Equations $ax = c$ and $xb = d$ in rings and rings with involution with applications to Hilbert space operators

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Abstract

This paper reviews the equations $ax = c$ and $xb = d$ from a new perspective by studying them in the setting of associative rings with or without involution. Results for rectangular matrices and operators between different Banach and Hilbert spaces are obtained by embedding the 'rectangles' into rings of square matrices or rings of operators acting on the same space. Necessary and sufficient conditions using generalized inverses are given for the existence of the hermitian, skew-hermitian, reflexive, antireflexive, positive and real-positive solutions, and the general solutions are described in terms of the original elements or operators. New results are obtained, and many results existing in the literature are recovered and corrected.

Key words: ring, ring with involution, equations in a ring, hermitian solution, reflexive solution, matrix equations, operator equations, Hilbert space operators, positive solution, real-positive solution

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1 Introduction

1.1 A brief survey

The equations $AX = C$ and $XB = D$ for square and rectangular matrices have a long history. In one of the early publications, Cecioni [6] in 1910 gave necessary and sufficient conditions for the existence of a common solution of these two equations. The general form of the common solution was described by Rao and Mitra [29] in 1971, and by Ben-Israel and Greville [2] in 1974. In 1956 the first equation was studied by Hodges [16] for matrices over finite fields; this was later followed by papers of Porter [27] in 1976 and Porter and Mousouris [28] in 1979. In 1987, Uhlig gave applications of the first equation to control theory. The rank of matrices plays an important role in these investigations.

Many authors addressed the question when the two equations have a solution belonging to a special class of matrices. There are two issues: (i) To find necessary and sufficient conditions for the existence of such solutions, and (ii) to describe the set of all such solutions. We give a brief (and perhaps somewhat selective) survey of these investigations.

(a) Hermitian, positive definite and positive semidefinite solutions for matrices were considered by Khatri and Mitra [18] in 1976, who gave necessary and sufficient conditions for the existence of solutions in these classes. They give explicit solutions based on generalized matrix inverses and the matrix rank. Further investigators were Don [11] in 1987, Higham [15] in 1988, Chu [7] in 1989 and Dai [9] in 1990. Phadke and Thakare [26] attempted to describe hermitian, positive definite and semidefinite solutions for Hilbert space operators in 1979, but several of their results are incorrect or have incorrect proofs. The literature for this type of problem is very extensive, including papers by Lei Zhang with or without coauthors (for instance [21,34]).

(b) In 1992 Wu [32] studied the real-positive definite solution to $AX = B$, and Wu and Cain [33] in 1996 the real-nonnegative solutions; Groß [12] gave a new derivation and a corrected version to some of Wu and Cain's results in 1999.

(c) In 2003, Peng and Hu [24] studied the reflexive and antireflexive solutions to the matrix equation $AX = B$, and supplied explicit formulae.

(d) Meng et al. [21] in 2005 described the skew-symmetric orthogonal solutions to $AX = B$, and provided a numerical algorithm for the solution.

(e) A recent paper [17] by Horn et al. introduced a new and useful concept of *congruence which enabled the authors to treat simultaneously solutions in the classes of hermitian, skew-hermitian, hermitian positive definite and semidefinite matrices, and matrices with positive definite and semidefinite symmetric part. An extension of their results to rings with involution and Hilbert space operators would be valuable.

Most of the research mentioned above utilizes special properties of finite dimensional vector spaces, and employs specialized matrix techniques, which include various matrix decompositions, block matrices and theory of rank.

1.2 A description of results

In this paper we turn our attention to the equations of the title when a, b, c, d and x are elements of a ring $\mathcal{R}$, with or without involution. This point of view emphasizes the purely algebraic nature of the problem without regard to specific properties of matrices or bounded linear operators, and reveals the intrinsic simplicity of the solutions. Thus the equations are studied in a greater generality and in a transparent environment.

A novel feature of our paper is that the results for finite rectangular matrices and bounded linear operators between Banach or Hilbert spaces are derived from theorems for rings using the method of embedding decribed in Section 4.

Having established preliminary results in Section 2, we study the equations $ax = c$ and $xb = d$ in Section 3 in the setting of rings, giving necessary and sufficient conditions for the existence, and the general form of these solutions, including the common solutions. In this section we perform the same analysis for the separate hermitian, skew-hermitian, reflexive and antireflexive solutions to these equations in rings with involution.

Section 4 is concerned with the extensions of the preceding results to finite rectangular matrices with entries in a ring, and to bounded linear operators between Banach or Hilbert spaces. This is achieved by embedding the rectangular matrices as 'blocks' into the ring of square matrices of the same order, and by embedding 'rectangular' operators via operator matrices into the ring of operators acting on the same space.

In Section 5 we apply the results for rectangular matrices to obtaining the common hermitian solutions to the equations of the title. We give an explicit expression in terms of the original elements to these solutions, which in the literature have only been given implicitly in terms of block matrices. It is interesting to observe that the generalized Schur complement of a 2×2 matrix (or a block matrix) arises naturally in the description of the solutions.

In Sections 6 and 7 we investigate respectively positive and real-positive solutions in rings with involution. There is no clear consensus in the literature as to what constitutes positive elements in an involutive ring. We have opted for the definition given in Berberian's monograph [3] on Baer $*$-rings, namely elements of the form $a = x_1^* x_1 + \cdots + x_n^* x_n$. This definition coincides with the usual one for square complex matrices, elements of C^*-algebras and Hilbert space operators, which are the most desirable cases. The key role in these sections is played by generalizations of Albert's lemma, originally characterizing positivity of matrices in 2×2 block form. It is interesting to observe how much this approach simplifies the derivation of the general positive solution to the equation $ax = c$ (and the common general solution to $ax = c$ and $xb = d$) by comparing our proof with that of Khatri and Mitra [18], and that of the present authors [10] for Hilbert space operators.

Again, the method of embedding extends the results of Sections 6 and 7 obtained for rings to the results for matrices and Hilbert space operators. The previous attempt by Phadke and Thakare in [26] to obtain positive solutions contains incorrect results and proofs which are rectified by the present paper. The general real-positive solution is given explicitly in terms of the original elements for the first time, new even for matrices when compared with [33].

2 Preliminaries

The word 'ring' will always mean an associative ring $\mathcal{R}$ with a unit $1 \neq 0$. An *involution* is a unary operation $a \mapsto a^*$ on $\mathcal{R}$ preserved by the addition ($(a+b)^* = a^* + b^*$), reversed by the multiplication ($(ab)^* = b^* a^*$) and satisfying $(a^*)^* = a$ and $1^* = 1$. The set of all *hermitian* elements ($a^* = a$) in a ring $\mathcal{R}$ with involution will be denoted by $\mathcal{R}^{\mathsf{h}}$. An element $a \in \mathcal{R}$ is *regular* (in the sense of von Neumann) if it possesses an *inner inverse* $a^- \in \mathcal{R}$ satisfying $aa^-a = a$. We observe that both aa^- and a^-a are idempotents. In this paper, a^-, $a^=$ will denote arbitrary inner inverses of a regular element a, and $\mathcal{R}^-$ will denote the set of all regular elements of $\mathcal{R}$. Every regular element $a \in \mathcal{R}$ possesses an inner inverse b which satisfies $bab = b$; for this take $b = a^-aa^-$, where a^- is any inner inverse of a. Such an inverse is called a *reflexive inverse*.

For any $a \in \mathcal{R}$ we define sets

$$a\mathcal{R} = \{ax : x \in \mathcal{R}\}, \qquad \mathcal{R}a = \{xa : x \in \mathcal{R}\},$$
$$a^\circ = \{x \in \mathcal{R} : ax = 0\}, \qquad {}^\circ a = \{x \in \mathcal{R} : xa = 0\}.$$

Lemma 2.1 *If $a, b \in \mathcal{R}^-$, then*

$$axb = 0 \Leftrightarrow x = u - a^-aubb^-, \quad u \in \mathcal{R}, \tag{2.1}$$

$$a^\circ = (1 - a^- a)\mathcal{R}, \quad {}^\circ b = \mathcal{R}(1 - bb^-), \tag{2.2}$$

$$a^\circ \cap {}^\circ b = (1 - a^- a)\mathcal{R}(1 - bb^-). \tag{2.3}$$

PROOF. Towards (2.1) suppose that $axb = 0$. Then also $a^- axbb^- = 0$, and $x = x - a^- axbb^-$. The converse is clear. Setting $b = 1$ and then $a = 1$ in (2.1) we get (2.2), which then gives rise to (2.3). □

If m, n are positive integers, we write $\mathcal{R}^{m\times n}$ for the set of all $m \times n$ matrices with entries in $\mathcal{R}$. The concept of regularity can be extended in a natural way to square or rectangular matrices over $\mathcal{R}$. A matrix $A \in \mathcal{R}^{m\times n}$ is *regular* if there exists a matrix $A^- \in \mathcal{R}^{n\times m}$ such that $AA^-A = A$; such a matrix A^- is an *inner inverse* of A. If $\mathcal{R}$ is a ring with involution and $A \in \mathcal{R}^{m\times n}$, then $A^* \in \mathcal{R}^{n\times m}$ denotes the involute transpose of A. A square matrix satisfying $A^* = A$ is called *hermitian* or *self-adjoint*.

The following lemma can be deduced from [23, Theorem 4], and will be needed in Section 5. We give a simplified proof for completeness.

Lemma 2.2 *Let $u, v \in \mathcal{R}^-$. Then*

$$v(1 - u^- u) \in \mathcal{R}^- \Leftrightarrow \begin{bmatrix} u \\ v \end{bmatrix} \text{ is regular} \Leftrightarrow \begin{bmatrix} v \\ u \end{bmatrix} \text{ is regular} \Leftrightarrow u(1 - v^- v) \in \mathcal{R}^-.$$

In this case two inner inverses of $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ are given by

$$\begin{bmatrix} u \\ v \end{bmatrix}^- = \begin{bmatrix} u^- - (1 - u^- u)m^- vu^- & (1 - u^- u)m^- \end{bmatrix} \tag{2.4}$$

and

$$\begin{bmatrix} u \\ v \end{bmatrix}^= = \begin{bmatrix} (1 - v^- v)n^- & v^- - (1 - v^- v)n^- uv^- \end{bmatrix}, \tag{2.5}$$

where m^- and n^- are inner inverses of $m = v(1 - u^- u)$ and $n = u(1 - v^- v)$.

PROOF. First we note that

$$\begin{bmatrix} u \\ v \end{bmatrix}^- = \begin{bmatrix} p \ q \end{bmatrix} \Rightarrow \begin{bmatrix} v \\ u \end{bmatrix}^- = \begin{bmatrix} q \ p \end{bmatrix}. \tag{2.6}$$

This implies that $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ is regular if and only if $\left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right]$ is regular, and gives the relation between the two inner inverses.

If m is regular, then a direct verification proves (2.4); hence $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ is regular.

Conversely, assume that $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ is regular with an inner inverse $\left[\begin{smallmatrix} p & q \end{smallmatrix}\right]$. Multiplying the equation

$$\begin{bmatrix} u \\ v \end{bmatrix} \begin{bmatrix} p \; q \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

by $\left[\begin{smallmatrix} 1-uu^- & 0 \\ -vu^- & 1 \end{smallmatrix}\right]$ from the left and by $1 - u^-u$ from the right, we get

$$\begin{bmatrix} 0 \\ v(1-u^-u)qv(1-u^-u) \end{bmatrix} = \begin{bmatrix} 0 \\ v(1-u^-u) \end{bmatrix},$$

that is, $m = v(1 - u^-u)$ is regular with an inner inverse q. The rest of the proof is obtained using (2.6). □

Remark 2.3 The preceding lemma and its proof can be cast in a ring setting by considering matrices $\left[\begin{smallmatrix} u & 0 \\ v & 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} p & q \\ 0 & 0 \end{smallmatrix}\right]$ instead of $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} p & q \end{smallmatrix}\right]$, etc.

