Range projections and the Moore-Penrose inverse in rings with involution

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Abstract

In this paper we study the existence of range projections in rings with involution, relating it to the existence of the Moore-Penrose inverse. The results are applied to the solution of the equation $xbx = x$ in rings with involution, extending the results of T. N. E. Greville for matrices. Simpler new proofs are given of the Moore-Penrose invertibility of regular elements in rings with involution, and of the Ljance's formula.

2000 Mathematics Subject Classification: 46L05, 46C05, 47A10, 47B15

Keywords: C^*-algebra, Moore-Penrose inverse, range projection, idempotent

1 Introduction

In this paper we characterize the existence of range projections of idempotents in rings with involution, and apply the results to the existence of the Moore-Penrose inverse, and to the solution of the equation $xbx = x$, extending the results of T. N. E. Greville [1]. We generalize the concept of range projection and of the Moore-Penrose inverse by introducing a new involution $x \mapsto a^{-1}x^*a$ for some symmetric invertible element a of the involutive ring.

We give a new proof of the Moore-Penrose invertibility of regular elements in rings with involution having the Gelfand-Naimark property based on the existence of range projections, starting with the Moore-Perose invertibility of idempotents.

Recently, the problem of the existence of an idempotent h in a C^*-algebra $\mathcal{A}$ satisfying $h\mathcal{A} = p\mathcal{A}$ and $(1-h)\mathcal{A} = q\mathcal{A}$, where p, q are given projections (symmetric idempotents) in $\mathcal{A}$, has been studied in [4, 5, 6]. In this paper we give an alternative formula (originally obtained by Greville [1] for matrices) for h in terms of p and q in a ring with involution, giving simpler new proofs.

Working in a ring with involution instead of a C^*-algebra clarifies the role played by the norm and the C^*-identity in derivation of various results. The algebraic substitute for the C^*-identity is the assumption that the ring is $*$-reducing ($a^*ax = 0 = yaa^*$ implies $ax = 0 = ya$ for all a). An interesting observation is that this often need not be assumed for all elements of the ring. Also, the Gelfand-Naimark property of C^*-algebras, asserting the invertibility of elements of the form $1 + a^*a$, will be seen as crucial in establishing the existence of range projections and the Moore-Penrose invertibility of regular elements.

An element $a \in \mathcal{R}$ is (*von Neumann*) *regular* if $a \in a\mathcal{R}a$; any solution x of the equation $axa = a$ is called an *inner inverse* of a. Any solution y to $yay = y$ is an *outer inverse* of a. The set of all regular elements in $\mathcal{R}$ will be denoted by $\widehat{\mathcal{R}}$; it obviously includes the invertible group $\mathcal{R}^{-1}$ of $\mathcal{R}$. Later in the paper we prove that in a ring with the GN-property the regular elements are $*$-cancellable.

The same argument that was used for matrices in [8] shows that for any $a \in \mathcal{R}$ there is at most one x such that the following equations hold

$$axa = a, \quad xax = x, \quad ax \in \mathcal{R}^{\mathsf{sym}} \quad \text{and} \quad xa \in \mathcal{R}^{\mathsf{sym}},$$

where $\mathcal{R}^{\mathsf{sym}}$ is the set of all symmetric elements of an involutive ring $\mathcal{R}$. The unique x (if it exists) is denoted by $a^\dagger$ and called the *Moore-Penrose inverse* of a.

A C^*-algebra $\mathcal{A}$ is a special case of a $*$-reducing ring with the GN-property. For an element $a \in \mathcal{A}$ we denote by $\sigma(a)$ the spectrum of a and by $r(a)$ the spectral radius of a. For any nonzero Moore-Penrose invertible element $a \in \mathcal{A}$ we define the *conorm* of a by $\gamma(a) = 1/\|a^\dagger\|$. For any such a,

$$\gamma(a)^2 = \inf\,(\sigma(a^*a) \setminus \{0\}) \tag{1.1}$$

(see [3, 7, 9]). Equation (1.1) is needed in the proof of Theorem 6.1.

