

Holomorphic and meromorphic properties of the g -Drazin inverse

by

J. J. Koliha* V. Rakočević

Abstract

If $A(z)$ is a function of a complex variable with values in the space $\mathcal{B}(X)$ of all bounded linear operators on a Banach space X with each $A(z)$ g -Drazin invertible, we study the conditions under which the g -Drazin inverse $A^D(z)$ is holomorphic and finite-meromorphic. From our results we recover a theorem of Steinberg on meromorphic families of compact operators.

1 Introduction and preliminaries

The g -Drazin inverse for operators [12] arises naturally in the context of isolated spectral points, and generalizes the conventional Drazin inverse introduced by Drazin [8]. This inverse is an important tool in theory of rings, and forms an extensive chapter in matrix analysis, where it features in many applications such as solutions to systems of linear equations, difference equations, optimal control, and even multibody system dynamics. In functional analysis it was applied to systems of singular linear differential equations between Banach spaces (see [15]).

When considering systems of singular linear differential equations, we often need to consider perturbations of the differential operators, which requires the differentiation of the g -Drazin inverses involved in the description of the solutions. If the changes involved in perturbations are of a higher degree of smoothness, we need to consider holomorphic and meromorphic properties of the Drazin inverse.

*Corresponding author

1991 *Mathematics Subject Classification*: 47A60, 47A05, 47A10

Key words and phrases: bounded linear operator, g -Drazin inverse, differentiable g -Drazin inverse, holomorphic g -Drazin inverse, finite-meromorphic g -Drazin inverse

In this section we sketch the necessary background of the Drazin inverse needed in the paper. Section 2 is concerned with holomorphic properties of the Drazin inverse, which are then extended in Section 3 to the study of meromorphic properties. Applications to the case of bounded Drazin index, and to compact operators are obtained as natural consequences of our results.

If A belongs to the class $\mathcal{B}(X)$ of all bounded linear operators on a complex Banach space X and 0 is not an accumulation point of the spectrum of A , then the *spectral projection* of A at 0 is the unique idempotent $P \in \mathcal{B}(X)$ such that

$$(1.1) \quad AP = PA \text{ is quasinilpotent, } A + P \text{ is invertible}$$

(see [12, Theorem 3.1]). Then there exists $r > 0$ such that $zI - (A + P)$ and $zI - AP$ are invertible if $0 < |z| < r$. From

$$zI - A = (zI - AP)P + (zI - (A + P))(I - P)$$

it follows that for any z satisfying $0 < |z| < r$,

$$\begin{aligned} (zI - A)^{-1} &= (zI - AP)^{-1}P + (zI - (A + P))^{-1}(I - P) \\ &= \sum_{n=0}^{\infty} z^{-n-1} A^n - \sum_{n=0}^{\infty} z^n (A + P)^{-n-1} (I - P) \\ &= \sum_{n=0}^{\infty} z^{-n-1} A^n - \sum_{n=0}^{\infty} z^n (A^{\text{D}})^{n+1}, \end{aligned}$$

where

$$(1.2) \quad A^{\text{D}} = (A + P)^{-1}(I - P)$$

is the *g-Drazin inverse* of A . We define the *Drazin index* $i(A)$ of A to be 0 if A is invertible, k if AP is nilpotent of index $k \geq 1$, and ∞ if AP is quasinilpotent but not nilpotent. We observe that $AA^{\text{D}} = A^{\text{D}}A = (A + P)^{-1}(A + P)(I - P) = I - P$, that is

$$(1.3) \quad P = I - AA^{\text{D}}.$$

An operator $A \in \mathcal{B}(X)$ for which 0 is not an accumulation spectral point will be called *g-Drazin invertible*. The *g-Drazin inverse* was introduced and investigated

in [12], its continuity properties were studied in [13], and the basic results on its differentiability were presented in [14].

A g -Drazin invertible $A \in \mathcal{B}(X)$ has a unique decomposition $A = C + Q$, where $C \in \mathcal{B}(X)$ is g -Drazin invertible of Drazin index 1 or 0, $Q \in \mathcal{B}(X)$ is quasinilpotent, and $CQ = 0 = CQ$. The operator C is called the *core part* of A .