Lemma 2.4 *Let $\mathcal{R}$ be a ring with involution and let $a, b \in \mathcal{R}$ be regular. Then*

$$a^\circ \cap \mathcal{R}^{\mathsf{h}} = (1 - a^-a)\mathcal{R}^{\mathsf{h}}(1 - a^-a)^*, \quad {}^\circ b \cap \mathcal{R}^{\mathsf{h}} = (1 - bb^-)^*\mathcal{R}^{\mathsf{h}}(1 - bb^-). \tag{2.7}$$

If in addition $m = b^*(1 - a^-a)$ (or equivalently $n = a(1 - bb^-)^*$) is regular, then

$$\begin{aligned} &a^\circ \cap {}^\circ b \cap \mathcal{R}^{\mathsf{h}} \\ &\qquad = (1 - (1 - a^-a)m^-b^*)\,(1 - a^-a)\mathcal{R}^{\mathsf{h}}[(1 - (1 - a^-a)m^-b^*)\,(1 - a^-a)]^* \end{aligned} \tag{2.8}$$

or

$$\begin{aligned} &a^\circ \cap {}^\circ b \cap \mathcal{R}^{\mathsf{h}} \\ &\qquad = (1 - (1 - bb^-)^*n^-a)\,(1 - bb^-)^*\mathcal{R}^{\mathsf{h}}[(1 - (1 - bb^-)^*n^-a)\,(1 - bb^-)]^*. \end{aligned} \tag{2.9}$$

PROOF. The inclusion $(1-a^-a)\mathcal{R}^{\mathsf{h}}(1-a^-a)^* \subset a^\circ \cap \mathcal{R}^{\mathsf{h}}$ is clear. Conversely, let $x \in a^\circ \cap \mathcal{R}^{\mathsf{h}}$. Then $x = (1 - a^-a)x = x(1 - a^-a)^*$. Multiplying $x = x(1 - a^-a)^*$ by $1 - a^-a$ from the left, we get $x = (1 - a^-a)x(1 - a^-a)^*$. The second equation in (2.7) is obtained similarly.

Towards (2.8) we observe that $x \in a^\circ \cap {}^\circ b \cap \mathcal{R}^{\mathsf{h}}$ if and only if $x \in \mathcal{R}^{\mathsf{h}}$ and

$$AX = \begin{bmatrix} a & 0 \\ b^* & 0 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

We can apply the result of the preceding part of this proof to the ring $\mathcal{R}^{2\times 2}$ using Lemma 2.2 and the fact that

$$\begin{bmatrix} p & q \end{bmatrix} = \begin{bmatrix} a \\ b^* \end{bmatrix}^- \Leftrightarrow \begin{bmatrix} p & q \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b^* & 0 \end{bmatrix}^- .$$

Thus

$$A^\circ \cap (\mathcal{R}^{2\times 2})^{\mathsf{h}} = (1 - A^- A)(\mathcal{R}^{2\times 2})^{\mathsf{h}}(1 - A^- A)^*$$

and

$$x \in a^\circ \cap {}^\circ b \cap \mathcal{R}^{\mathsf{h}} \Leftrightarrow \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} zsz^* & 0 \\ 0 & 0 \end{bmatrix}$$

where $s \in \mathcal{R}^{\mathsf{h}}$ and $z = (1 - (1 - a^- a)m^- b^*)\,(1 - a^- a)$ is calculated from (2.4). This proves (2.8). Equation (2.9) is derived from (2.5). □

The following result will be used in later parts of the paper. For block matrices the first formula for M^- was given in [5, Corollary 6.3.5]. The element $f := d - ca^- b$ (respectively $g := a - bd^- c$) in the following lemma is known as a *first* (respectively *second*) *generalized Schur complement* (see [30] for a discussion of this complement for matrices). The classical Schur complements have inverses in place of inner inverses. In the later part of this paper we use the notation

$$s(M) = d - ca^- b \tag{2.10}$$

for the first Schur complement of a matrix M given by (2.11).

Lemma 2.5 *Let $a, b, c, d \in \mathcal{R}$ and let a (respectively d) be regular. Let further $aa^- b = b$ and $ca^- a = c$ (respectively $bd^- d = b$ and $dd^- c = c$). Then*

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2.11}$$

is regular in $\mathcal{R}^{2\times 2}$ if and only if $f := d - ca^- b$ (respectively $g := a - bd^- c$) is regular. An inner inverse of M is given by

$$M^- = \begin{bmatrix} a^- + a^- bf^- ca^- & -a^- bf^- \\ -f^- ca^- & f^- \end{bmatrix} \tag{2.12}$$

(*respectively by*

$$M^{=} = \begin{bmatrix} g^- & -g^-bd^- \\ -d^-cg^- & d^- + d^-cg^-bd^- \end{bmatrix}). \tag{2.13}$$

PROOF. Suppose that a is regular and set

$$S = \begin{bmatrix} 1 & 0 \\ -ca^- & 1 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & -a^-b \\ 0 & 1 \end{bmatrix}; \tag{2.14}$$

then $SMT = \operatorname{diag}(a, f)$. Since S, T are invertible, M is regular if and only if $\operatorname{diag}(a, f)$ is regular, which occurs if and only if f is regular. Then (2.12) is calculated from the equation $M^- = T \operatorname{diag}(a^-, f^-)S$.

Let d be regular. Applying the preceding part of the proof to N given by

$$N = \begin{bmatrix} d & c \\ b & a \end{bmatrix} \quad \left(= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} M \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = ZMZ^{-1} \right)$$

and observing that $M^{=} = Z^{-1}N^-Z$, we get (2.13). □

Remark 2.6 We observe that under the conditions $aa^-b = b$ and $ca^-a = c$ the Schur complement $f = s(M) = d - ca^-b$ is independent of the choice of the inner inverse a^-. Indeed, if $a^=$ is another inner inverse of a, then $a^= = a^- + u - a^-auaa^-$, where u is an arbitrary element of $\mathcal{R}$, and

$$\begin{aligned} d - ca^=b &= d - c(a^- + u - a^-auaa^-)b \\ &= d - ca^-b - cub + ca^-auaa^-b \\ &= d - ca^-b. \end{aligned}$$

A similar comment applies to the second Schur complement.

For future reference we note that under the hypotheses of Theorem 2.5 we have

$$MM^- = \begin{bmatrix} aa^- & 0 \\ (1 - ff^-)ca^- & ff^- \end{bmatrix}, \tag{2.15}$$

observing that $MM^- = S^{-1}(SMT)(T^{-1}M^-S^{-1})S = S^{-1}\operatorname{diag}(aa^-, ff^-)S$.

A ring $\mathcal{R}$ is called *regular* if every element of $\mathcal{R}$ is regular. By a well known theorem of von Neumann, the ring $\mathcal{R}^{n\times n}$ of all $n \times n$ matrices over a regular ring $\mathcal{R}$ is regular. Also, every $m \times n$ matrix over a regular ring is regular.

If $\mathcal{R}$ is a ring with involution and $a \in \mathcal{R}$, we say that $b \in \mathcal{R}$ is a *Moore-Penrose inverse* of a, or *MP-inverse* for short, if it satisfies the *Penrose equations* [25]:

$$aba = a, \quad bab = b, \quad (ab)^* = ab, \quad (ba)^* = ba. \tag{2.16}$$

The Moore-Penrose inverse of a is unique if it exists, and is denoted by $a^\dagger$. If the Moore-Penrose inverse $a^\dagger$ of a exists, we say that a is *Moore-Penrose invertible*, or *MP-invertible* for short.

Lemma 2.7 *A hermitian MP-invertible element $a \in \mathcal{R}$ commutes with $a^\dagger$.*

PROOF. From the definition and uniqueness of the MP-inverse it follows that $a^\dagger$ is hermitian. Then $aa^\dagger = (aa^\dagger)^* = (a^\dagger)^* a^* = a^\dagger a$. □

The concept of the Moore-Penrose inverse can be extended in a natural way to square or rectangular matrices over $\mathcal{R}$. We say that an element $a \in \mathcal{R}$ is $*$-*cancellable* if for every $x \in \mathcal{R}$,

$$a^*ax = 0 \Rightarrow ax = 0 \quad \text{and} \quad xaa^* = 0 \Rightarrow xa = 0.$$

It was proved in [19, Theorem 5.4] that $a \in \mathcal{R}$ is Moore-Penrose invertible if and only if a is $*$-cancellable and both aa^* and a^*a are regular. A ring in which $a^*a = 0$ implies $a = 0$ is called $*$-*reducing*. In a $*$-reducing ring every element is $*$-cancellable as $a^*ax = 0$ implies $(ax)^*ax = 0$. The matrix rings $\mathcal{R}^{n\times n}$ are $*$-reducing if and only if for every $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} a_i^* a_i = 0 \;\Rightarrow\; a_i = 0, \quad i = 1, \ldots, n.$$

Rings with this property are called *strongly $*$-reducing*. If $\mathcal{R}$ is strongly $*$-reducing, then also every set $\mathcal{R}^{m\times n}$ of rectangular matrices over $\mathcal{R}$ is $*$-reducing.

Lemma 2.8 *Let $a \in \mathcal{R}$ be $*$-cancellable. If a^*a is regular, then so is a with $a^- = (a^*a)^- a^*$. If a^*a is MP-invertible, then so is a with $a^\dagger = (a^*a)^\dagger a^*$.*

PROOF. Let a^*a be regular and let $b = (a^*a)^- a^*$. Then

$$a^*aba = a^*a(a^*a)^- a^*a = a^*a.$$

Since a is $*$-cancellable, $aba = a$. This proves the regularity of a.

Let a^*a be MP-invertible. We can similarly verify that $b = (a^*a)^\dagger a^*$ satisfies the Penrose equations. $\square$

The preceding result holds also for rectangular matrices over $\mathcal{R}$.

2.2 The real part of an element

Finally we introduce the following concept in rings with involution:

Definition 2.9 Let $\mathcal{R}$ be a ring with involution such that 2 is invertible in $\mathcal{R}$. The *real* or *hermitian part* of $a \in \mathcal{R}$, $\operatorname{Re} a$, is defined by

$$\operatorname{Re} a = \tfrac{1}{2}(a + a^*). \tag{2.17}$$

The following properties of the real part will be useful in the sequel:

(i) $\operatorname{Re} a^* = \operatorname{Re} a$ is hermitian,

(ii) $\operatorname{Re}(x^*ax) = x^*(\operatorname{Re} a)x$ for any $x \in \mathcal{R}$,

(iii) $\operatorname{Re}(a \pm b) = \operatorname{Re} a \pm \operatorname{Re} b$,

(iv) $\operatorname{Re} a = 0$ if a is skew-hermitian ($a^* = -a$).

See the monograph [4] by K. P. S. Bhaskara Rao for more details on generalized inverses in associative (commutative and non-commutative) rings.

3 Equations $ax = c$ and $xb = d$ in a ring

In this section, $\mathcal{R}$ denotes a ring or a ring with involution. The following theorem is a standard result (see for instance Theorem 1 in [2, p. 52] for matrices) whose proof we include for completeness.

Theorem 3.1 *Let $a, b, c \in \mathcal{R}$, and let a and b be regular. The equation*

$$axb = c \tag{3.1}$$

has a solution if and only if $aa^-cb^-b = c$. The general solution is of the form

$$x = a^-cb^- + u - a^-aubb^-, \quad u \in \mathcal{R}. \tag{3.2}$$

PROOF. Let $aa^-cb^-b = c$. Then $x_0 = a^-cb^-$ is a particular solution of $axb = c$. Conversely assume that x satisfies $axb = c$. Then $aa^-cb^-b = aa^-axbb^-b = axb = c$. Further, $a(x - x_0)b = 0$, and the general form of the solution is obtained from Lemma 2.1. □

Setting first $b = 1$ and then $a = 1$ in the preceding theorem, we get the following two corollaries well known for matrices (for instance, Cecioni [6]).

Corollary 3.2 *Let $a, c \in \mathcal{R}$, and let a be regular. The equation*

$$ax = c \tag{3.3}$$

has a solution if and only if $aa^-c = c$. The general solution is of the form

$$x = a^-c + (1 - a^-a)u, \quad u \in \mathcal{R}. \tag{3.4}$$

Corollary 3.3 *Let $b, d \in \mathcal{R}$, and let b be regular. The equation*

$$xb = d \tag{3.5}$$

has a solution if and only if $db^-b = d$. The general solution is of the form

$$x = db^- + v(1 - bb^-), \quad v \in \mathcal{R}. \tag{3.6}$$

Remark 3.4 We observe that

$$aa^-c = c \Leftrightarrow c\mathcal{R} \subset a\mathcal{R} \quad \text{and} \quad db^-b = d \Leftrightarrow \mathcal{R}d \subset \mathcal{R}b. \tag{3.7}$$

3.1 Hermitian and skew-hermitian solutions

We now assume that $\mathcal{R}$ is a ring with involution, and turn our attention to hermitian solutions.