2 Range projections

We denote by $\mathcal{R}$ a ring with involution $x \mapsto x^*$ and unit 1; by $\mathcal{R}^{-1}$, $\mathcal{R}^{\mathsf{sym}}$ and $\mathcal{R}^{\mathsf{proj}}$ the set of all invertible, symmetric and simultaneously symmetric and idempotent elements of $\mathcal{R}$, respectively. The elements of $\mathcal{R}^{\mathsf{proj}}$ are called *projections.* An element a of $\mathcal{R}$ is *$*$-cancellable* if

$$a^*ax = 0 \implies ax = 0 \quad \text{and} \quad xaa^* = 0 \implies xa = 0.$$

If all elements in $\mathcal{R}$ are $*$-cancellable, the ring is call *$*$-reducing.* We say that a ring $\mathcal{R}$ with involution has the *Gelfand-Naimark property* (or GN-*property*) if

$$1 + x^*x \in \mathcal{R}^{-1} \quad \text{for all} \quad x \in \mathcal{R}.$$

Every C^*-algebra is a $*$-reducing ring with the GN-property.

Note that the term projection is reserved for an element p of a ring $\mathcal{R}$ which is symmetric and idempotent, that is, $p^* = p = p^2$. If f, $g \in \mathcal{R}$ are idempotents, then $f\mathcal{R} \subset g\mathcal{R} \iff gf = f$; consequently

$$f\mathcal{R} = g\mathcal{R} \iff gf = f \text{ and } fg = g. \tag{2.1}$$

This provides a geometrical motivation for the definition of the range projection. Let $f \in \mathcal{R}$ be an idempotent. Generalizing the definition in Koliha [4], we say that $p \in \mathcal{R}$ is a *range projection* of f if p is a projection satisfying

$$pf = f \text{ and } fp = p. \tag{2.2}$$

The range projection of an idempotent f will be denoted by $f^\perp$.

Theorem 2.1. *Let f be an idempotent in a ring $\mathcal{R}$ with involution. Then the following conditions are equivalent*:

(i) $f + f^* - 1$ *is invertible in* $\mathcal{R}$.

(ii) $f^\perp$ *and* $(f^*)^\perp$ *exist.*

The range projections are unique, given by the formulae

$$f^\perp = f(f + f^* - 1)^{-1}, \quad (f^*)^\perp = (f + f^* - 1)^{-1} f. \tag{2.3}$$

Proof. (i) $\Longrightarrow$ (ii). Let $g = f + f^* - 1$ be invertible in $\mathcal{R}$, and let $p = fg^{-1} = g^{-1}f^*$. We need to show that p is idempotent and symmetric, and $fp = p$, $pf = f$:

$$p^2 = (fg^{-1})(fg^{-1}) = (g^{-1}f^*f)g^{-1} = fg^{-1} = p, \;\; p^* = (fg^{-1})^* = g^{-1}f^* = fg^{-1} = p,$$
$$fp = f^2g^{-1} = fg^{-1} = p, \quad pf = fg^{-1}f = ff^*g^{-1} = f.$$

This proves that $f^\perp = p$. Interchanging f and f^* in the preceding argument, we get $(f^*)^\perp = f^*(f + f^* - 1)^{-1} = (f + f^* - 1)^{-1} f$ since $(f^*)^\perp$ is symmetric.

(ii) $\Longrightarrow$ (i). Let $p = f^\perp$ and $q = (f^*)^\perp$. Then

$$\begin{aligned}(p + q - 1)(f + f^* - 1) &= pf + (fp)^* - p + (fq)^* + qf^* - q - f - f^* + 1\\ &= f + p - p + q + f^* - q - f - f^* + 1 = 1.\end{aligned}$$

Similarly, $(f+f^*-1)(p+q-1)=1$, which proves that $f+f^*-1$ is invertible with

$$(f+f^*-1)^{-1}=f^{\perp}+(f^*)^{\perp}-1. \tag{2.4}$$

To prove the uniqueness of $f^{\perp}$, assume that there exists a projection u such that $fu=u$ and $uf=f$. Then

$$u(f+f^*-1)=uf+(fu)^*-u=f+u-u=f;$$

since $f+f^*-1\in\mathcal{R}^{-1}$, we have $u=f(f+f^*-1)^{-1}$. The result for $(f^*)^{\perp}$ follows by symmetry. ☐

Corollary 2.2. *If an involutive ring $\mathcal{R}$ has the* GN-*property, then every idempotent has a unique range projection given by* (2.3).