We denote the nullspace and range of $T \in \mathcal{B}(X)$ by $N(T)$ and $R(T)$, respectively, and define the *reduced minimum modulus* $\gamma(T)$ of T by $\gamma(T) = \infty$ if $T = 0$ and

$$(1.4) \quad \gamma(T) = \inf \{ \|Tu\| / \text{dist}(u, N(T)) : \text{dist}(u, N(T)) > 0 \}$$

if $T \neq 0$. We recall that $\gamma(T) > 0$ if and only if $R(T)$ is closed [11, p. 231].

In the following, Ω is an open subset of the complex plane, $z_0 \in \Omega$, and $A(z)$ is a $\mathcal{B}(X)$ -valued function. By $A'(z)$ we denote the derivative of $A(z)$ at z , and by $A^D(z)$ the g -Drazin inverse $A(z)^D$. The following result was obtained in [14]:

Theorem 1.1. *Let A be a $\mathcal{B}(X)$ -valued function defined on U such that $A(z)$ is g -Drazin invertible for all $z \in U$, and differentiable at $z_0 \in U$. Then $A^D(z)$ is differentiable at z_0 if and only if $A^D(z)$ is continuous at z_0 . In this case the derivative $(A^D)'(z_0)$ is given by*

$$(1.5) \quad (A^D)'(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{-1} (\lambda - A(z_0))^{-1} A'(z_0) (\lambda - A(z_0))^{-1} d\lambda,$$

where Γ is a cycle in $\mathbb{C} \setminus \{0\}$ surrounding the nonzero part of the spectrum of $A(z_0)$.

We give a brief sketch of the proof whose full details appear in [14]. Assume that $A^D(z)$ is continuous at z_0 . First we express the g -Drazin inverse of $A(z)$ as the integral

$$A^D(z) = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{-1} (\lambda - A(z))^{-1} d\lambda$$

with a cycle Γ surrounding the nonzero part of the spectra of all the $A(z)$ with $|z - z_0| < \delta$ for a suitable $\delta > 0$ (the existence of such δ is guaranteed by the continuity of $A^D(z)$) (see condition (1.7) below). Using the second resolvent equation for the integrand, we calculate the limit of

$$\frac{A^D(z) - A^D(z_0)}{z - z_0}$$

as $z \rightarrow z_0$. This gives (1.5).

[13, Theorem 4.1] gives several conditions equivalent to the continuity of $A^{\text{D}}(z)$ at z_0 . We state some of them which will be needed in the following sections. Here $C(z)$ is the core of $A(z)$ and $P(z)$ the spectral projection of $A(z)$.

$$(1.6) \quad \sup \{ \|A^{\text{D}}(z)\| : |z - z_0| < \delta \} < \infty \text{ for some } \delta > 0,$$

$$(1.7) \quad \sup \{ r(A^{\text{D}}(z)) : |z - z_0| < \delta \} < \infty \text{ for some } \delta > 0,$$

$$(1.8) \quad A^{\text{D}}(z)A(z) \rightarrow A^{\text{D}}(z_0)A(z_0) \text{ as } z \rightarrow z_0,$$

$$(1.9) \quad P(z) \rightarrow P(z_0) \text{ as } z \rightarrow z_0.$$

$$(1.10) \quad C(z) \rightarrow C(z_0) \text{ and } \gamma(C(z)) \rightarrow \gamma(C(z_0)) \text{ as } z \rightarrow z_0.$$

We say that $B \in \mathcal{B}(X)$ is an *inner inverse* of $A \in \mathcal{B}(X)$ if $ABA = A$. The following result originally due to Greville for matrices will be often used in the paper. A proof is given for the reader's convenience.

Lemma 1.2. *Suppose that $A \in \mathcal{B}(X)$ is g -Drazin invertible, $i(A)$ is finite and $l \geq i(A)$ is an integer. If A^{2l+1} has an inner inverse G , then $A^{\text{D}} = A^l G A^l$.*