Theorem 3.5 *Let $a, c \in \mathcal{R}$, and let a be regular. Then equation $ax = c$ has a hermitian solution x if and only if $aa^-c = c$ and ac^* is hermitian. The general hermitian solution is of the form*

$$x = a^-c + (1 - a^-a)(a^-c)^* + (1 - a^-a)s(1 - a^-a)^*, \quad s \in \mathcal{R}^{\mathsf{h}}. \tag{3.8}$$

PROOF. By a^- we denote an inner inverse of a.

Assume that $aa^-c = c$ and ac^* is hermitian. By Corollary 3.2, $x = a^-c + (1 - a^-a)(a^-c)^*$ is a solution to $ax = c$, and $x = a^-c + (a^-c)^* - a^-ac^*(a^-)^*$ is hermitian.

Conversely suppose that $ax = c$ has a hermitian solution x. By Corollary 3.2, $aa^-c = c$; also, $ac^* = a(ax)^* = axa^*$ is hermitian. We already know that $y = a^-c + (1 - a^-a)(a^-c)^*$ is a hermitian solution to $ay = c$. The element $x - y$ belongs to $a^\circ \cap \mathcal{R}^{\mathsf{h}}$ which is equal to $(1 - a^-a)\mathcal{R}^{\mathsf{h}}(1 - a^-a)^*$ by Lemma 2.4. □

Remark 3.6 We can express the set of all hermitian solutions to $ax = c$ as a subset of all solutions:

$$x = a^-c + (1 - a^-a)u, \quad u = (a^-c)^* + s(1 - a^-a)^*, \ s \in \mathcal{R}^{\mathsf{h}}.$$

We have the following corollary.

Corollary 3.7 *Let $a, c \in \mathcal{R}$ and let a be right invertible. Then the equation $ax = c$ has a hermitian solution if and only if ac^* is hermitian.*

PROOF. Let $a' \in \mathcal{R}$ be a right inverse of a. Then a' is also an inner inverse of a, and $aa'c = c$. The result then follows from Theorem 3.5. □

A general method for converting results for the equation $ax = c$ to results for $xb = d$ is to consider the ring $\mathcal{R}$ with the opposite multiplication. If $\mathcal{R}$ has involution, then alternatively we may use the fact that $xb = d$ is equivalent to $b^*x^* = d^*$. Applying Theorem 3.5 we obtain the following result.

Theorem 3.8 *Let $b, d \in \mathcal{R}$, and let b be regular. Then equation $xb = d$ has a hermitian solution x if and only if $db^-b = d$ and d^*b is hermitian. The general hermitian solution is of the form*

$$x = db^- + (db^-)^*(1 - bb^-) + (1 - bb^-)^*t(1 - bb^-), \quad t \in \mathcal{R}^{\mathsf{h}}. \tag{3.9}$$

Remark 3.9 The hermitian solutions as a subset of all solutions:

$$x = db^- + v(1 - bb^-), \quad v = (db^-)^* + (1 - bb^-)^*t, \quad t \in \mathcal{R}^{\mathsf{h}}.$$

In analogy with Corollary 3.7, we have the following result.

Corollary 3.10 *Let $b, d \in \mathcal{R}$, and let b be left invertible. Then the equation $xb = d$ has a hermitian solution if and only if d^*b is hermitian.*

We now turn our attention to the common solution of the equations of the title. For finite real or complex matrices this result was obtained by Cecioni in [6]. See also [2,14,22].

Theorem 3.11 *Let $a, b, c, d \in \mathcal{R}$, and let a, b be regular. A common solution to $ax = c$ and $xb = d$ exists if and only if*

$$aa^- c = c, \quad db^- b = d \quad \textit{and} \quad ad = cb. \tag{3.10}$$

The general common solution is of the form

$$x = a^- c + db^- - a^- adb^- + (1 - a^- a)u(1 - bb^-), \quad u \in \mathcal{R}. \tag{3.11}$$

PROOF. If a common solution x exists, then $ad = a(xb) = (ax)b = cb$; the remaining equations in (3.10) follow from Corollaries 3.2 and 3.3. Conversely, let (3.10) hold. By Corollary 3.2, equation $ax = c$ has a general solution $x = a^- c + (1 - a^- a)w$ for $w \in \mathcal{R}$. Choosing $w = db^-$, which is a solution of $wb = d$, we check that $y = a^- c + (1 - a^- a)db^-$ is also a solution to $yb = d$:

$$\begin{aligned} yb &= a^- cb + (1 - a^- a)db^- b = a^- cb + (1 - a^- a)d \\ &= a^- cb + d - a^- ad = a^- cb + d - a^- cb = d. \end{aligned}$$

Let x be an arbitrary common solution to the two equations. We have shown that $y = a^- c + (1 - a^- a)db^-$ is also a common solution, so that $x - y \in a^\circ \cap {}^\circ b = (1 - aa^-)\mathcal{R}(1 - bb^-)$ by Lemma 2.1. □

An element a of a ring $\mathcal{R}$ with involution is called *skew-hermitian* if $a^* = -a$; the set of all skew-hermitian elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^{\mathsf{sh}}$. We have this counterpart to Theorem 3.5.

Theorem 3.12 *Let $a, c \in \mathcal{R}$, where $\mathcal{R}$ is a ring with involution, and let a be regular. Then equation $ax = c$ has a skew-hermitian solution x if and only if $a^- ac = c$ and ac^* is skew-hermitian. The general form of a skew-hermitian solution is given by*

$$x = a^- c - (1 - a^- a)(a^- c)^* + (1 - a^- a)u(1 - a^- a)^*, \quad u \in \mathcal{R}^{\mathsf{sh}}.$$

PROOF. The proof is analogous to the proof of Theorem 3.5 when we observe that $x_0 = a^- c - (a^- c)^* + a^- a(a^- c)^*$ is a particular skew-hermitian solution and that $a^\circ \cap \mathcal{R}^{\mathsf{sh}} = (1 - a^- a)\mathcal{R}^{\mathsf{sh}}(1 - a^- a)^*$. □

Similarly we obtain a counterpart to Theorem 3.8.

In this subsection we assume that $\mathcal{R}$ is a ring with involution with the property that 2 is invertible in $\mathcal{R}$. For a motivation suppose that $W \in \mathcal{B}(H)$ is a reflection in the closed subspace M of the Hilbert space H. The reflection of a point $x \in H$ in M is the point Wx such that the orthogonal projection Px of x onto M is the midpoint of x and Wx. From $Px = \frac{1}{2}(x + Wx)$ we obtain $Wx = 2Px - x$ for all x, that is, $W = 2P - I$. Since $P^* = P = P^2$, it is easy to check that $W^* = W$ and $W^2 = I$. For rings with involution we generalize the concept of a reflection as follows.

Definition 3.13 An element w of $\mathcal{R}$ is called a *reflection* if w is hermitian and unitary, that is, $w^* = w$ and $w^2 = 1$; equivalently, $w = 2p - 1$, where p is a *projection* in $\mathcal{R}$ ($p^* = p = p^2$). We say that $a \in \mathcal{R}$ is *w-reflexive* if $waw = a$.

Lemma 3.14 *Let $\mathcal{R}$ be a ring with involution such that 2 is invertible in $\mathcal{R}$, and let $w \in \mathcal{R}$ be a reflection. Then $x \in \mathcal{R}$ is w-reflexive if and only if*

$$px(1-p) = 0 = (1-p)xp, \ \textit{equivalently} \ \ x = pxp + (1-p)x(1-p), \quad (3.12)$$

where p is the projection $p = \frac{1}{2}(w + 1)$.

PROOF. Suppose first that $wxw = x$. We note that p and $w = 2p - 1$ commute, $pw = p$ and $w(1 - p) = p - 1$. Then $px(1 - p) = -pwxw(1 - p) = -px(1 - p)$ and $(1 - p)xp = (1 - p)wxwp = -(1 - p)xp$. This proves (3.12).

Conversely, if (3.12) holds, then

$$wxw = wpxpw + w(1-p)x(1-p)w = pxp + (1-p)x(1-p) = x. \qquad \square$$

Theorem 3.15 *Let $\mathcal{R}$ be a ring with involution such that 2 is invertible in $\mathcal{R}$, let $w = 2p - 1$ be a reflection, and let $a, c \in \mathcal{R}$ be such that $a_1 = ap$ and $a_2 = a(1 - p)$ are regular. Then $ax = c$ has a w-reflexive solution if and only if $a_1a_1^-c_1 = c_1$ and $a_2a_2^-c_2 = c_2$, where $c_1 = cp$ and $c_2 = c(1 - p)$. The general w-reflexive solution x has the form*

$$x = pa_1^-c_1 + p(1-a_1^-a_1)u_1p + (1-p)a_2^-c_2 + (1-p)(1-a_2^-a_2)u_2(1-p), \quad (3.13)$$

with $u_1, u_2 \in \mathcal{R}$ arbitrary.

PROOF. Suppose that $a_1a_1^-c_1 = c_1$ and $a_2a_2^-c_2 = c_2$, and set $x_0 = pa_1^-c_1 + (1 - p)a_2^-c_2$. Then $ax_0 = a_1a_1^-c_1 + a_2a_2^-c_2 = c_1 + c_2 = c$. Further, $px_0(1 - p) =$

$pa_1^- c_1(1-p) = 0$, and $(1-p)x_0 p = (1-p)a_2^- c_2 p = 0$. By Lemma 3.14, x_0 is w-reflexive.

Let x be a w-reflexive solution to $ax = c$. By Lemma 3.14, $x = x_1 + x_2$, where $x_1 = pxp$ and $x_2 = (1-p)x(1-p)$. From $ax = c$ we obtain $(a_1+a_2)(x_1+x_2) = a_1x_1 + a_2x_2 = c_1 + c_2$. Postmultiplying by p and $1-p$, we get $a_1x_1 = c_1$ and $a_2x_2 = c_2$. Hence $a_1a_1^- c_1 = c_1$ and $a_2a_2^- c_2 = c_2$. Defining x_0 as in the preceding part of the proof, we see that $y = x - x_0$ is a w-reflexive solution to $ay = 0$; the latter equation is equivalent to $a_1y_1 = 0$ and $a_2y_2 = 0$ with y_1 in the subring $p\mathcal{R}p$ and y_2 in the subring $(1-p)\mathcal{R}(1-p)$ of $\mathcal{R}$. The general solutions are $y_1 = p(1-a_1^- a_1)u_1 p$ and $y_2 = (1-p)(1-a_2^- a_2)u_2(1-p)$ with arbitrary $u_1, u_2 \in \mathcal{R}$. □

The general solution in the preceding theorem can be given as a matrix under the correspondence

$$x \leftrightarrow \begin{bmatrix} pxp & px(1-p) \\ (1-p)xp & (1-p)x(1-p) \end{bmatrix}.$$

Equation (3.13) takes the form

$$x = \begin{bmatrix} pa_1^- c_1 + p(1-a_1^- a_1)u_1 p & 0 \\ 0 & (1-p)a_2^- c_2 + (1-p)(1-a_2^- a_2)u_2(1-p) \end{bmatrix} \tag{3.14}$$

(compare with the matrix equation (8) in [24]).

In the next section we will see that the preceding theorem implies the general version of [24, Theorem 1] for rectangular matrices.

By reversing the multiplication in the ring, we get the following analogue of the preceding theorem.

Theorem 3.16 *Let $\mathcal{R}$ be a ring with involution such that 2 is invertible in $\mathcal{R}$, let $w = 2p - 1$ be a reflection, and let $b, d \in \mathcal{R}$ be such that $b_1 = pb$ and $b_2 = (1-p)b$ are regular. Then $xb = d$ has a w-reflexive solution if and only if $d_1b_1^- b_1 = d_1$ and $d_2b_2^- b_2 = d_2$, where $d_1 = pd$ and $d_2 = (1-p)d$. The general w-reflexive solution x has the form*

$$x = d_1b_1^- p + pv_1(1-b_1b_1^-)p + d_2b_2^-(1-p) + (1-p)v_2(1-b_2b_2^-)(1-p), \tag{3.15}$$

with $v_1, v_2 \in \mathcal{R}$ arbitrary.

Let $w \in \mathcal{R}$ be a reflection. We say that $x \in \mathcal{R}$ is *w-antireflexive* if $wxw = -x$. We have the following counterpart of Theorem 3.15.