Proof. If $\mathcal{R}$ has the GN-property, then for any idempotent f,

$$(f+f^*-1)^2=1+(f^*-f)^*(f^*-f)\in\mathcal{R}^{-1},$$

which implies that $f+f^*-1$ is invertible in $\mathcal{R}$. ☐

Specializing $\mathcal{R}$ to a C^*-algebra, we recover [4, Theorem 1.3].

In a ring $\mathcal{R}$ with involution, $p^{\perp}=p$ for any projection p. For any idempotent $f\in\mathcal{R}$ which possesses the range projection, in agreement with [4, Proposition 1.4] we have

$$(f^*)^{\perp}=1-(1-f)^{\perp}\quad\text{and}\quad(1-f^*)^{\perp}=1-f^{\perp}. \tag{2.5}$$

Definition 2.3. Let e, $f\in\mathcal{R}$ be idempotents. By $\pi(e,f)$ we denote an idempotent $h\in\mathcal{R}$ (if it exists) satisfying the conditions

$$h^{\perp}=e^{\perp},\quad(1-h)^{\perp}=f^{\perp}. \tag{2.6}$$

We note that by (2.5),

$$h^*=(\pi(e,f))^*=\pi(1-f,1-e). \tag{2.7}$$

3 More on Moore-Penrose inverse

Let $\mathcal{R}^\dagger$ denote the set of all elements in $\mathcal{R}$ which have Moore-Penrose inverses. For C^*-algebras it was proved in [2, Theorem 6] and [4, Theorem 3.1] that $\mathcal{R}^\dagger = \widehat{\mathcal{R}}$. We show that this also holds in an involutory ring with the GN-property.

First a characterization of Moore-Penrose invertible idempotents in a ring with involution.

Theorem 3.1. *Let $\mathcal{R}$ be a ring with involution and let $f \in \mathcal{R}$ be an idempotent. Then f is Moore-Penrose invertible if and only if $f + f^* - 1$ is invertible. In this case $f^\dagger = (f^*)^\perp f^\perp$, and $ff^\dagger = f^\perp$, $f^\dagger f = (f^*)^\perp$.*

Proof. Suppose that $f + f^* - 1$ is invertible. By Theorem 2.1 the range projections $f^\perp$ and $(f^*)^\perp$ exist. Let $u = (f^*)^\perp f^\perp$:

$$\begin{aligned}
fuf &= f(f^*)^\perp f^\perp f = ((f^*)^\perp f^*)^* f = f^2 = f,\\
ufu &= (f^*)^\perp f^\perp f (f^*)^\perp f^\perp = (f^*)^\perp f^\perp f f^\perp = (f^*)^\perp f^\perp = u,\\
fu &= f(f^*)^\perp f^\perp = ff^\perp = f^\perp \in \mathcal{R}^{\mathsf{sym}},\\
uf &= (f^*)^\perp f^\perp f = (f^*)^\perp f = (f^*)^\perp \in \mathcal{R}^{\mathsf{sym}}.
\end{aligned}$$

Hence $u = f^\dagger$.

Conversely assume that f is Moore Penrose invertible. A direct verification of the properties of range projections shows that

$$ff^\dagger = f^\perp, \quad f^\dagger f = (f^*)^\perp. \qquad \square$$

Theorem 3.2. *Let $\mathcal{R}$ be an involutive ring with the* GN*-property. Then every regular element of $\mathcal{R}$ is Moore-Penrose invertible.*