Proof. For any positive integer k , $A^{l+k}(A^{\text{D}})^k = A^l$. Then

$$A^l G A^l = (A^{\text{D}})^{l+1} A^{2l+1} G A^{2l+1} (A^{\text{D}})^{l+1} = (A^{\text{D}})^{2l+1} A^{2l+1} = A^{\text{D}},$$

which proves the result. \square

In some situations, while a g -Drazin invertible function $A(z)$ will depend analytically on z , so will the spectral projection $P(z)$. We can sometimes reduce the general case to the situation when $P(z)$ is locally constant. This is done using the so-called *stabilizing transformation*: Suppose that projections $Q(z)$ are analytic in an open connected set U . Select $z_0 \in U$, and define

$$(1.11) \quad E(z) = Q(z)Q(z_0) + (I - Q(z))(I - Q(z_0)), \quad z \in U.$$

Since $E(z_0) = I$, there exists $\varepsilon > 0$ such that $E(z)$ is invertible for $|z - z_0| < \varepsilon$. Then

$$E(z)Q(z_0) = Q(z)E(z), \text{ that is, } Q(z) = E(z)^{-1}Q(z_0)E(z),$$

which means that $Q(z)$ is analytically similar to a single projection $Q(z_0)$. A standard analytic continuation argument shows that the analytic similarity holds in all of U .

2 Holomorphic properties of $A^D(z)$

Throughout this section Ω is an open set in the complex plane, and $A(z)$ a continuous function on Ω whose values are bounded linear operators on X . If the operators $A(z)$ are assumed g -Drazin invertible, we shall use $A^D(z)$ for the Drazin inverse of $A(z)$, $C(z)$ for the core of $A(z)$ and $P(z)$ for the spectral projection of $A(z)$. This notation will be used throughout this section.

Theorem 1.1 gives necessary and sufficient conditions for the complex differentiability of $A(z)$ at z_0 .

Recall [1, 16, 18] that a family $A(z)$ of operators with closed ranges is *uniformly regular* at $z_0 \in \Omega$ if the function $\gamma(A(z))$ is continuous at z_0 . This is equivalent to $\gamma(A(z))$ being bounded away from zero on some neighbourhood of z_0 .

From Theorem 1.1 we obtain the following main result on holomorphic behavior of $A^D(z)$.

Theorem 2.1. *Let $A(z)$ be holomorphic at $z_0 \in \Omega$ and g -Drazin invertible for all $z \in \Omega_0 \subset \Omega$, where Ω_0 is a neighbourhood of z_0 . Then the following conditions are equivalent:*

- (i) $A^D(z)$ is a holomorphic function at z_0 ,
- (ii) $A^D(z)$ is a continuous function at z_0 ,
- (iii) the family $C(z)$ is holomorphic and uniformly regular at z_0 ,
- (iv) the family $C(z)$ is uniformly regular at z_0 ,
- (v) there exist closed subspaces M_{z_0} and N_{z_0} of X and a neighbourhood $U_{z_0} \subset \Omega_0$ of the point z_0 such that $R(C(z)) \oplus M_{z_0} = X$ and $N(C(z)) \oplus N_{z_0} = X$ for all $z \in U_{z_0}$.

Proof. By Theorem 1.1, [13, Theorem 4.1] and [18, Theorem 2.1] (see also [1, 16]). □

In the special case when the spectral projections are of finite rank we have the following theorem.

Theorem 2.2. *Let $A(z)$ be holomorphic and g -Drazin invertible for all $z \in \Omega$ with the spectral projections $P(z)$ of finite rank. Then the following conditions are equivalent:*

- (i) $A^D(z)$ is holomorphic,
- (ii) $\dim R(P(z))$ is constant on each connected component of Ω .

Proof. To prove that (i) implies (ii), suppose that $A^D(z)$ is a holomorphic function in Ω . Then so is $P(z) = I - A(z)A^D(z)$. Using a stabilizing transformation (1.11), it is not difficult to deduce that $\dim R(P(z))$ is constant on every connected component of Ω .

To prove that (ii) implies (i), suppose that $\dim R(P(z))$ is constant on each connected component of Ω . Let Ω_0 be a connected component of Ω . Then Ω_0 is open, $\dim N(C(z)) = \dim R(P(z))$ is constant (say c_0) on Ω_0 , and the Drazin index of $A(z)$ satisfies the inequality $i(A(z)) \leq c_0$ for all $z \in \Omega_0$. Hence the Drazin index of $A(z)$ is uniformly bounded on Ω_0 . According to [13, Theorem 5.1], $A^D(z)$ is continuous in Ω_0 , and hence holomorphic by Theorem 1.1. \square

From Theorem 2.1 we obtain the following result.

