(I - A^\dagger A) + (I - A^\dagger A)E^*X_0X_0^\dagger \\ & + (I - A^\dagger A)E^*X_0^\dagger E(I - A^\dagger A) + (I - A^\dagger A)F(I - A^\dagger A), \end{aligned} \tag{2.6}$$

where $X_0 = A^\dagger C(A^\dagger)^*$ is a particular positive solution, $E \in \mathcal{B}(H)$ is arbitrary and $F \in \mathcal{B}(H)$ is arbitrary positive.

Proof. We embed all the operators into the C^*-algebra $\mathcal{A} = \mathcal{B}(H \oplus K)$ as operator matrices

$$a = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}, \ a^* = \begin{bmatrix} 0 & A^* \\ 0 & 0 \end{bmatrix}, \ a^\dagger = \begin{bmatrix} 0 & A^\dagger \\ 0 & 0 \end{bmatrix}, \ c = \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix}. \tag{2.7}$$

In $\mathcal{A}$ we solve the equation $axa^* = c$, where $x = \left[\begin{smallmatrix} X_1 & X_2 \\ X_3 & X_4 \end{smallmatrix}\right]$. Then $axa^* = c$ in $\mathcal{A}$ if and only if $AX_1A^* = C$. Thus we can restrict ourselves to the x of the form $x = \left[\begin{smallmatrix} X & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Applying Lemma 2.1 in the C^*-algebra $\mathcal{A}$, we conclude that a positive solution to $AXA^* = C$ exists if and only if C is positive.

According to (2.2), the general solution is given by

$$\begin{aligned} x = x_0 &+ x_0x_0^\dagger e(1 - a^\dagger a) + (1 - a^\dagger a)e^* x_0 x_0^\dagger \\ &+ (1 - a^\dagger a)e^* x_0^\dagger e(1 - a^\dagger a) + (1 - a^\dagger a)f(1 - a^\dagger a), \end{aligned}$$

where $x_0 = a^\dagger c(a^\dagger)^*$ is a particular positive solution, $e \in \mathcal{A}$ is arbitrary and $f \in \mathcal{A}$ is positive. Here $x_0 = \left[\begin{smallmatrix} X_0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ with $X_0 = A^\dagger C(A^\dagger)^*$, and $e = \left[\begin{smallmatrix} E_1 & E_2 \\ E_3 & E_4 \end{smallmatrix}\right]$, $f = \left[\begin{smallmatrix} F_1 & F_2 \\ F_3 & F_4 \end{smallmatrix}\right]$. We find that

$$x_0x_0^\dagger e(1 - a^\dagger a) = \begin{bmatrix} X_0X_0^\dagger E_1(I - A^\dagger A) & * \\ 0 & 0 \end{bmatrix},$$

$$(1 - a^\dagger a)e^* x_0 x_0^\dagger = \begin{bmatrix} (I - A^\dagger A)E_1^* X_0 X_0^\dagger & 0 \\ * & 0 \end{bmatrix},$$

$$(1 - a^\dagger a)e^* x_0^\dagger e(1 - a^\dagger a) = \begin{bmatrix} (I - A^\dagger A)E_1^* X_0^\dagger E_1(I - A^\dagger A) & * \\ * & * \end{bmatrix},$$

$$(1 - a^\dagger a)f(1 - a^\dagger a) = \begin{bmatrix} (I - A^\dagger A)F_1(I - A^\dagger A) & * \\ * & * \end{bmatrix},$$

where the asterisks denote entries which are not necessarily zero. Only the entries in position (1,1) count for the solution. Since f is positive, so is F_1. Equation (2.6) then follows when we write $E = E_1$ and $F = F_1$. □

If the Hilbert spaces in the preceding corollary are finite dimensional, we get the corresponding result for matrices. We write $\mathbb{C}^{m \times n}$ for the set of all $m \times n$ complex matrices.

Corollary 2.4. *Let* $A \in \mathbb{C}^{m \times n}$*, and let* $C \in \mathbb{C}^{m \times m}$ *be hermitian and such that* $AA^\dagger C = C$*. Then there exists a positive matrix* $X \in \mathbb{C}^{n \times n}$ *such that* $AXA^* = C$ *if* *and only if C is positive. The general solution is given by the equation* (2.6) *with $X_0 = A^\dagger C(A^\dagger)^*$, an arbitrary $E \in \mathbb{C}^{n\times n}$ and an arbitrary positive $F \in \mathbb{C}^{n\times n}$.*