Proof. Let $a \in \mathcal{R}$ be regular; then there exists $b \in \mathcal{R}$ such that $aba = a$ and $bab = b$. Set $f = ba$ and $g = ab$. Then f, g are idempotents, $af = a$, $ga = a$, and the Moore-Penrose inverses $f^\dagger = (ba)^\dagger$ and $g^\dagger = (ab)^\dagger$ exist by the preceding theorem. Set $r = f^\dagger f$, $s = gg^\dagger$ and $x = rbs$. We verify that $x = a^\dagger$. First note that $ar = (af)r = aff^\dagger f = af = a$. Similarly, $sa = a$. Then

$$axa = (ar)b(sa) = aba = a, \quad xax = (rbs)a(rbs) = rb(sar)bs = rbabs = rbs = x,$$
$$xa = rbsa = rba = f^\dagger f = (f^*)^\perp f^\perp f = (f^*)^\perp f = (f^*)^\perp \in \mathcal{R}^{\mathsf{sym}},$$
$$ax = arbs = abs = gg^\dagger = g(g^*)^\perp g^\perp = gg^\perp = g^\perp \in \mathcal{R}^{\mathsf{sym}}.$$

This proves the Moore-Penrose invertibility of a. □

The preceding theorem gives an alternative proof of the Moore-Penrose invertibility of regular elements in C^*-algebras.

Corollary 3.3. *In an involutive ring $\mathcal{R}$ with the* GN*-property every regular element is $*$-cancellable.*

Proof. Let $a \in \mathcal{R}$ be regular. By the preceding theorem, the Moore-Penrose $a^\dagger$ of a exists. Suppose that $a^*ax = 0$. Then

$$ax = aa^\dagger ax = (aa^\dagger)^* ax = (a^\dagger)^* a^* ax = 0. \qquad \square$$

Let us remark that for each $a \in \mathcal{R}^\dagger$ we have

$$a^\dagger \mathcal{R} = a^* \mathcal{R} \quad \text{and} \quad \mathcal{R} a^\dagger = \mathcal{R} a^*. \tag{3.1}$$

For example, to prove $a^\dagger \mathcal{R} = a^* \mathcal{R}$, we note that

$$a^\dagger \mathcal{R} = a^\dagger a a^\dagger \mathcal{R} \subset a^\dagger a \mathcal{R} = (a^\dagger a)^* \mathcal{R} = a^* a^{\dagger *} \mathcal{R} \subset a^* \mathcal{R},$$

and

$$a^* \mathcal{R} = a^* a^{\dagger *} a^* \mathcal{R} \subset a^* a^{\dagger *} \mathcal{R} = (a^\dagger a) \mathcal{R} \subset a^\dagger \mathcal{R}.$$

4 Range a-projections and the a-Moore-Penrose inverse

In this section a is a fixed element of $\mathcal{R}^{-1} \cap \mathcal{R}^{\mathsf{sym}}$. Range projections in rings with involution can be generalized by introducing the *a-involution*: The operation $x \mapsto x^{*,a} = a^{-1} x^* a\, \mathcal{R}$ is an involution in $\mathcal{R}$; we denote the ring $\mathcal{R}$ with this involution by $\mathcal{R}_a$. A *range a-projection* of an idempotent $f \in \mathcal{R}$ is the range projection of f in the ring $\mathcal{R}_a$, that is, an idempotent $p \in \mathcal{R}$ satisfying

$$p^* a = ap, \quad fp = p, \quad pf = f; \tag{4.1}$$

we write $p = f^{\perp,a}$.

Lemma 4.1. *Let $\mathcal{R}$ be an involutive ring.*

(i) *An idempotent $p \in \mathcal{R}$ is the range a-projection of f in $\mathcal{R}$ if and only if p is the range projection of f in $\mathcal{R}_a$.*

(ii) *An idempotent* $h \in \mathcal{R}$ *is the range* a^{-1}*-projection of* f^* *in* $\mathcal{R}$ *if and only if* h^* *is the range projection of* $f^{*,a}$ *in* $\mathcal{R}_a$.

Proof. (i) An idempotent $p \in \mathcal{R}$ is the range projection of f in $\mathcal{R}_a$ if and only if $p^*a = ap$ and $fp = p$, $pf = f$, that is, if and only if p is the range a-projection of f.