Theorem 2.3. *Let $A(z)$ be holomorphic in Ω and let $A(z)$ be a finite rank operator for all $z \in \Omega$ so that $A(z)$ has a finite Drazin index for all $z \in \Omega$. Then the following conditions are equivalent:*

- (i) $A^D(z)$ is holomorphic,
- (ii) $\dim R(C(z))$ is constant on each connected component of Ω .

Proof. To prove that (i) implies (ii), suppose that $A^D(z)$ is a holomorphic function in Ω . Then $\dim R(C(z)) < \infty$ for all $z \in \Omega$, and by Theorem 2.1 (v) it is easy to see that $\dim R(C(z))$ is constant on each connected component of Ω .

Now let us prove that (ii) implies (i). Suppose that $\dim R(C(z))$ is constant on each connected component of Ω . Let Ω_0 be a connected component of Ω and let $z_0 \in \Omega_0$. Then $\dim R(C(z))$ is constant (say c_0) on Ω_0 . Since $A(z)$ is holomorphic in Ω and $\dim R(A(z)) < \infty$ for all $z \in \Omega$, by [10, Theorem 1] there is an integer r_0 such that

$$\sup_{z \in \Omega_0} \dim R(A(z)) = r_0.$$

The condition $\dim R(C(z)) = c_0$ for all $z \in \Omega_0$ implies that $\text{des}(A(z)) \leq r_0 - c_0$ for all $z \in \Omega_0$. Thus the Drazin index $i(A(z))$ of $A(z)$ is uniformly bounded by $r_0 - c_0$ on Ω_0 . Let $m \geq r_0 - c_0$. Hence

$$\dim R(A^{2m+1}(z)) = \dim R(C(z)) = c_0 \quad \text{for each } z \in \Omega_0.$$

By [3, Theorem 4.2], $A^{2m+1}(z)$ has a holomorphic inner inverse $G(z)$ on Ω_0 , and by Lemma 1.2,

$$A^D(z) = A^m(z)G(z)A^m(z).$$

Thus $A^D(z)$ is a holomorphic function at z_0 . □

When the family $A(z)$ is commuting, we have the following specialization.

Theorem 2.4. *Let Ω be open, bounded and connected, and let $A(z)$ be holomorphic and g -Drazin invertible for all $z \in \Omega$. Suppose further that the family $A(z)$ is commuting, that is, $A(z)A(\mu) = A(\mu)A(z)$ for all $z, \mu \in \Omega$. Then the following conditions are equivalent:*

- (i) $A^D(z)$ is holomorphic,
- (ii) $R(C(z))$ and $N(C(z))$ are constant on Ω .

Proof. Clearly, by Theorem 2.1, (ii) implies (i).

Conversely, suppose that (i) is true. Then $C(z) = A^2(z)A^D(z)$ is a holomorphic function from Ω into $\mathcal{B}(X)$. Further, since the family $A(z)$ is commuting, by [12, Theorem 4.4], the family $C(z)$ is commuting also. Let us remark that by [12, Theorem 6.4], the Drazin index $i(C(z))$ of $C(z)$ does not exceed 1. Hence $\text{asc}(C(z)) = \text{des}(C(z)) = i(C(z)) \leq 1$. Finally, [1, Theorem 2] and condition (1.10) of Theorem 1.1 (see also [13, condition (4.14) of Theorem 4.1]) implies (ii). □

3 Meromorphic properties of $A^D(z)$

In this section we study conditions under which $A^D(z)$ is a finite-meromorphic operator valued function, and recover the well known Steinberg's theorem [17, Theorem 1].

First we recall some notation (see [2, 3, 4, 5, 18]). Let X, Y, Z be Banach spaces, and $\mathcal{B}(X, Y)$ the set of all bounded linear operators from X into Y . We write $\mathcal{H}(z_0, Z)$ for the set of all Z -valued functions that are holomorphic on a punctured neighborhood of $z_0 \in \mathbb{C}$. Any function $\phi \in \mathcal{H}(z_0, Z)$ has a (unique) Laurent expansion at z_0 :