Theorem 3.17 *Let $\mathcal{R}$ be a ring with involution such that 2 is invertible in $\mathcal{R}$, let $w = 2p - 1$ be a reflection and let $a, c \in \mathcal{R}$ be such that $a_1 = ap$ and $a_2 = a(1-p)$ are regular. Then $ax = c$ has an w-antireflexive solution if and only if $a_1 a_1^- c_2 = c_2$ and $a_2 a_2^- c_1 = c_1$, where $c_1 = cp$ and $c_2 = c(1-p)$. The general w-antireflexive solution x has the form*

$$x = pa_1^- c_2 + p(1 - a_2^- a_2)u_1(1-p) + (1-p)a_2^- c_1 + (1-p)(1 - a_2^- a_2)u_2 p, \quad (3.16)$$

with $u_1, u_2 \in \mathcal{R}$ arbitrary.

Transcribing (3.16) in the matrix form, we get

$$x = \begin{bmatrix} 0 & pa_1^- c_2 + p(1 - a_2^- a_2)u_1(1-p) \\ (1-p)a_2^- c_1 + (1-p)(1 - a_2^- a_2)u_2 p & 0 \end{bmatrix}. \quad (3.17)$$

Theorem 3.16 has an antireflexive counterpart whose formulation is left to the reader.

4 The embedding: From rings to rectangular matrices and operators

In this section we describe a construction that will enable us to extend the results for elements of a ring $\mathcal{R}$ to the square and rectangular matrices over $\mathcal{R}$, and to bounded linear operators between complex Banach or Hilbert spaces.

The results of the preceding section apply to square matrices of the same order n over a ring $\mathcal{R}$ as these form a ring $\mathcal{R}^{n \times n}$ under the usual matrix operations. If $\mathcal{R}$ is a ring with involution, we define the involution in $\mathcal{R}^{n \times n}$ as the involute transpose. In order to extends our theorems to rectangular matrices, we embed the rectangular matrices as 'blocks' into square matrices of the same order, and then interpret the results for the original rectangular matrices. We now describe this embedding procedure.

Suppose that $A, C \in \mathcal{R}^{m \times n}$, and consider equation $AX = C$ for $X \in \mathcal{R}^{n \times n}$. The transfer of the equation to a ring is achieved by defining a and c in the ring $\mathcal{R}^{k \times k}$, where $k = m + n$, by

$$a = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}, \qquad c = \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}. \quad (4.1)$$

If A^- is an inner inverse of A, then we can check that

$$a^- = \begin{bmatrix} 0 & A^- \\ 0 & 0 \end{bmatrix}$$

is an inner inverse of a. From

$$aa^-c = \begin{bmatrix} 0 & 0 \\ AA^-C & 0 \end{bmatrix}$$

we conclude that

$$aa^-c = c \Leftrightarrow AA^-C = C.$$

This means that the matrix equation $AX = C$ is consistent if and only if the equation $ax = c$ is consistent in the ring $\mathcal{R}^{k\times k}$. We note that the solution x is of the form

$$x = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}, \tag{4.2}$$

where $X_{11} \in \mathcal{R}^{n\times n}$, $X_{12} \in \mathcal{R}^{n\times m}$, $X_{21} \in \mathcal{R}^{n\times m}$ and $X_{22} \in \mathcal{R}^{m\times m}$. A straightforward check reveals that X is a solution to $AX = C$ if and only if x of the form (4.2) is a solution to $ax = c$ with $X_{11} = X$. This leads to the following

Lemma 4.1 *Let $A, C \in \mathcal{R}^{m\times n}$, let a, c be defined by (4.1), x by (4.2), and let $k = m + n$. Then the equation $AX = C$ has a solution $X \in \mathcal{R}^{n\times n}$ if and only if the equation $ax = c$ has a solution x in the ring $\mathcal{R}^{k\times k}$ with $X_{11} = X$. In this case there is a one-to-one correspondence between the solutions X of $AX = C$ and the solutions x of $ax = c$ of the form*

$$x = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.3}$$

As a sample of the embedding procedure we give a proof of the matrix version of Theorem 3.5. For $\mathcal{R} = \mathbb{C}$ we recover Theorem 2.1 of [18]. First we make the following observation:

$$AA^-C = C \Leftrightarrow R(C) \subset R(A) \quad \text{and} \quad DB^-B = D \Leftrightarrow N(B) \subset N(D),$$

where $R(T)$ denotes the range and $N(T)$ the nullspace of a matrix T.

Theorem 4.2 *Let $\mathcal{R}$ be a ring with involution, let $A, C \in \mathcal{R}^{m\times n}$ and let A be regular. Then the equation $AX = C$ has a hermitian solution $X \in \mathcal{R}^{n\times n}$ if and only if $R(C) \subset R(A)$ and $AC^* \in \mathcal{R}^{m\times m}$ is hermitian. The general hermitian* *solution is of the form*

$$X = A^-C + (I - A^-A)(A^-C)^* + (I - A^-A)S(I - A^-A)^*,$$
$$S^* = S \in \mathcal{R}^{n\times n}. \tag{4.4}$$

PROOF. Let $k = m + n$ and define a and c in $\mathcal{R}^{k\times k}$ by (4.1). Then

$$ac^* = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} \begin{bmatrix} 0 & C^* \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & AC^* \end{bmatrix},$$

so that

$$(ac^*)^* = ac^* \Leftrightarrow (AC^*)^* = AC^*.$$

Taking into account Lemma 4.1 and applying Theorem 3.5 in the ring $\mathcal{R}^{k\times k}$ we conclude that $AX = C$ has a hermitian solution if and only if $AA^-C = C$ and $AC^* \in \mathcal{R}^{m\times m}$ is hermitian. The equation for the general hermitian solution to $AX = C$ is obtained from Theorem 3.5 considering only solutions x to $ax = c$ of the form (4.3). □

An argument similar to the one given in the proof of Theorem 4.2 can be used to extend Corollaries 3.2, 3.3, Theorem 3.8 and Corollaries 3.7, 3.10 to finite matrices over $\mathcal{R}$. Details are left to the reader.

Next we discuss common solutions of the equations $AX = C$ and $XB = D$ for (rectangular) matrices over $\mathcal{R}$. If $A, C \in \mathcal{R}^{m\times n}$, $B, D \in \mathcal{R}^{n\times p}$, the embedding $A \mapsto a$, $C \mapsto c$, $B \mapsto b$ and $D \mapsto d$ takes the form

$$a = \begin{bmatrix} 0 & 0 & 0 \\ A & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 0 & 0 \\ C & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad d = \begin{bmatrix} 0 & 0 & D \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{4.5}$$

where a, b, c and d are elements of the ring $\mathcal{R}^{k\times k}$ with $k = m+n+p$. If A and B are regular, we find that $AX = C$ and $XB = D$ have a common solution X if and only if the equations $ax = c$ and $xb = d$ have a common solution x, where

$$x = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}$$

and $X_{11} = X$. There is a one-to-one correspondence between the common solutions X to $AX = C$ and $XB = D$, and the common solutions to $ax = c$

and $xb = d$, where x is of the form

$$x = \begin{bmatrix} X & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Applying this embedding and Theorem 3.11 in the ring $\mathcal{R}^{k\times k}$, we obtain the following result.

Theorem 4.3 *Let $A, C \in \mathcal{R}^{m\times n}$, $B, D \in \mathcal{R}^{n\times p}$, and let A and B be regular. The equations*

$$AX = C, \quad XB = D$$

have a common solution $X \in \mathcal{R}^{n\times n}$ if and only if $R(C) \subset R(A)$, $N(B) \subset N(D)$, and $AD = CB$. The general solution is of the form

$$X = A^{-}C + DB^{-} - A^{-}ADB^{-} + (I - A^{-}A)U(I - BB^{-}), \quad U \in \mathcal{R}^{n\times n}.$$

We now discuss the reflexive and antireflexive solutions to the matrix equation $AX = C$ for a ring $\mathcal{R}$ with involution such that 2 is invertible in $\mathcal{R}$. Suppose that A and C are $m \times n$ matrices and X is $n \times n$. Let W be an $n \times n$ reflection matrix, that is, $W^* = W$ and $W^2 = I$, and let $P = \frac{1}{2}(W + I_n)$. Let $k = m + n$ and define

$$a = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}, \quad w = \begin{bmatrix} W & 0 \\ 0 & I_m \end{bmatrix}, \quad p = \begin{bmatrix} P & 0 \\ 0 & I_m \end{bmatrix}$$

in the ring $\mathcal{R}^{k\times k}$ with the unit $1 = I_k$. We observe that w is a reflection in $\mathcal{R}^{k\times k}$ and that $p = \frac{1}{2}(w + 1)$. Under this correspondence between A, C, W, P and their ring counterparts a, c, w, p we obtain Theorems 1 and 2 of [24] for rectangular matrices as an application of our Theorems 3.15 and 3.17 with $\mathcal{R} = \mathbb{C}$. For comparison, our AP and $A(I - P)$ are $\begin{bmatrix} A_1 & 0 \end{bmatrix} U^*$ and $\begin{bmatrix} 0 & A_2 \end{bmatrix} U^*$ in the notation of [24]. The details are left to the reader.

Remark 4.4 Equation (12) in [24] giving the general w-antireflexive solution to the matrix equation $AX = B$ is incorrect. Comparing it with our equation (3.17), we conclude that the matrix in equation (12) of [24] should be

$$\begin{bmatrix} 0 & A_1^{+}B_2 + (I_r - A_1^{+}A_1)G_1 \\ A_2^{+}B_1 + (I_{n-r} - A_2^{+}A_2)G_2 & 0 \end{bmatrix}.$$

Most results of Section 3 for rings without involution extend in a natural way to Banach spaces operators. We write $\mathcal{B}(E,F)$ for the set of all bounded linear operators between complex Banach spaces E,F; if $E=F$, we write $\mathcal{B}(E)$. The crucial fact is that the space $\mathcal{B}(E)$ is a ring. We give an example of an embedding of 'rectangular' operators into a ring of operators on the same space in the case of the common solution to the equations $AX=C$ and $XB=D$.

Theorem 4.5 *Let E,F,G be Banach spaces, let $A,C\in\mathcal{B}(E,F)$, $B,D\in\mathcal{B}(G,E)$, and let A and B be regular. The equations*

$$AX=C,\quad XB=D$$

have a common solution $X\in\mathcal{B}(E)$ if and only if $R(C)\subset R(A)$, $N(B)\subset N(D)$ and $AD=CB$. The general solution is of the form

$$X=A^-C+DB^- - A^-ADB^- + (I-A^-A)U(I-BB^-),\quad U\in\mathcal{B}(E).$$

To apply Theorem 3.11, we embed the operators A,C,B and D into the space $\mathcal{B}(E\oplus F\oplus G)$, a ring, by defining operator matrices a,c,b and d as in (4.5), all acting on the space $E\oplus F\oplus G$ in the usual manner.

In the following, H and K denote complex Hilbert spaces. For any bounded linear operator $A\in\mathcal{B}(H,K)$,

$$A \text{ is regular } \Leftrightarrow A \text{ has a closed range } \Leftrightarrow A \text{ is MP-invertible.}$$

Theorem 3.5 is transcribed for Hilbert space operators as follows.

Theorem 4.6 *Let H,K be Hilbert spaces, let $A,C\in\mathcal{B}(H,K)$, and let A be a closed range operator. Then the equation $AX=C$ has a hermitian solution $X\in\mathcal{B}(H)$ if and only if $R(C)\subset R(A)$ and $AC^*\in\mathcal{B}(K)$ is hermitian. The general hermitian solution is of the form*

$$\begin{aligned}X&=A^-C+(I-A^-A)(A^-C)^*+(I-A^-A)S(I-A^-A)^*,\\ &\quad S^*=S\in\mathcal{B}(H).\end{aligned}\tag{4.6}$$

PROOF. We transfer the theorem to a ring, and apply Theorem 3.5. To this end we define operator matrices a and c as in (4.1); as operators they belong to the ring $\mathcal{B}(H\oplus K)$. Then we proceed as in the proof of Theorem 5.1. □

Remark 4.7 For finite complex matrices we recover Khatri and Mitra's result [18, Theorem 2.1]. Phadke and Thakare [26, Theorem 2.1] state the theorem for bounded linear operators on a Hilbert space to itself, however they omit to include the condition $AA^-C = C$ or equivalently $R(C) \subset R(A)$. Further, they do not prove the general form of the solution, but merely show that operators of the form (4.6) are hermitian solutions.