We recall some representations of the general solution in the existing literature. Khatri and Mitra in [7, Lemma 2.1] gave the general positive solution in the form

$$X = A^\dagger C(A^\dagger)^* + (I - A^\dagger A)T(I - A^\dagger A)^*,$$

where $A^\dagger$ is an arbitrary inner inverse on A and T is an arbitrary positive matrix. It was however pointed out by Baksalary [2] that this result does not produce all the positive solutions. (Note that our solution needs two parameters E and F.) He proposed a general solution of the form

$$X = YY^*, \quad Y = A^\dagger D + (I - A^\dagger A)Z,$$

where $A^\dagger$ is an arbitrary inner inverse of A, $D \in \mathbb{C}^{m\times n}$ is any matrix with $C = DD^*$, and $Z \in \mathbb{C}^{n\times n}$ is arbitrary. Groß in [5] gave a representation of the form

$$X = A^= C(A^=)^* + (I - A^\dagger A)UU^*(I - A^\dagger A)^*,$$

where

$$A^= = A^\dagger + (I - A^\dagger A)Z(C^{1/2})^-,$$

with $A^\dagger$ and $(C^{1/2})^-$ arbitrary but fixed inner inverses of A and $C^{1/2}$, $U \in \mathbb{C}^{n\times(n-r)}$ arbitrary ($r = \operatorname{rank}(C)$), and $Z \in \mathbb{C}^{n\times m}$ arbitrary.

There are many other papers dealing with the equation $AXA^* = C$ for matrices, notably Groß [4], Zhang [9, 10], Zhang and Cheng [11, 12], and Zhou [13].

3 Real-positive solutions

The advantage of the method employed in the preceding section is that it can be applied to the finding of the real-positive solutions of the equation $axa^* = c$.

Theorem 3.1. *Let $a, c \in \mathcal{A}$, let a be regular, and let the equation $axa^* = c$ be solvable. Then $axa^* = c$ has a real-positive solution if and only if c is* Re*-positive. If, in addition,* Re c *is regular, then the general* Re*-positive solution is of the form*

$$\begin{aligned} x = a^\dagger c(a^\dagger)^* + hh^\dagger e(1 - a^\dagger a) + \tfrac{1}{4}(1 - a^\dagger a)e^* h^\dagger e(1 - a^\dagger a) \\ + (1 - a^\dagger a)ya^\dagger a - a^\dagger a y^*(1 - a^\dagger a) + (1 - a^\dagger a)f(1 - a^\dagger a), \end{aligned} \tag{3.1}$$

where $h = a^\dagger(\operatorname{Re} c)(a^\dagger)^*$, and $e, y, f \in \mathcal{A}$ are arbitrary with f Re*-positive.*

Proof. If $\operatorname{Re} c \geq 0$, then $x_0 = a^\dagger c(a^\dagger)^*$ is a Re-positive solution to $axa^* = c$. Conversely suppose that x is a Re-positive solution to $axa^* = c$; then $\operatorname{Re} c = a(\operatorname{Re} x)a^*$ is positive.

Assume that $\operatorname{Re} c$ is regular and x is a Re-positive solution to $axa^* = c$. Proceeding as in the proof of Theorem 2.2 we define p_i, q_i, $i = 1, 2$, by (2.3). Then equations (2.4) and (2.5) hold. We represent the solution x in the matrix form (1.2) relative to the orthogonal system (q_1, q_2).

Let us write down the conditions for $\operatorname{Re} x \geq 0$. First,

$$\operatorname{Re} x = \begin{bmatrix} \operatorname{Re} x_1 & \frac{1}{2}(x_2 + x_3^*) \\ \frac{1}{2}(x_2^* + x_3) & \operatorname{Re} x_4 \end{bmatrix}.$$

We need

(i) $h = \operatorname{Re} x_1 = a^\dagger(\operatorname{Re} c)(a^\dagger)^*$ is positive and regular,

(ii) $hh^\dagger(x_2 + x_3^*) = x_2 + x_3^*$,

(iii) $\operatorname{Re} x_4 - \frac{1}{4}(x_2^* + x_3)h^\dagger(x_2 + x_3^*) \geq 0$.

We show that the positivity and regularity of h follow from the hypotheses. From $\operatorname{Re} c \geq 0$ we get $h = a^\dagger(\operatorname{Re} c)(a^\dagger)^* \geq 0$. To verify that h is regular, we observe that $\operatorname{Re} c = p_1 \operatorname{Re} c = \operatorname{Re} c\, p_1$ with $\operatorname{Re} c$ regular. Then

$$\begin{aligned} h(a^*(\operatorname{Re} c)^\dagger a)h &= a^\dagger \operatorname{Re} c\,(a^\dagger)^* a^* (\operatorname{Re} c)^\dagger a a^\dagger \operatorname{Re} c\,(a^\dagger)^* \\ &= a^\dagger \operatorname{Re} c\, p_1 (\operatorname{Re} c)^\dagger p_1 \operatorname{Re} c\,(a^\dagger)^* \\ &= a^\dagger \operatorname{Re} c\,(\operatorname{Re} c)^\dagger \operatorname{Re} c\,(a^\dagger)^* = h, \end{aligned}$$

which shows that $u = a^*(\operatorname{Re} c)^\dagger a$ is an inner inverse of h.

Condition (ii) is equivalent to $x_2 + x_3^* = hh^\dagger \tilde{e}$, where $\tilde{e} \in \mathcal{A}q_2$.