(ii) Let h be the range a^{-1}-projection of f^* in $\mathcal{R}$. Then $a^{-1}ha = h^*$, $f^*h = h$ and $hf^* = f^*$, which implies

$$(a^{-1}f^*a)(a^{-1}ha) = a^{-1}ha, \qquad (a^{-1}ha)(a^{-1}f^*a) = a^{-1}f^*a,$$

that is, $f^{*,a}h^* = h^*$, $h^*f^{*,a} = f^{*,a}$. Since also $(h^*)^{*,a} = h^*$, h^* is the range projection of $f^{*,a}$ in $\mathcal{R}_a$.

For the converse assume that q is the range projection of $f^{*,a}$ in $\mathcal{R}_a$, and write $h = q^*$. The preceding argument can be reversed to show that h is the range a^{-1}-projection of f^* in $\mathcal{R}$. □

We have the following analogue of Theorem 2.1:

Theorem 4.2. *Let* $\mathcal{R}$ *be an involutive ring, let* $a \in \mathcal{R}^{-1} \cap \mathcal{R}^{\mathsf{sym}}$ *and let* f *be an idempotent. Then the following conditions are equivalent*:

(i) $af + f^*a - a \in \mathcal{R}^{-1}$.

(ii) $f^{\perp,a}$ *and* $(f^*)^{\perp,a^{-1}}$ *exist.*

The range projections are unique, given by the formulae

$$f^{\perp,a} = f(af + f^*a - a)^{-1}a, \quad (f^*)^{\perp,a^{-1}} = f^*a(af + f^*a - a)^{-1}. \tag{4.2}$$

Proof. (i) $\iff$ (iii). We observe that $af + f^*a - a$ is invertible in $\mathcal{R}$ if and only if $f + f^{*,a} - 1$ is invertible in $\mathcal{R}_a$. By Theorem 2.1 this is equivalent to the existence of the range projections of f and $f^{*,a}$ in $\mathcal{R}_a$. By Lemma 4.1, this is in turn equivalent to the existence of the range a-projection of f and of the range a^{-1}-projection of f^* in $\mathcal{R}$.

From Theorem 2.1 and Lemma 4.1 (i) we obtain

$$f^{\perp,a} = f(f + f^{*,a} - 1)^{-1} = f(a^{-1}(af + f^*a - a))^{-1} = f(af + f^*a - a)^{-1}a.$$

The second formula in (4.2) is obtained from this equation by replacing f by f^* and a by a^{-1}, and doing a little algebra. □

Considering the Moore-Penrose inverse in the involutive ring $\mathcal{R}_a$, we are led to the concept of the a-Moore-Penrose inverse: Let $x \in \mathcal{R}$. Then $y \in \mathcal{R}$ is the *a-Moore-Penrose inverse* of x, written $y = x^{\dagger,a}$, if

$$xyx = x, \quad yxy = y, \quad (xy)^* a = axy, \quad (yx)^* a = ayx.$$

Let f be an idempotent in $\mathcal{R}$. According to the results of this section and Theorem 3.1, $f^{\dagger,a}$ exists if and only if $af + f^*a - a$ is invertible; in this case

$$f^{\dagger,a} = [(f^*)^{\perp,a^{-1}}]^* f^{\perp,a} = (af + f^*a - a)^{-1} af (af + f^*a - a)^{-1} a, \tag{4.3}$$

and

$$ff^{\dagger,a} = f^{\perp,a} \qquad f^{\dagger,a} f = [(f^*)^{\perp,a^{-1}}]^*. \tag{4.4}$$

We say that $\mathcal{R}$ has the a-GN-*property* if every element of the form $a + x^*ax$ is invertible in $\mathcal{R}$. We have the following corolary of Theorem 3.2.

Theorem 4.3. *Let $\mathcal{R}$ be an involutive ring with the a-GN-property. Then every regular element of $\mathcal{R}$ is a-Moore-Penrose invertible.*

Proof. Observe that $\mathcal{R}$ has the a-GN-property if and only if $\mathcal{R}_a$ has the GN-property. Theorem 3.2 does the rest. □

5 Equation $xbx = x$ in rings

The following result is a generalization to rings with involution of Greville's theorem [1] for matrices.