We leave it to the reader to interpret other results of Section 3 for Banach space operators or Hilbert space operators when the involution is present.

5 Common hermitian solution to $ax = c$ and $xb = d$

As an application of Theorem 4.2 we give a proof of the following theorem for rings with involution, which is one of the main results of the paper.

Theorem 5.1 *Let $\mathcal{R}$ be a ring with involution, let $a, b, c, d \in \mathcal{R}$, and let a, b and $m = b^*(1 - a^-a)$ be regular. A common hermitian solution to $ax = c$ and $xb = d$ exists if and only if*

$$aa^-c = c, \quad ad = cb; \quad ac^*,\ b^*d \in \mathcal{R}^{\mathsf{h}}, \tag{5.1}$$

and

$$(1 - mm^-)d^* = (1 - mm^-)b^*a^-c \tag{5.2}$$

hold.

PROOF. Let us write

$$A = \begin{bmatrix} a \\ b^* \end{bmatrix}, \quad C = \begin{bmatrix} c \\ d^* \end{bmatrix}, \quad Q := AC^* = \begin{bmatrix} ac^* & ad \\ (cb)^* & b^*d \end{bmatrix}.$$

The equations $ax = c$ and $xb = d$ have a common hermitian solution x if and only the matrix equation

$$Ax = C \tag{5.3}$$

has a hermitian solution x. By Theorem 4.2, this happens if and only if $AA^-C = C$ and AC^* is hermitian. By Lemma 2.2, A is regular with

$$A^- = \begin{bmatrix} a^- - (1 - a^-a)m^-b^*a^- & (1 - a^-a)m^- \end{bmatrix}. \tag{5.4}$$

Then

$$AA^-C = \begin{bmatrix} a \\ b^* \end{bmatrix} \begin{bmatrix} a^- - (1 - a^-a)m^-b^*a^- & (1 - a^-a)m^- \end{bmatrix} \begin{bmatrix} c \\ d^* \end{bmatrix}$$

$$= \begin{bmatrix} aa^-c \\ (1-mm^-)b^*a^-c + mm^-d^* \end{bmatrix}. \tag{5.5}$$

Hence $AA^-C = C$ if and only if $aa^-c = c$ and $(1-mm^-)d^* = (1-mm^-)b^*a^-c$.

Finally, AC^* is hermitian if and only if ac^* and b^*d are hermitian, and $ad = cb$. This proves the theorem. □

Remark 5.2 We observe that condition (5.2) can be expressed in terms of the generalized Schur complement of the matrix

$$M = \begin{bmatrix} a & c \\ b^* & d^* \end{bmatrix}$$

as

$$(1-mm^-)s(M) = (1-mm^-)(d^* - b^*a^-c) = 0.$$

Further, in view of Lemma 2.2, conditions (5.1) and (5.2) can be replaced by

$$db^-b = d, \quad ad = cb; \quad ac^*,\ b^*d \in \mathcal{R}^{\mathsf{h}},$$

and

$$(1-nn^-)c = (1-nn^-)a(b^-)^*d^*$$

where $n = a(1-bb^-)^*$. We take this opportunity to correct Theorem 4.2 in [10]: In addition to the conditions of that theorem, an operator version of equation (5.2) should be included.

Using the embedding of rectangular blocks into square matrices of the same order as in Section 4, we can extend the preceding theorem to rectangular matrices over a ring $\mathcal{R}$ with involution. When we set $\mathcal{R} = \mathbb{C}$, from the matrix version of Theorem 5.1 we recover Theorem 2.3 of Khatri and Mitra [18]; in this case the matrices are always regular. Khatri and Mitra proved the theorem using the matrices A, C and Q as above, and gave the general solution by the equation (4.4) for the block matrices, without resolving it for the original matrices. We derive the explicit formula now.

By Theorem 4.2, this solution is given by

$$x = A^-C + (1-A^-A)(A^-C)^* + (1-A^-A)u(1-A^-A)^*, \quad u \in \mathcal{R}^{\mathsf{h}}. \tag{5.6}$$

By Lemma 2.2, an inner inverse A^- of A is given by (5.4), or by

$$A^= = \begin{bmatrix} a \\ b^* \end{bmatrix}^= = \begin{bmatrix} (1-bb^-)^*n^- & \left(b^-\right)^* - (1-bb^-)^*n^-a\left(b^-\right)^* \end{bmatrix},$$

where $n = a(1-bb^-)^*$. Substituting for A^- (respectively $A^=$) in (5.6), we get explicit expressions for x in terms of a, c, b, d, $m = b^*(1-a^-a)$ (respectively $n = a(1-bb^-)^*$), and requisite inner inverses. For the sake of succinctness, we express the general solution using the first Schur complements $s(M)$, $s(N)$, of the matrices

$$M = \begin{bmatrix} a & c \\ b^* & d^* \end{bmatrix}, \qquad N = \begin{bmatrix} b^* & d^* \\ a & c \end{bmatrix}.$$

Theorem 5.3 *Under the hypotheses of Theorem* 5.1*, the general common hermitian solution to $ax = c$ and $xb = d$ is given by*

$$\begin{aligned} x = {} & a^-c + (1-a^-a)m^- \, s(M) \\ & + (1-a^-a)(1-m^-m)[a^-c + (1-a^-a)m^- \, s(M)]^* \\ & + (1-a^-a)(1-m^-m)u(1-m^-m)^*(1-a^-a)^*, \\ & u \in \mathcal{R}^{\mathsf{h}}. \end{aligned} \tag{5.7}$$

Alternatively,

$$\begin{aligned} x = {} & (b^-)^*d^* + (1-bb^-)^*n^- \, s(N) \\ & + (1-bb^-)^*(1-n^-n)[(b^-)^*d^* + (1-bb^-)^*n^- \, s(N)]^* \\ & + (1-bb^-)^*(1-n^-n)v(1-n^-n)^*(1-bb^-), \\ & v \in \mathcal{R}^{\mathsf{h}}. \end{aligned} \tag{5.8}$$

PROOF. Write $p = 1 - a^-a$ for brevity. Then

$$\begin{aligned} 1 - A^-A &= 1 - \begin{bmatrix} a^- - pm^-b^*a^- & pm^- \end{bmatrix} \begin{bmatrix} a \\ b^* \end{bmatrix} \\ &= 1 - a^-a + pm^-b^*a^-a - pm^-b^* \\ &= p - pm^-b^*(1-a^-a) = p(1-m^-m) \\ &= (1-a^-a)(1-m^-m); \end{aligned}$$

$$\begin{aligned} A^-C &= \begin{bmatrix} a^- - pm^-b^*a^- & pm^- \end{bmatrix} \begin{bmatrix} c \\ d^* \end{bmatrix} \\ &= a^-c - pm^-b^*a^-c + pm^-d^* \\ &= a^-c + pm^-(d^* - b^*a^-c) \\ &= a^-c + pm^-s(M) \\ &= a^-c + (1-a^-a)(1-m^-m)s(M). \end{aligned}$$

Hence by (5.6),

$$x = a^-c + pm^-s(M) + p(1-m^-m)(a^-c + pm^-s(M))^*$$

$$+ (1 - a^- a)(1 - m^- m)u(1 - m^- m)^*(1 - a^- a)^*, \; u \in \mathcal{R}^{\mathsf{h}}.$$

The alternative formula (5.8) is obtained from (5.7) by interchanging the equations $ax = c$ and $b^* x = d^*$. □

Formulae (5.7) and (5.8) are new even for matrices.

The following result is obtained by applying Theorem 5.1 to square operator matrices formed as in (4.5). It describes the existence of common hermitian solutions for the equations $AX = C$ and $XB = D$ for operators between Hilbert spaces H, K, L. For matrices we recover Khatri and Mitra's result [18, Theorem 2.3].

Theorem 5.4 *Let H, K, L be Hilbert spaces. Let $A, C \in \mathcal{B}(H, K)$, $B, D \in \mathcal{B}(L, H)$, and let the operators A, B and $M = B^*(I - A^- A)$ have closed range. Then the equations*

$$AX = C, \quad XB = D$$

have a common hermitian solution $X \in \mathcal{B}(H)$ if and only if

$$R(C) \subset R(A), \quad AD = CB; \quad AC^*,\ B^* D \text{ are hermitian} \tag{5.9}$$

and

$$(I - MM^-)D^* = (I - MM^-)B^* A^- C. \tag{5.10}$$

The general common hermitian solution is given by (5.7) *or* (5.8) *interpreted for operators.*

Again, conditions (5.9) and (5.10) can be replaced by

$$N(B) \subset N(D), \quad AD = CB; \quad AC^*,\ B^* D \text{ are hermitian}$$

and

$$(I - NN^-)C = (I - NN^-)A(B^-)^* D^*,$$

where $N = A(I - BB^-)^*$.

Remark 5.5 In [26], Phadke and Thakare state a version of the preceding theorem as Theorem 2.3 (I). They assume only the regularity of A and B, and a condition equivalent to the self-adjointness of AC^* and $B^* D$. Simple counterexamples show that these conditions are not sufficient for the existence of a common hermitian solution.

6 Positive solutions

In a C^*-algebra, positive elements are the hermitian elements with the nonnegative spectrum. By a classical result obtained independently by Kaplansky and Fukamiya, an element a in a C^*-algebra is positive if and only if it is of the form $a = x^*x$. From this it follows that the sum of two positive elements is again positive. Further, every positive element a possesses a unique positive square root. Since the space of all bounded linear operators on a Hilbert space is a C^*-algebra, all these facts apply also to Hilbert space operators, and as a special case, to square complex matrices.

In rings with involution the situation is different since in these rings the concept of the spectrum is not available. If we define elements to be positive when they are of the form $a = x^*x$, we have no guarantee that the positivity is preserved under addition. Hence we have to adopt a stronger definition, but equivalent to the one for C^*-algebras. The results concerning the positivity of elements in rings with involution can be found in Berberian's monograph [3].

Let $\mathcal{R}$ be a ring with involution. We say that $a \in \mathcal{R}$ is *positive*, written $a \geq 0$, if $a = x_1^*x_1 + \cdots + x_n^*x_n$ for suitable elements $x_1, \ldots, x_n \in \mathcal{R}$. We note that every positive element of $\mathcal{R}$ is hermitian, that x^*ax is positive for all $x \in \mathcal{R}$ if $a \geq 0$, and $a + b$ is positive if $a \geq 0$ and $b \geq 0$. We say that $\mathcal{R}$ obeys the *positive square root axiom* if for each positive $a \in \mathcal{R}$ there exists a unique positive $b \in \mathcal{R}$ such that b double commutes with a and $b^2 = a$. In a ring satisfying the positive square root axiom every positive element a is of the form $a = x^*x$ for some $x \in \mathcal{R}$. Positivity is defined in the same way in rings of square matrices over $\mathcal{R}$. It is not difficult to show that a diagonal matrix $A \in \mathcal{R}^{n\times n}$ is positive in $\mathcal{R}^{n\times n}$ if and only if every diagonal element of A is positive in $\mathcal{R}$.

Lemma 6.1 *Let $\mathcal{R}$ be a ring with involution. If $a \in \mathcal{R}$ is hermitian (positive) and regular, then there exists a hermitian (positive) inner inverse for a.*

PROOF. Let a be regular with an inner inverse b. If a is hermitian, then $c = \operatorname{Re} b$ is a hermitian inner inverse of a. If a is also positive, then $d = cac$ is a positive inner inverse for a. □

We say the $\mathcal{R}$ has the *Gelfand-Naimark property* (*GN-property* for short) if $1 + a$ is invertible for each positive element a. This property is enjoyed by C^*-algebras, and therefore by operators on the same Hilbert space.

Remark 6.2 All the results of this section are valid for C^*-algebras since any C^*-algebra $\mathcal{A}$ is strongly $*$-reducing, $\mathcal{A}^{2\times 2}$ obeys the positive square root axiom, and $\mathcal{A}$ has the GN-property.

For us the most important consequence of the GN-property is the following result.