$$\phi(z) = \sum_{n=-\infty}^{\infty} (z - z_0)^n \phi_n.$$

Set $\nu(\phi; z_0) = \inf\{n : \phi_n \neq 0\}$; if $\phi_n = 0$ for all n , then $\nu(\phi; z_0) = \infty$. An operator valued function $A \in \mathcal{H}(z_0, \mathcal{B}(X, Y))$ is *finite-meromorphic* at z_0 if $\nu(A; z_0) > -\infty$ and if the coefficients of the principal part of the Laurent expansion of $A(z)$ at z_0 are all finite rank operators, that is, if $\nu(A; z_0) = -p$, where $p > 0$, then

$$A(z) = \sum_{n=-p}^{\infty} (z - z_0)^n A_n,$$

with $A_{-p}, A_{-p+1}, \dots, A_{-1}$ of finite rank; $A(z)$ is finite-meromorphic in an open set Ω if it is finite-meromorphic at z for every $z \in \Omega$.

If U is a nonempty connected open subset of \mathbb{C} and $A(z)$ an operator valued function such that $A(z) \in \mathcal{H}(z, \mathcal{B}(X, Y))$ for each z in U , then there exists a discrete subset G of U (G has no accumulation points in U), the so-called *singular set* of $A(z)$ in U , consisting of all points $z_0 \in U$ such that $A(z)$ is not holomorphic at z_0 (see [2, 3, 4, 5, 18]).

For a given $A \in \mathcal{H}(z_0, \mathcal{B}(X, Y))$ and an integer m , let $H_m[A; z_0]$ be the set of all $x \in X$ for which there exists $\phi \in \mathcal{H}(z_0, X)$ such that $\phi(z) \rightarrow x$ as $z \rightarrow z_0$ and $\nu(A\phi; z_0) \geq m + 1$ (note that $(A\phi)(z) \equiv A(z)\phi(z)$). Set

$$H[A; z_0] = \bigcap_m H_m[A; z_0].$$

The *stability number* $k(A; z_0)$ of $A(z)$ at z_0 is defined to be the dimension of the quotient space $H_0[A; z_0]/H[A; z_0]$.

We can now present the main theorem of this section.

Theorem 3.1. *Let $A(z)$ be a $\mathcal{B}(X)$ -valued function $A(z)$ holomorphic in Ω , such that $A(z)$ is g -Drazin invertible with the spectral projection $P(z)$ of finite rank for each $z \in \Omega$. Then $A^D(z)$ is a finite-meromorphic function in Ω .*

Proof. It is enough to prove the theorem under the assumption that Ω is a connected open set. As in Corollary 2.2 we know that $A(z)$ is a Fredholm operator function. Hence $\dim N(A^k(z))$ is finite in Ω for each integer k , and by [9, Theorem 1] there are discrete (possibly empty) subsets Ω_k of Ω and integers c_k , $k = 1, 2, \dots$, such that $\dim N(A^k(z)) = c_k$ for each $z \in \Omega \setminus \Omega_k$, and $\dim N(A^k(\mu)) > c_k$ for each $\mu \in \Omega_k$, $k = 1, 2, \dots$.

Next we prove a constancy result for the Drazin index. Set $\Omega_\infty = \bigcup_{k=1}^\infty \Omega_k$, and choose $\mu_0 \in \Omega \setminus \Omega_\infty$ such that the Drazin index of $A(\mu_0)$ has the following minimum property:

$$(3.1) \quad k_0 = i(A(\mu_0)) = \min_{z \in \Omega \setminus \Omega_\infty} i(A(z)).$$

Observe that $\bigcup_{k=1}^{k_0+1} \Omega_k \equiv \Omega_d$ is a discrete subset of Ω , and $\dim N(A^k(z)) = \dim N(A^k(\mu_0))$ for each $z \in \Omega_a = \Omega \setminus \Omega_d$ and each $k \in \{1, \dots, k_0 + 1\}$ (see also the proof of [7, Theorem 2]). Hence $\text{asc}(A(z)) \leq k_0$, that is, $i(A(z)) \leq k_0$, for each $z \in \Omega_a$. We see that the Drazin index $i(A(z))$ is uniformly bounded by, and in fact equal to, k_0 on Ω_a . Thus $\dim N(A^k(z)) = \dim N(A^{k_0}(z)) = c_{k_0}$ for each $z \in \Omega_a$. Hence $\dim R(P(z)) = c_{k_0}$ is constant for each $z \in \Omega_a$, and by Corollary 2.2 we conclude that $A^D(z)$ is a holomorphic function in Ω_a .