Theorem 5.1. *Let $\mathcal{R}$ be an involutive ring with the GN-property, and let $b \in \mathcal{R}$. Then $x \in \mathcal{R}$ satisfies $x = xbx$ if and only if it is expressible in the form*

$$x = (ebf)^\dagger, \tag{5.1}$$

where e and f are projections in $\mathcal{R}$.

Proof. First assume that $xbx = x$. By Theorem 3.2 there exists $x^\dagger$ and

$$x^\dagger = x^\dagger b x^\dagger = (x^\dagger x) b (x x^\dagger) = ebf,$$

where $e = x^\dagger x$ and $f = xx^\dagger$ are projections.

To prove the converse, suppose that x satisfies (5.1). First, let us prove that $fx = x$ and $xe = x$. By (3.1) there exists $u \in \mathcal{R}$ such that $(ebf)^\dagger = (ebf)^*u = f(b^*eu)$. Now $fx = ff(b^*eu) = f(b^*eu) = x$.

In a similar way, there exists $v \in \mathcal{R}$ such that $(ebf)^\dagger = v(ebf)^* = (vfb^*)e$. So $xe = (vfb^*)ee = (vfb^*)e = x$.

Now, finally,

$$xbx = (xe)b(fx) = x(ebf)x = xx^\dagger x = x. \qquad \square$$

The theorem has the following useful corollary (Penrose [8] and Koliha [4]).

Corollary 5.2. *Let $\mathcal{R}$ be an involutive ring with the* GN*-property. An element $h \in \mathcal{R}$ is idempotent if and only if it is expressible in the form*

$$h = (ef)^\dagger, \tag{5.2}$$

where e and f are projections in $\mathcal{R}$.

Proof. Letting $b = 1$ in the above theorem we obtain the corollary. $\square$

The following theorem is a generalization of Greville's result [1] for matrices.

Theorem 5.3. *Let $\mathcal{R}$ be an involutive ring with the* GN*-property, and let $h \in \mathcal{R}$ be idempotent. Then*

$$h = h^\perp u (h^*)^\perp, \tag{5.3}$$

where u is an arbitrary inner inverse of $h^\dagger$.

Proof. By Theorem 3.1, h is Moore-Penrose invertible with $h^\dagger = (h^*)^\perp h^\perp$, while $hh^\dagger = h^\perp$ and $h^\dagger h = (h^*)^\perp$. Suppose that u is an arbitrary inner inverse of $h^\dagger$. Then

$$h^\perp u (h^*)^\perp = hh^\dagger u h^\dagger h = hh^\dagger h = h. \qquad \square$$

Applying the preceding theorem to the a-involution, we get a corollary:

Corollary 5.4. *Let $\mathcal{R}$ be an involutive ring with the a-*GN*-property for some $a \in \mathcal{R}^{-1} \cap \mathcal{R}^{\mathsf{sym}}$, and let $h \in \mathcal{R}$ be idempotent. Then*

$$h = h^{\perp,a} u \, [(h^*)^{\perp,a^{-1}}]^*, \tag{5.4}$$

where u is an arbitrary inner inverse of $h^{\dagger,a}$.

Recall that, for projections $p, q \in \mathcal{R}$, $\pi(p,q)$ denotes an idempotent $h \in \mathcal{R}$ satisfying $p = h^{\perp}$ and $q = (1-h)^{\perp}$. In [6] we have proved that in a ring with involution the idempotent $\pi(p,q)$ is given by the formulae

$$\pi(p,q) = (1-pqp)^{-1}(p-pq) = (p-q)^{-1}(1-q). \tag{5.5}$$

We give an alternative expression for $\pi(p,q)$ which for matrices reduces to the one obtained by Greville [1].