Lemma 6.3 [20, Theorem 3.2] *Let $\mathcal{R}$ be a ring with involution with the GN-property. Then every regular element $a \in \mathcal{R}$ is MP-invertible.*

Lemma 6.4 (Generalized Albert's lemma I) *Let $\mathcal{R}$ be a strongly $*$-reducing ring with involution and let $\mathcal{R}^{2\times 2}$ obey the positive square root axiom. Let*

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \tag{6.1}$$

be hermitian and let x_{11} be regular. Then X is positive in the ring $\mathcal{R}^{2\times 2}$ if and only if for any hermitian inner inverse x_{11}^- of x_{11} the following conditions are satisfied in $\mathcal{R}$:

(i) *x_{11} is positive,*

(ii) *$x_{11}x_{11}^-x_{12} = x_{12}$,*

(iii) *$s(X) = x_{22} - x_{21}x_{11}^-x_{12}$ is positive.*

PROOF. Since X is hermitian, then so is x_{11}. By Lemma 6.1 we can choose a hermitian inner inverse x_{11}^- of x_{11}.

Assume first that conditions (i)–(iii) are satisfied, and set

$$S = \begin{bmatrix} 1 & -x_{11}^-x_{12} \\ 0 & 1 \end{bmatrix}.$$

Then the diagonal matrix

$$S^*XS = \begin{bmatrix} x_{11} & 0 \\ 0 & s(X) \end{bmatrix} \tag{6.2}$$

is positive, and hence $X = (S^{-1})^*(S^*XS)S^{-1}$ is also positive.

Conversely assume that $X = Y^*Y$ for some $Y \in \mathcal{R}^{2\times 2}$ (the positive square root axiom). Write $Y = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}$ where $Y_i \in \mathcal{R}^{2\times 1}$. We have

$$X = Y^*Y = \begin{bmatrix} Y_1^* \\ Y_2^* \end{bmatrix} \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} = \begin{bmatrix} Y_1^*Y_1 & Y_1^*Y_2 \\ Y_2^*Y_1 & Y_2^*Y_2 \end{bmatrix}.$$

Elements of $\mathcal{R}^{2\times 1}$ are $*$-cancellable as $\mathcal{R}$ is strongly $*$-reducing. Since $Y_1^*Y_1 = x_{11}$ is regular, then so is Y_1^* by Lemma 2.8 with $(Y_1^*)^- = Y_1(Y_1^*Y_1)^-$, where $(Y_1^*Y_1)^-$ is chosen to be hermitian (Lemma 6.1). Then

$$x_{11}x_{11}^-x_{12} = Y_1^*[Y_1(Y_1^*Y_1)^-]Y_1^*Y_2 = Y_1^*(Y_1^*)^-Y_1^*Y_2 = Y_1^*Y_2 = x_{12},$$

that is, condition (ii) is satisfied.

To prove conditions (i) and (iii), define S as in the first part of the proof. Then the diagonal matrix S^*XS given by (6.2) is positive, and so both x_{11} and $s(X)$ are positive. □

The preceding lemma is a strong generalization of the result of Albert [1] for matrices, as well as of the result [8, Theorem 2.2] obtained by Cvetković-Ilić et al. for C^*-algebras and the MP-inverse. The latter result is recovered from Lemma 6.4 when we observe that every C^*-algebra $\mathcal{R}$ is strongly $*$-reducing and obeys the positive square root axiom; $\mathcal{R}^{2\times 2}$ is by definition also a C^*-algebra and therefore obeys the positive square root axiom.

Let X be hermitian and let x_{11}^- be a hermitian inner inverse for x_{11} satisfying $x_{11}x_{11}^-x_{12} = x_{12}$. An arbitrary inner inverse $x_{11}^=$ of x_{11} is given by $x_{11}^= = x_{11}^- + u - x_{11}^-x_{11}ux_{11}x_{11}^-$ for some $u \in \mathcal{R}$. Then

$$\begin{aligned} x_{21}(x_{11}^= - x_{11}^-)x_{12} &= x_{21}(u - x_{11}^-x_{11}ux_{11}x_{11}^-)x_{12} \\ &= x_{12}^*ux_{12} - x_{12}^*x_{11}^-x_{11}ux_{11}x_{11}^-x_{12} \\ &= x_{12}^*ux_{12} - (x_{11}x_{11}^-x_{12})^*ux_{11}x_{11}^-x_{12} \\ &= x_{12}^*ux_{12} - x_{12}^*ux_{12} = 0, \end{aligned}$$

and $x_{22} - x_{21}x_{11}^=x_{12} = x_{22} - x_{21}x_{11}^-x_{12}$. Further, $x_{11}x_{11}^-x_{12} = x_{12}$ is equivalent to $x_{11}x_{11}^=x_{12} = x_{12}$ for any inner inverse $x_{11}^=$. This proves the following lemma.

Lemma 6.5 *If the hypotheses of the preceding lemma hold and $x_{11}x_{11}^-x_{12} = x_{12}$ for some hermitian inner inverse x_{11}^- of x_{11}, then the Schur complement of X is independent of the choice of a hermitian inner inverse x_{11}^-.*

If the matrix X in (6.1) has a special form, we can relate the positivity of X to the positivity of a certain element x in $\mathcal{R}$.

Lemma 6.6 (Generalized Albert's lemma II) *Let $\mathcal{R}$ be a strongly $*$-reducing ring with involution and let $\mathcal{R}^{2\times 2}$ obey the positive square root axiom. Let p_1 be a projection in $\mathcal{R}$, and let $p_2 = 1 - p_1$. For each $x \in \mathcal{R}$ define $\Phi(x)$ as the 2×2 matrix X with the entries $x_{ij} = p_ixp_j$ $(i, j = 1, 2)$. Let $x \in \mathcal{R}$ be hermitian and let x_{11} be regular. Then $x \geq 0$ in $\mathcal{R}$ if and only if $\Phi(x) \geq 0$ in $\mathcal{R}^{2\times 2}$, that is, if and only if conditions* (i), (ii), (iii) *of Lemma* 6.4 *hold.*

PROOF. First we observe that $\Phi\colon \mathcal{R} \to \mathcal{R}^{2\times 2}$ is a $*$-monomorphism of rings (with the unit of $\mathcal{R}$ mapped onto $\left[\begin{smallmatrix} p_1 & 0 \\ 0 & p_2 \end{smallmatrix}\right]$). Let $x \geq 0$ be of the form $x = \sum_{i=1}^n y_i^* y_i$. Then

$$\Phi(x) = \Phi\Big(\sum_{i=1}^n y_i^* y_i\Big) = \sum_{i=1}^n \Phi(y_i)^* \Phi(y_i) \geq 0.$$

(For this part the regularity of x_{11} is not required.)

Conversely, let $\Phi(x) \geq 0$ in $\mathcal{R}^{2\times 2}$. Then conditions (i)–(iii) of Lemma 6.4 are satisfied for $X = \Phi(x)$. Since $\Phi(x)$ is hermitian, so are x_{22} and x_{11}. We can choose a hermitian inner inverse x_{11}^- of x_{11}. Then $f := x_{22} - x_{21}x_{11}^- x_{12} \geq 0$. From

$$\begin{aligned}
\Phi[(1 - x_{21}x_{11}^-)x(1 - x_{21}x_{11}^-)^*] &= \Phi(1 - x_{21}x_{11}^-)\Phi(x)\Phi(1 - x_{21}x_{11}^-)^* \\
&= \begin{bmatrix} p_1 & 0 \\ -x_{21}x_{11}^- p_1 & p_2 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} p_1 & -p_1 x_{11}^- x_{12} \\ 0 & p_2 \end{bmatrix} \\
&= \begin{bmatrix} x_{11} & 0 \\ 0 & f \end{bmatrix} = \Phi(x_{11} + f)
\end{aligned}$$

and the injectivity of Φ we infer that

$$(1 - x_{21}x_{11}^-)x(1 - x_{21}x_{11}^-)^* = x_{11} + f \geq 0.$$

Since $\Phi(1 - x_{21}x_{11}^-)$ is invertible with the inverse $\Phi(1 + x_{21}x_{11}^-)$, $1 - x_{21}x_{11}^-$ is invertible with the inverse $1 + x_{21}x_{11}^-$. Hence

$$x = (1 + x_{21}x_{11}^-)(x_{11} + f)(1 + x_{21}x_{11}^-)^* \geq 0,$$

which completes the proof. □

In the following theorem we give necessary and sufficient conditions for the existence of a positive solution to $ax = c$ and express the general positive solution in terms of inner inverses, however, we need to assume the MP-invertibility of a. The reason for this is that in order to apply Lemma 6.6 we must have projections, that is, *hermitian* idempotents of the form $a^\dagger a$ and $1 - a^\dagger a$.

Theorem 6.7 *Let $\mathcal{R}$ be a strongly $*$-reducing ring with involution and let $\mathcal{R}^{2\times 2}$ obey the positive square root axiom. Let $a, c \in \mathcal{R}$ be such that a is MP-invertible and ca^* regular. Then $ax = c$ has a positive solution $x \in \mathcal{R}$ if and only if*

$$ca^* \geq 0 \quad \text{and} \quad (ca^*)(ca^*)^- c = c. \tag{6.3}$$

The general positive solution is given by

$$x = c^*(ca^*)^- c + (1 - a^- a)s(1 - a^- a)^*, \quad s \geq 0, \tag{6.4}$$

where $(ca^*)^-$ *is a positive inner inverse of* ca^* *and* a^- *an arbitrary inner inverse of* a*. Then* $x_0 = c^*(ca^*)^- c$ *is a particular positive solution to* $ax = c$ *which is independent of the choice of a positive inner inverse of* ca^**.*

PROOF. Suppose first that x is a positive solution to $ax = c$. Then $ca^* = axa^*$ is positive. Let $p_1 = a^\dagger a$ and $p_2 = 1 - a^\dagger a$, and let $X = \Phi(x)$ be defined as in Lemma 6.6. We observe that x_{11} is regular with an inner inverse $x_{11}^- = a^*(ca^*)^- a$. By Lemma 6.1 we can choose $(ca^*)^-$ positive. Then x_{11}^- is also positive. By Lemma 6.6, $\Phi(x)$ is positive and the entries x_{ij} of X satisfy the conditions of Lemma 6.4, in particular, $x_{11}x_{11}^- x_{12} = x_{12}$ with a chosen hermitian inner inverse x_{11}^-. This implies $a^\dagger c p_1 z = a^\dagger c p_2$ with $z = x_{11}^- x_{12}$. Applying a from the left to the penultimate equation, we get $cp_1 z = cp_2$. Then $c = cp_1 + cp_2 = cp_1 + cp_1 z = cp_1 y = ca^\dagger a y = ca^*(a^\dagger)^* y$. Hence

$$ca^*(ca^*)^- c = ca^*(ca^*)^- ca^*(a^\dagger)^* y = ca^*(a^\dagger)^* y = c.$$

Conversely, let (6.3) hold. Set $x_0 = c^*(ca^*)^- c$ with $(ca^*)^-$ positive . Then x_0 is positive, and $ax_0 = ac^*(ca^*)^- c = ca^*(ca^*)^- c = c$.

Now we derive the general form of a positive solution x. We abbreviate ca^* as w and note that w^- is positive. The last condition in Lemma 6.4 requires that the Schur complement $s(X) = x_{22} - x_{21}x_{11}^- x_{12}$ is positive:

$$\begin{aligned} x_{21}x_{11}^- x_{12} &= p_2 x p_1 a^* w^- a p_1 x p_2 \\ &= p_2 x a^* w^- a x p_2 = p_2 c^* w^- c p_2 \\ &= p_2 x_0 p_2. \end{aligned}$$

Then $s(X)$ is positive if and only if $x_{22} = p_2 x_0 p_2 + p_2 u p_2$, where $u \geq 0$. Note that $p_1 x = a^\dagger c = a^\dagger w w^- c = p_1 c^* w^- c = p_1 x_0$, which gives

$$x_{11} = p_1 x_0 p_1, \quad x_{12} = p_1 x_0 p_2.$$

Hence $\Phi(x) = \Phi(x_0 + p_2 u p_2)$ with $u \geq 0$; the injectivity of Φ gives $x = x_0 + p_2 u p_2$. Set $s = p_2 u p_2$. Then $s \geq 0$, and for any choice a^- of an inner inverse of a, $(1 - a^- a)p_2 = (1 - a^- a)(1 - a^\dagger a) = 1 - a^\dagger a$. Thus $p_2 u p_2$ can be written as $(1 - a^- a)s(1 - a^- a)^*$ with some $s \geq 0$, and x is of the form (6.4).