Condition (iii) implies

$$\operatorname{Re} x_4 - \tfrac{1}{4}(x_2^* + x_3)h^\dagger(x_2 + x_3^*) = \operatorname{Re} x_4 - \tfrac{1}{4}\tilde{e}^* h^\dagger h h^\dagger h h^\dagger \tilde{e} = \operatorname{Re} x_4 - \tfrac{1}{4}\tilde{e}^* h^\dagger \tilde{e} \geq 0,$$

that is, $x_4 = \frac{1}{4}\tilde{e}^* h^\dagger \tilde{e} + \tilde{f}$, where $\tilde{f} \in q_2\mathcal{A}q_2$ is Re-positive. Thus we have

$$x = \begin{bmatrix} a^\dagger c(a^\dagger)^* & hh^\dagger\tilde{e} - x_3^* \\ x_3 & \frac{1}{4}\tilde{e}^* h^\dagger \tilde{e} + \tilde{f} \end{bmatrix}, \tag{3.2}$$

where $\tilde{e} \in \mathcal{A}q_2$, $\tilde{f} \in q_2\mathcal{A}q_2$ and $\operatorname{Re} \tilde{f} \geq 0$, and $x_3 \in q_2\mathcal{A}q_1$. Set

$$\tilde{e} = eq_2, \quad \tilde{f} = q_2 f q_2, \quad x_3 = q_2 y q_1,$$

where $e, f, y \in \mathcal{A}$ and f is Re-positive. Observing that $x = x_1 + x_2 + x_3 + x_4$, we obtain (3.1).

Conversely, let x be of the form (3.1). Then $axa^* = c$ by direct verification. To prove that x is Re-positive, we first find the representation of x as a matrix (1.2) relative to (q_1, q_2):

$$\begin{aligned}
x_1 &= q_1xq_1 = a^\dagger c(a^\dagger)^*,\\
x_2 &= q_1xq_2 = -a^\dagger ay^*(1-a^\dagger a) + hh^\dagger e(1-a^\dagger a),\\
x_3 &= q_2xq_1 = (1-a^\dagger a)ya^\dagger a,\\
x_4 &= q_2xq_2 = \tfrac{1}{4}(1-a^\dagger a)e^*h^\dagger e(1-a^\dagger a) + (1-a^\dagger a)f(1-a^\dagger a).
\end{aligned}$$

If $\operatorname{Re} x = \left[\begin{smallmatrix} z_1 & z_2 \\ z_3 & z_4 \end{smallmatrix}\right]$ relative to (q_1, q_2), then

$$\begin{aligned}
z_1 &= a^\dagger(\operatorname{Re} c)(a^\dagger)^* = h \geq 0,\\
z_2 &= \tfrac{1}{2}(x_2 + x_3^*) = \tfrac{1}{2}hh^\dagger e(1-a^\dagger a),\\
z_3 &= \tfrac{1}{2}(1-a^\dagger a)e^*h^\dagger h,\\
z_4 &= \tfrac{1}{4}(1-a^\dagger a)e^*h^\dagger e(1-a^\dagger a) + (1-a^\dagger a)f(1-a^\dagger a).
\end{aligned}$$

We then verify that the matrix for $\operatorname{Re} x$ satisfies the conditions of Lemma 1.1, which implies $\operatorname{Re} x \geq 0$. $\square$

We can apply the preceding theorem to bounded linear operators between different Hilbert spaces using the embedding procedure from the proof of Corollary 2.3.

Corollary 3.2. *Let H, K be Hilbert spaces, and let $A \in \mathcal{B}(H, K)$, $C \in \mathcal{B}(K)$ be such that A has closed range, and the equation $AXA^* = C$ is consistent. Then $AXA^* = C$ has a* Re-*positive solution if and only if C is* Re-*positive. If, in addition,* $\operatorname{Re} C$ *has closed range, then the general* Re-*positive solution is of the form*

$$\begin{aligned}
X = A^\dagger C(A^\dagger)^* &+ GG^\dagger E(I-A^\dagger A) + \tfrac{1}{4}(I-A^\dagger A)E^*G^\dagger E(I-A^\dagger A)\\
&+ (I-A^\dagger A)YA^\dagger A - A^\dagger AY^*(I-A^\dagger A) + (I-A^\dagger A)F(I-A^\dagger A), \qquad (3.3)
\end{aligned}$$

where $G = A^\dagger(\operatorname{Re} C)(A^\dagger)^*$, $E, Y \in \mathcal{B}(H)$ are arbitrary and $F \in \mathcal{B}(H)$ is arbitrary Re-*positive.*

For finite dimensional Hilbert spaces we get this result for complex matrices:

Corollary 3.3. *Let* $A \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{m\times m}$, *and let* $AA^\dagger CAA^\dagger = C$. *Then there exists a* Re*-positive matrix* $X \in \mathbb{C}^{n\times n}$ *such that* $AXA^* = C$ *if and only if* C *is* Re*-positive. The general* Re*-positive solution is given by the equation* (3.3) *where* $G = A^\dagger(\operatorname{Re} C)(A^\dagger)^*$, $E, Y \in \mathbb{C}^{n\times n}$ *are arbitrary, and* $F \in \mathbb{C}^{n\times n}$ *is arbitrary* Re*-positive.*

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