Theorem 5.5. *Let $\mathcal{R}$ be a ring with involution, let p, $q \in \mathcal{R}$ be projections, and let $\pi(p,q)$ be an idempotent $h \in \mathcal{R}$ satisfying $p = h^{\perp}$ and $q = (1-h)^{\perp}$. Then*

$$\pi(p,q) = (1-qp)^{-1}(1-q) = p(p+q-qp)^{-1}. \tag{5.6}$$

Proof. By [6, Theorem 4.4] we know that an idempotent $h = \pi(p,q)$ exists if and only if $p-q$ is invertible, while $(p-q)^{-1} = h + h^* - 1$ and $h = (p-q)^{-1}(1-q)$. This is in turn equivalent to

$$1-pq \in \mathcal{R}^{-1} \quad \text{and} \quad p+q-pq \in \mathcal{R}^{-1}.$$

By the definition of the range projection, $q(1-h) = 1-h$, and so $(1-q)h = 1-q$. Further, $ph = h$, and $(1-qp)h = 1-q$. Hence

$$h = \pi(p,q) = (1-qp)^{-1}(1-q). \tag{5.7}$$

We observe that by (2.7),

$$(h^*)^{\perp} = 1-q, \qquad (1-h^*)^{\perp} = 1-p.$$

As $q-p = (1-p)-(1-q)$ is invertible, we can apply (5.7) with appropriate substitutions:

$$h^* = \pi(1-q, 1-p) = (p+q-pq)^{-1}p,$$

from which the second equality in (5.6) follows. □

6 An application to C^*-algebras

We give an application of our preceding results to C^*-algebras. In our recent paper [5] we proved a formula (originally obtained by Ljance [7] for Hilbert space operators) for the norm of a nontrivial idempotent in a C^*-algebra. This result was obtained by exploiting the spectral properties of the difference of two orthogonal projections in a C^*-algebra. We give a new proof based on results of preceding sections and more elementary observations.

Theorem 6.1. *Let $\mathcal{A}$ be a C^*-algebra and $h \in \mathcal{A}$ a nontrivial idempotent. Then*

$$\|h\| = \frac{1}{\sqrt{1 - \|h^\perp(1-h)^\perp\|^2}}. \tag{6.1}$$

Proof. By Theorem 3.1,

$$h^\perp = hh^\dagger \quad \text{and} \quad (h^*)^\perp = h^\dagger h.$$

Therefore,

$$h = \big(h^\dagger h h^\dagger\big)^\dagger = \big((h^\dagger h)(hh^\dagger)\big)^\dagger = ((h^*)^\perp h^\perp)^\dagger = \big((1-(1-h)^\perp)h^\perp\big)^\dagger.$$

Set $p = h^\perp$ and $q = (1-h)^\perp$. Let us recall that $\|pq\| < 1$, $r(pq) = r(pqp) = \|pq\|^2$. Furthermore, $\sigma(pq) = \sigma_{p\mathcal{A}p}(pqp) \cup \{0\}$, where $\sigma_{p\mathcal{A}p}(x)$ stands for the spectrum of $x \in p\mathcal{A}p$ in the algebra $p\mathcal{A}p$. It is well known that $\sigma(pqp) = \sigma_{p\mathcal{A}p}(pqp) \cup \{0\}$, and $\sigma(pq) = \sigma(pqp)$ [5, Lemma 2.4]. Using (1.1), we obtain

$$\begin{aligned}
\|h\| &= \big\|\big((1-q)p\big)^\dagger\big\| = \frac{1}{\gamma\big((1-q)p\big)} \\
&= \frac{1}{\sqrt{\inf\big(\sigma[((1-q)p)^*(1-q)p] \setminus \{0\}\big)}} = \frac{1}{\sqrt{\inf\big(\sigma(p(1-q)p) \setminus \{0\}\big)}} \\
&= \frac{1}{\sqrt{\inf(\sigma_{p\mathcal{A}p}(p(1-q)p) \setminus \{0\}}} = \frac{1}{\sqrt{\inf(1 - \sigma_{p\mathcal{A}p}(pqp)) \setminus \{0\}}} \\
&= \frac{1}{\sqrt{1 - \sup \sigma_{p\mathcal{A}p}(pqp)}} = \frac{1}{\sqrt{1 - \|pq\|^2}}.
\end{aligned}$$

□

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