Let w^- be a positive inner inverse of w. Any other inner inverse of w is of the form $w^= = w^- + y - w^- w y w w^-$ for some $y \in \mathcal{R}$. Then

$$c^* w^= c = c^*(w^- + y - w^- w y w w^-) c$$

$$\begin{aligned}
&= c^*w^-c + c^*yc - c^*w^-wyww^-c \\
&= c^*w^-c + c^*yc - (ww^-c)^*yww^-c \\
&= c^*w^-c + c^*yc - c^*yc \\
&= c^*w^-c,
\end{aligned}$$

which shows that the definition of x_0 is independent of the choice of a positive inner inverse of w.

Finally, it is not difficult to check that any x of the form (6.4) is a positive solution to $ax = c$. □

Interpreting the preceding theorem for bounded linear operators on a Hilbert space, we get the following result.

Theorem 6.8 *Let H, K be Hilbert spaces, let $A, C \in \mathcal{B}(H, K)$ and let A and CA^* be closed range operators. Then $AX = C$ has a positive solution $X \in \mathcal{B}(H)$ if and only if*

$$CA^* \geq 0 \quad \text{and} \quad R(C) \subset R(CA^*). \tag{6.5}$$

The general positive solution is given by

$$X = C^*(CA^*)^-C + (I - A^-A)S(I - A^-A)^*, \quad S \geq 0, \tag{6.6}$$

where $(CA^*)^-$ is a positive inverse of CA^* and A^- is an arbitrary inner inverse of A. Then $X_0 = C^*(CA^*)^-C$ is a particular positive solution to $AX = C$, independent of the choice of a positive inner inverse $(CA^*)^-$.

PROOF. Using equations (4.1), we use the embedding $A \mapsto a$ and $C \mapsto c$ into the C^*-algebra $\boldsymbol{\mathcal{R}} = \mathcal{B}(H \oplus K)$ which is a strongly $*$-reducing ring. The set of all 2×2 matrices over $\boldsymbol{\mathcal{R}}$ is also a C^*-algebra, and as such it obeys the positive square root axiom. Recall that by Lemma 4.1 we only need to consider solutions x to $ax = c$ of the form (4.3). We can then apply the preceding theorem observing that

$$X \geq 0 \Leftrightarrow \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \geq 0. \qquad \square$$

Remark 6.9 If the Hilbert spaces in the preceding theorem are finite dimensional, we recover [18, Theorem 2.2] of Khatri and Mitra. Phadke and Thakare [26, Theorem 2.2] sketch a proof of the sufficiency of conditions (6.5). In [26, Corollary 2.2] they give a formula for the general positive solution of the equation $AX = C$, show that every operator of the form (6.6) is a positive solution of $AX = C$, but do not prove that all positive solutions are of the form (6.6).

In [10], the present authors gave a proof of the preceding theorem using strictly Hilbert space operator techniques. By comparison, the proof given here demonstrates the advantages of the ring theoretical setting as it is considerably simpler and shorter and clearly reveals what makes the theorem work.

6.1 Common positive solutions

We address the existence of a common positive solution to equations $ax = c$ and $xb = d$. The formulation utilizing block matrices and their inner inverses owes its concisesness and elegance to Theorem 2.3 obtained by Khatri and Mitra [18] for finite matrices, but lacks explicit expressions in terms of the original operators. We fill this gap in Theorem 6.13 obtaining results which are new even for finite matrices.

Theorem 6.10 *Let $\mathcal{R}$ be a strongly $*$-reducing ring with involution and let $\mathcal{R}^{2\times 2}$ obey the positive square root axiom. Let $a, c, b, d \in \mathcal{R}$, let*

$$A = \begin{bmatrix} a \\ b^* \end{bmatrix}, \quad C = \begin{bmatrix} c \\ d^* \end{bmatrix}, \quad Q = \begin{bmatrix} ca^* & cb \\ (ad)^* & d^*b \end{bmatrix}, \tag{6.7}$$

and let A be MP-invertible and Q regular. The equations $ax = c$ and $xb = d$ have a common positive solution $x \in \mathcal{R}$ if and only if $R(C) \subset R(Q)$ and Q is positive. The general common positive solution is given by

$$x = C^*Q^-C + (1 - A^-A)s(1 - A^-A)^*, \quad s \geq 0, \tag{6.8}$$

where Q^- is an arbitrary positive inner inverse of Q and A^- an arbitrary inner inverse of A. Then $x_0 = C^*Q^-C$ is a particular common positive solution independent of the choice of a positive inner inverse Q^- of Q.

PROOF. Using the embedding into the ring of matrices of the same order, we can verify that Theorem 6.7 holds for rectangular matrices over $\mathcal{R}$. We observe that the equations $ax = c$ and $xb = d$ have a common positive solution in $\mathcal{R}$ if and only if $Ax = C$ has a positive solution. Theorem 6.7 then applies with $Q = CA^*$. Note that the 1×1 matrices C^*Q^-C and A^-A are identified with elements of $\mathcal{R}$. □

This result translates into the following theorem for Hilbert space operators, in which the common solution to $AX = C$ and $XB = D$ is interpreted as the solution to the equation $\boldsymbol{A}X = \boldsymbol{C}$.

Theorem 6.11 *Let H, K be Hilbert spaces, let $A, C \in \mathcal{B}(H, K)$, $B, D \in \mathcal{B}(L, H)$, let*

$$\boldsymbol{A} = \begin{bmatrix} A \\ B^* \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} C \\ D^* \end{bmatrix}, \quad \boldsymbol{Q} = \begin{bmatrix} CA^* & CB \\ (AD)^* & D^*B \end{bmatrix},$$

and let $\boldsymbol{A} \in \mathcal{B}(H, K \oplus L)$ and $\boldsymbol{Q} = \mathcal{B}(K \oplus L)$ be closed range operators. The equations $AX = C$ and $XB = D$ have a common positive solution $X \in \mathcal{B}(H)$ if and only if $\boldsymbol{Q}$ is positive and $R(\boldsymbol{C}) \subset R(\boldsymbol{Q})$. The general common positive solution is given by

$$X = \boldsymbol{C}^*\boldsymbol{Q}^-\boldsymbol{C} + (I - \boldsymbol{A}^-\boldsymbol{A})T(I - \boldsymbol{A}^-\boldsymbol{A})^*, \quad T \geq 0,$$

where $\boldsymbol{Q}^-$ is an arbitrary positive inner inverse of $\boldsymbol{Q}$ and $\boldsymbol{A}^-$ an arbitrary inner inverse of $\boldsymbol{A}$. Then $X_0 = \boldsymbol{C}^*\boldsymbol{Q}^-\boldsymbol{C}$ is a particular common positive solution independent of the choice of a positive inner inverse for $\boldsymbol{Q}$.

Remark 6.12 Theorem 2.3 (II) of Phadke and Thakare [26] assumes merely that $\left[\begin{smallmatrix} CA^* & 0 \\ 0 & D^*B \end{smallmatrix}\right]$ is positive definite, which is not sufficient, and the formula given for the general solution is incorrect.

In the next theorem we describe conditions for the existence of common positive solutions of $ax = c$ and $xb = d$ expressed in terms of the original elements. These conditions will be an explicit transcription of the matrix requirements that A and Q are regular, Q is positive and $R(C) \subset R(Q)$. To do this we use Lemmas 2.2 and 2.5, and Lemma 6.4, and present an explicit form of the general common solution. In the proof of the theorem, A, C and Q are the matrices defined in (6.7). We also define two Schur complements

$$g := s\left(\begin{bmatrix} ca^* & c \\ (cb)^* & d^* \end{bmatrix}\right) = d^* - (cb)^*(ca^*)^-c,$$

$$f := s\left(\begin{bmatrix} ca^* & cb \\ (cb)^* & d^*b \end{bmatrix}\right) = d^*b - (cb)^*(ca^*)^-cb = gb,$$

which will be used in the statement of the theorem.

Theorem 6.13 *Let $\mathcal{R}$ be a strongly $*$-reducing ring with involution and let $\mathcal{R}^{2\times 2}$ have the GN-property and obey the positive square root axiom. Let a, b,* *c, d in $\mathcal{R}$ satisfy the conditions*

$$a, b, m = b^*(1 - a^- a),\ ca^*,\ f \text{ are MP-invertible};\ ca^*(ca^*)^- cb = cb \tag{6.9}$$

with a choice of hermitian inner inverse $(ca^*)^-$. Then $ax = c$ and $xb = d$ have a common positive solution if and only if

(i) *d^*b is hermitian and $cb = ad$,*

(ii) *ca^* and f are positive,*

(iii) *$ca^*(ca^*)^- c = c$ and $ff^- g = g$.*

The general common positive solution is given by

$$\begin{aligned} x &= c^*(ca^*)^- c + g^* f^- g \\ &\quad + (1 - a^- a)(1 - m^- m)t(1 - m^- m)^*(1 - a^- a)^*, \quad t \geq 0. \end{aligned} \tag{6.10}$$

PROOF. In order to apply Theorem 6.10, we need A to be MP-invertible and Q regular. According to Lemma 2.2 and 2.5, conditions (6.9) ensure the regularity of A and Q. The MP-invertibility of A (and of Q) then follows from Lemma 6.3.

In view of Lemma 6.4, (i) and (ii) are equivalent to the positivity of Q. Applying Equation (2.15), we show that condition (iii) is equivalent to $QQ^- C = C$:

$$QQ^- C = \begin{bmatrix} ca^*(ca^*)^- & 0 \\ (1 - ff^-)(cb)^*(ca^*)^- & ff^- \end{bmatrix} \begin{bmatrix} c \\ d^* \end{bmatrix} = \begin{bmatrix} ca^*(ca^*)^- c \\ d^* - (1 - ff^-)g \end{bmatrix} = \begin{bmatrix} c \\ d^* \end{bmatrix}.$$

From Theorem 6.10 we know that $x_0 = C^* Q^- C$ is a particular common positive solution. According to Lemma 2.5, Q^- is of the form

$$Q^- = \begin{bmatrix} (ca^*)^- + (ca^*)^- cbf^-(cb)^*(ca^*)^- & -(ca^*)^- cbf^- \\ -f^-(cb)^*(ca^*)^- & f^- \end{bmatrix}.$$

After a calculation we obtain

$$x_0 = c^*(ca^*)^- c + g^* f^- g.$$

The expression for $1 - A^- A$ is derived as in the proof of Theorem 5.3. Writing $m = b^*(1 - a^- a)$, we obtain

$$1 - A^- A = (1 - a^- a)(1 - m^- m).$$

Substituting this into (6.8) we obtain (6.10). □

Using the regularity of $n = a(1 - bb^-)^*$ and the second Schur complements in the preceding theorem, we obtain an alternative expression for the general common positive solution. This is left to the reader.

The preceding theorem can be applied to Hilbert space operators using the embedding to produce a result, new even in the case of finite matrices.

Theorem 6.14 *Let H, K be Hilbert spaces. Let $A, C \in \mathcal{B}(H, K)$, $B, D \in \mathcal{B}(L, H)$. Let A, B, $M = B^*(I - A^-A)$, CA^* and $F = D^*B - (CB)^*(CA^*)^-CB$ be closed range operators, and let $R(CB) \subset R(CA^*)$. Then the equations $AX = C$ and $XB = D$ have a common positive solution $X \in \mathcal{B}(H)$ if and only if*

(i) *D^*B is hermitian and $AD = CB$,*

(ii) *CA^* and F are positive,*

(iii) *$R(C) \subset R(CA^*)$ and $R(G) \subset R(F)$,*

where $G = D^* - (CB)^*(CA^*)^-C$. The general common positive solution is given by

$$X = C^*(CA^*)^-C + G^*F^-G \\ + (I - A^-A)(I - M^-M)T(I - M^-M)^*(I - A^-A)^*, \quad T \geq 0. \quad (6.11)$$

Remark 6.15 The range inclusions in condition (iii) are in fact equalities. The preceding result gives a corrected version of Theorem 6.3 in [10], which gives the inclusion $R(D^*) \subset R(D^*B)$ instead of the correct $R(G) \subset R(F)$.

7 Real-positive solutions

In this section we assume that 2 is invertible in the given involutive ring $\mathcal{R}$, and recall that the real part of $a \in \mathcal{R}$ is defined by $\operatorname{Re} a = \frac{1}{2}(a + a^*)$. Some useful properties of the real part are listed after Definition 2.9. We say that a is *real-positive* if $\operatorname{Re} a \geq 0$. In this last section we investigate the real-positive solutions to $ax = c$.