Let $z_0 \in \Omega_d$, and let l be an integer such that $l \geq \max\{i(A(z_0)), k_0\}$.

We observe that there exists $\delta > 0$ such that the open disc $D(z_0, \delta) = \{z \in \mathbb{C} : |z - z_0| < \delta\}$ is contained in $\Omega_a \cup \{z_0\}$. Now $A^{2l+1}(z)$ is a holomorphic function in $D(z_0, \delta)$ and the stability number $k(A^{2l+1}; z)$ is finite for each $z \in D(z_0, \delta)$; more precisely $k(A^{2l+1}; z) = 0$ for each $z \in D(z_0, \delta) \setminus \{z_0\}$, and $k(A^{2l+1}; z_0) > 0$ (see [2, 3, 4, 5, 18]). By [6, Theorem 2.2] there exists a finite-meromorphic inner inverse of A^{2l+1} on $D(z_0, \delta)$, that is, there exists a finite-meromorphic function G defined on $D(z_0, \delta)$ with values in $\mathcal{B}(X)$ such that

$$A^{2l+1}(z)G(z)A^{2l+1}(z) = A^{2l+1}(z) \quad \text{for all } z \in D(z_0, \delta).$$

Hence by Lemma 1.2,

$$(3.2) \quad A^D(z) = A^l(z)G(z)A^l(z) \quad \text{for each } z \in D(z_0, \delta),$$

and $A^D(z)$ is finite-meromorphic at z_0 . The proof of the theorem is complete. \square

As a corollary we obtain Steinberg's theorem [17, Theorem 1].

Corollary 3.2. *Let Ω be an open connected subset of the complex plane, and T a holomorphic function on Ω whose values $T(z)$ are compact operators for $z \in \Omega$. Then either $I - T(z)$ is nowhere invertible in Ω or else $(I - T(z))^{-1}$ is finite-meromorphic in Ω .*

Proof. Set $A(z) = I - T(z)$, $z \in \Omega$. Clearly $A(z)$ satisfies the conditions of Theorem 3.1; hence $A^D(z)$ is a finite-meromorphic function in Ω . Suppose that there is $z_0 \in \Omega$ such that $A(z_0)$ is invertible; then $i(A(z_0)) = 0$. Let Ω_d be the discrete set defined in the proof of Theorem 3.1, and let $\Omega_a = \Omega \setminus \Omega_d$. By the constancy result for the Drazin index established in the proof of Theorem 3.1, $i(A(z)) = 0$ on Ω_a , and $A^D(z) = A^{-1}(z)$ for all $z \in \Omega_a$. \square

As a consequence of Theorems 2.3 and 3.1 we have the following result.

Theorem 3.3. *Let $A(z)$ be an $\mathcal{B}(X)$ valued function holomorphic in Ω and let $A(z)$ be a finite rank operator for all $z \in \Omega$ so that $A(z)$ has a finite index Drazin inverse for all $z \in \Omega$. Then A^D is a finite-meromorphic function in Ω .*

Proof. It is enough to prove the theorem under the assumption that Ω is a connected open set.

Since $A(z)$ is holomorphic in Ω , and $\dim R(A(z)) < \infty$ for all $z \in \Omega$, by [10, Theorem 1], there is an integer r_0 such that

$$\sup_{z \in \Omega} \dim R(A(z)) = r_0.$$

Hence $\text{des}(A(z)) \leq r_0$, that is, $i(A(z)) \leq r_0$, for each $z \in \Omega$. Let l be an integer such that $l \geq r_0$. Now $A^{2l+1}(z)$ is a holomorphic function in Ω and the stability number $k(A^{2l+1}; z)$ is finite for each $z \in \Omega$ [5, Theorem 7.1]. Hence, by [6, Theorem 2.2] there exists a finite-meromorphic inner inverse $G(z)$ of $A^{2l+1}(z)$ on Ω . By Lemma 1.2,

$$(3.3) \quad A^D(z) = A^l(z)G(z)A^l(z) \quad \text{for each } z \in \Omega,$$

and A^D is finite-meromorphic in Ω . The proof of the theorem is complete. \square