Theorem 7.1 *Let $\mathcal{R}$ be a strongly $*$-reducing ring with involution and let $\mathcal{R}^{2\times 2}$ obey the positive square root axiom. Let $a, c \in \mathcal{R}$ with a regular. Then* *the equation* $ax = c$ *has a real-positive solution* $x \in \mathcal{R}$ *if and only if*

$$aa^{-}c = c \quad \textit{and} \quad \mathrm{Re}\,(ac^{*}) \geq 0. \tag{7.1}$$

The element $x_0 = a^{-}c - (a^{-}c)^{*} + a^{-}a(a^{-}c)^{*}$ *is a particular real-positive solution.*

PROOF. If x is a real-positive solution, then $aa^{-}c = c$. Further, $ca^{*} = axa^{*}$ and $\mathrm{Re}\,(ac^{*}) = \mathrm{Re}\,(ca^{*}) = a(\mathrm{Re}\,x)a^{*} \geq 0$.

Conversely, let $aa^{-}c = c$ and let ca^{*} be real-positive. With x_0 defined as in the theorem, we have $ax_0 = c$. Writing $x_0 = a^{-}c - (a^{-}c)^{*} + a^{-}ac^{*}(a^{-})^{*}$, we see that $\mathrm{Re}\,x_0 = a^{-}\mathrm{Re}\,(ac^{*})(a^{-})^{*} \geq 0$. □

Wu [32] and Wu and Cain [33] investigated real-positive solutions to $AX = C$ for matrices addressing the existence and the general form of the solutions; Groß [12] corrected [33, Theorem 5]. The following Corollary to the preceding theorem generalizes [12, Theorem 1] obtained by Groß for matrices.

Corollary 7.2 *Let* $A, C \in \mathcal{B}(H, K)$ *with* A *a closed range operator. Then the equation* $AX = C$ *has a real-positive solution* $X \in \mathcal{B}(H)$ *if and only if*

$$R(C) \subset R(A) \quad \textit{and} \quad \mathrm{Re}\,AC^{*} \geq 0. \tag{7.2}$$

As observed by Groß in [12] for matrices, any x given by

$$x = x_0 + (1 - a^{-}a)t(1 - a^{-}a)^{*}, \quad \mathrm{Re}\,t \geq 0, \tag{7.3}$$

is also a real-positive solution to $ax = c$. It is tempting to think that this equation gives the general real-positive solution, but this is not the case.

Theorem 7.3 *Let* $\mathcal{R}$ *be a strongly* $*$*-reducing ring with involution and let* $\mathcal{R}^{2\times 2}$ *obey the positive square root axiom. Let* $a, c \in \mathcal{R}$, *let* a *and* $\mathrm{Re}\,(ac^{*})$ *be MP-invertible,* $\mathrm{Re}\,(ac^{*})$ *positive, and let* $aa^{\dagger}c = c$. *Then the general real-positive solution of the equation* $ax = c$ *is of the form*

$$\begin{aligned} x = x_0 &+ (1 - a^{\dagger}a)sa^{*}(\mathrm{Re}\,(ac^{*}))^{\dagger}\left\{(\mathrm{Re}\,(ac^{*}))(a^{\dagger})^{*} + \tfrac{1}{4}as^{*}(1 - a^{\dagger}a)\right\} \\ &+ (1 - a^{\dagger}a)t(1 - a^{\dagger}a) \quad s, t \in \mathcal{R},\ \mathrm{Re}\,t \geq 0, \end{aligned} \tag{7.4}$$

where $x_0 = a^{\dagger}c - (a^{\dagger}c)^{*} + a^{\dagger}a(a^{\dagger}c)^{*}$ *is a particular real-positive solution.*

PROOF. According to Theorem 7.1, the conditions of the theorem are sufficient for the existence of real-positive solutions to $ax = c$. Set

$$p_1 = aa^\dagger, \quad p_2 = 1 - aa^\dagger, \quad q_1 = a^\dagger a, \quad q_2 = 1 - a^\dagger a.$$

Then

$$p_1 a = a q_1 = p_1 a q_1 = a, \quad q_1 a^\dagger = a^\dagger p_1 = q_1 a^\dagger p_1 = a^\dagger.$$

Let x be a real-positive solution to $ax = c$ and let $\Phi(x)$ be the matrix with the entries

$$x_{ij} = q_i x q_j, \quad i, j = 1, 2.$$

Then $M := \Phi(\operatorname{Re} x) = \operatorname{Re}\Phi(x)$ is given by

$$M = \begin{bmatrix} \operatorname{Re} x_{11} & \frac{1}{2}(x_{12} + x_{21}^*) \\ \frac{1}{2}(x_{21} + x_{12}^*) & \operatorname{Re} x_{22} \end{bmatrix}. \tag{7.5}$$

From $ax = c$ it follows that $a^\dagger a x = a^\dagger c$ and $q_1 x = a^\dagger c$. Then

$$\begin{aligned} q_1(x + x^*)q_1 &= q_1 x q_1 + q_1 x^* q_1 = a^\dagger c q_1 + q_1 c^* (a^\dagger)^* \\ &= a^\dagger (ca^* + ac^*)(a^\dagger)^* = 2a^\dagger (\operatorname{Re}(ca^*))(a^\dagger)^*. \end{aligned}$$

Since $ax = c$ is solvable by a real-positive x, ac^* is real-positive by Theorem 7.1. Hence $\operatorname{Re} x_{11} = q_1(\operatorname{Re} x)q_1 = a^\dagger(\operatorname{Re}(ca^*))(a^\dagger)^* \geq 0$. Next we show that $\operatorname{Re} x_{11}$ is regular with an inner inverse $g := a^*(\operatorname{Re}(ca^*))^\dagger a$: Write $w := \operatorname{Re}(ca^*)$. Then $p_1 w = w = w p_1$, and

$$\begin{aligned} (\operatorname{Re} x_{11}) g (\operatorname{Re} x_{11}) &= a^\dagger w (a^\dagger)^* a^* w^\dagger a a^\dagger w (a^\dagger)^* = a^\dagger w p_1 w^\dagger p_1 w (a^\dagger)^* \\ &= a^\dagger w w^\dagger w (a^\dagger)^* = a^\dagger w (a^\dagger)^* = \operatorname{Re} x_{11}. \end{aligned}$$

According to Lemma 6.6, the matrix M given by (7.5) is positive, and therefore satisfies the conditions of the generalized Albert's lemma (Lemma 6.4). We have already checked that $\operatorname{Re} x_{11}$ is regular and $\operatorname{Re} x_{11} \geq 0$; then $\operatorname{Re} x_{11}$ has a hermitian inner inverse $(\operatorname{Re} x_{11})^-$. Further, M satisfies

$$\operatorname{Re} x_{11} (\operatorname{Re} x_{11})^- (x_{12} + x_{21}^*) = x_{12} + x_{21}^*. \tag{7.6}$$

Write $y := x_{12} + x_{21}^*$. On application of the relations derived above, (7.6) becomes $a^\dagger w w^\dagger a y = y$, where $a^\dagger w w^\dagger a$ is an idempotent and $y \in q_1 \mathcal{R} q_2$; the general solution is $y = a^\dagger w w^\dagger a u$, where $u \in \mathcal{R} q_2$. Thus

$$x_{12} + x_{21}^* = a^\dagger w\, w^\dagger a u, \quad u \in \mathcal{R} q_2. \tag{7.7}$$

Finally M satisfies the condition

$$\operatorname{Re} x_{22} - \tfrac{1}{4}(x_{21} + x_{12}^*)(\operatorname{Re} x_{11})^-(x_{12} + x_{21}^*) \geq 0.$$

We observe that w and $w^\dagger$ commute being hermitian. Then

$$\begin{aligned}(x_{21}+x_{12}^*)(\operatorname{Re} x_{11})^-(x_{12}+x_{21}^*) &= (u^*a^*ww^\dagger(a^\dagger)^*)a^*w^\dagger a(a^\dagger w\, w^\dagger au)\\ &= u^*a^*ww^\dagger p_1 w^\dagger p_1 ww^\dagger au\\ &= u^*a^*w^\dagger wp_1 w^\dagger p_1 ww^\dagger au\\ &= u^*a^*w^\dagger au, \quad u\in \mathcal{R}q_2.\end{aligned}$$

Thus $\operatorname{Re} x_{22} - \frac{1}{4}u^*a^*w^\dagger au \geq 0$ with an arbitrary $u \in \mathcal{R}q_2$. This is equivalent to

$$x_{22} = \tfrac{1}{4}u^*a^*w^\dagger au + z, \quad z\in q_2\mathcal{R}q_2, \quad \operatorname{Re} z \geq 0.$$

We note that any $u \in \mathcal{R}q_2$ can be written as $s^*(1-a^\dagger a)$ for some $s\in\mathcal{R}$, and any $z\in q_2\mathcal{R}q_2$ with $\operatorname{Re} z\geq 0$ can be expressed as $(1-a^\dagger a)t(1-a^\dagger a)$ for some $t\in\mathcal{R}$ with $\operatorname{Re} t\geq 0$ (we can choose $t=z$).

Summarizing:

$$\begin{aligned} x_{11} &= a^\dagger c a^\dagger a, & x_{12} &= a^\dagger c(1-a^\dagger a)\\ x_{21} &= -x_{12}^* + u^*a^*ww^\dagger(a^\dagger)^*, & x_{22} &= \tfrac{1}{4}u^*a^*w^\dagger au + z.\end{aligned}$$

The general real-positive solution x is then given by

$$\begin{aligned} x &= x_{11}+x_{12}+x_{21}+x_{22}\\ &= a^\dagger ca^\dagger a + a^\dagger c(1-a^\dagger a) - (1-a^\dagger a)c^*(a^\dagger)^* + u^*a^*ww^\dagger(a^\dagger)^*\\ &\quad + \tfrac{1}{4}u^*a^*w^\dagger au + z\\ &= a^\dagger c - (a^\dagger c)^* + a^\dagger a(a^\dagger c)^* + u^*a^*w^\dagger w(a^\dagger)^* + \tfrac{1}{4}u^*a^*w^\dagger au + z\\ &= x_0 + (1-a^\dagger a)sa^*(\operatorname{Re}(ac^*))^\dagger \left\{\operatorname{Re}(ac^*)(a^\dagger)^* + \tfrac{1}{4}as^*(1-a^\dagger a)\right\}\\ &\quad + (1-a^\dagger a)t(1-a^\dagger a)\end{aligned}$$

where $x_0 = a^\dagger c - (a^\dagger c)^* + a^\dagger a(a^\dagger c)^*$, $s\in\mathcal{R}$ is arbitrary, and $t\in\mathcal{R}$ is any element of $\mathcal{R}$ satisfying $\operatorname{Re} t\geq 0$. From Theorem 7.1 we know that $x_0 = a^\dagger c - (a^\dagger c)^* + a^\dagger a(a^\dagger c)^*$ is a particular real-positive solution to $ax=c$. □

We note that equation (7.4) for the general real-positive solution to $ax=c$ is new, and gives an alternative form of the general solution for finite matrices. When we use the embedding to interpret the preceding theorem for bounded linear operators between Hilbert spaces, we obtain the following result.

Corollary 7.4 *Let H, K be Hilbert spaces, $A, C\in\mathcal{B}(H,K)$, let A be a closed range operator,* $\operatorname{Re}(AC^*)$ *a closed range positive operator, and let $R(C)\subset R(A)$. Then the general real-positive solution of the equation $AX=C$ is of the form*

$$X = X_0 + (I-A^\dagger A)SA^*(\operatorname{Re}(AC^*))^\dagger\left\{\operatorname{Re}(AC^*)(A^*)^\dagger + \tfrac{1}{4}AS^*(I-A^\dagger A)\right\}$$

$$+ (I - A^{\dagger}A)T(I - A^{\dagger}A), \quad S, T \in \mathcal{B}(H), \ \operatorname{Re} T \geq 0, \tag{7.8}$$

where $X_0 = A^{\dagger}C - (A^{\dagger}C)^* + A^{\dagger}A(A^{\dagger}C)^*$ *is a particular real-positive solution.*

Let us remark that the method used previously for the investigation of the common positive solutions cannot be applied to the common real-positive solutions to the equations of the title, since that method does not work for solutions which are not necessarily hermitian.

Remark 7.5 After this paper was completed, the authors were alerted to the preprint [35], 'Common hermitian and positive solutions to the adjointable operator equations $AX = C$, $XB = D$' by Qingxiang Xu. In this paper the author extends the results of [10] to the setting of Hilbert C^*-modules, and corrects two mistakes in that work which are also corrected in the present paper.

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