Note 3.4. In this note we explore a local type formula for the g -Drazin inverse of a continuous operator valued function $A(z)$, $z \in \Omega$. Suppose that $z_0 \in \Omega$. By

[12, Lemma 2.4] we know that

$$(3.4) \quad A^D(z) = (A(z) + P(z))^{-1}(I - P(z)) \quad \text{for each } z \in \Omega,$$

where $P(z)$ is the spectral projection of $A(z)$ corresponding to $z = 0$. Hence there exists a neighborhood of z_0 (say $U \subset \Omega$) such that $A(z) + P(z_0)$ is invertible for each $z \in U$. Hence by (3.4) we have the following local type formula for the g -Drazin inverse valid for all $z \in U$:

$$(3.5) \quad A^D(z) = (A_z + P_0)^{-1}[I + (P_z - P_0)(A_z - P_0)^{-1}]^{-1}(I - P_z),$$

where we write A_z for $A(z)$, P_z for $P(z)$ and P_0 for $P(z_0)$. It is clear that the formula (3.5) could be of practical use when investigating the continuity and differentiability of the g -Drazin inverse.

Acknowledgement The authors are indebted to an anonymous referee of an earlier version of the paper for a number of useful comments that led to several improvements. They are also grateful to the referee of the current version for his/her suggestions on a presentation of this paper.

References

- [1] B. Aupetit and J. Zemánek, Uniformly regular families of commuting operators, *J. Funct. Anal.* **78** (1988), 24–30.
- [2] H. Bart, *Meromorphic operator valued functions*, Mathematical Centrum, Amsterdam, 1973.
- [3] ———, Holomorphic relative inverses of operator valued functions, *Math. Ann.* **208** (1974), 179–194.
- [4] H. Bart and W. Kabbalo, Local invertibility of meromorphic operator functions, *Proc. Royal Irish Acad.* **78A** (1978), 37–50.
- [5] H. Bart, M. A. Kaashoek and D. C. Lay, Stability properties of finite meromorphic operator functions I, II, III, *Indag. Math.* **36** (1974), 217–259.
- [6] ———, Relative inverses of meromorphic operator functions and associated holomorphic projection functions, *Math. Ann.* **218** (1975), 199–210.
- [7] S. Caradus, Operators with finite ascent and descent, *Pacific J. Math.*, **18** (1966), 437–449.

- [8] M. P. Drazin, Pseudo-inverse in associative rings nad semigroups, *Amer. Math. Monthly* **65** (1958), 506–514.
- [9] I. C. Gohberg, On linear operators depending analytically on a parameter, *Doklady Akad. Nauk SSSR (N.S.)* **78** (1951), 629–632 (Russian).
- [10] J. S. Howland, Analyticity of determinants of operators on a Banach space, *Proc. Amer. Math. Soc.* **28** (1971), 177–180.
- [11] T. Kato, *Perturbation Theory for Linear Operators*, 2nd edition, Springer, Berlin, 1980.
- [12] J. J. Koliha, A generalized Drazin inverse, *Glasgow Math. J.* **38** (1996), 367–381.
- [13] J. J. Koliha and V. Rakočević, Continuity of the Drazin inverse II, *Studia Math.* **131** (1998), 167–177.
- [14] ———, Differentiability of the g -Drazin inverse, *Studia Math.*, to appear
- [15] J. J. Koliha and T. D. Tran, Semistable operators and singularly perturbed differential equations, *J. Math. Anal. Appl.* **231** (1999), 446–458.
- [16] Z. Słodkowski, Operators with closed ranges in spaces of analytic vector valued functions, *J. Funct. Anal.* **69** (1986), 155–177.
- [17] S. Steinberg, Meromorphic families of compact operators, *Arch. Rat. Mech. Anal.* **31** (1968), 372–380.
- [18] G. Ph. A. Thijsse, *Decompositions theorems for finite-meromorphic operator functions*, Thesis, Vrije Universiteit te Amsterdam, 1978.

J. J. Koliha

Department of Mathematics and Statistics

University of Melbourne

VIC 3010

Australia

E-mail: j.koliha@ms.unimelb.edu.au

V. Rakočević

Faculty of Science and Mathematics

University of Niš

18000 Niš

Serbia and Montenegro

E-mail: vrankoc@bankerinter.net