Differentiability of the g-Drazin inverse

by

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Abstract. If $A(z)$ is a function of a real or complex variable with values in the space $B(X)$ of all bounded linear operators on a Banach space X with each $A(z)$ g-Drazin invertible, we study conditions under which the g-Drazin inverse $A^{\mathsf{D}}(z)$ is differentiable. From our results we recover a theorem due to Campbell on the differentiability of the Drazin inverse of a matrix valued function and a result on differentiation of the Moore–Penrose inverse in Hilbert spaces.

1 Introduction and preliminaries

The Drazin inverse defined originally for semigroups in [4] in 1958 is an important theoretical and practical tool in algebra and analysis. When $\mathcal{A}$ is an algebra and $a \in \mathcal{A}$, then $b \in \mathcal{A}$ is the *Drazin inverse* of a if

(1.1) $$ab = ba, \quad bab = b, \quad aba = a + u \quad \text{where } u \text{ is nilpotent.}$$

It was observed by Harte [7, 8] and by first author in [11] that in Banach algebras it is more natural to replace the nilpotent element u in (1.1) by a quasinilpotent element. If u in (1.1) is allowed to be quasinilpotent, we call b the *g-Drazin inverse* of a.

The g-Drazin inverse introduced in [11] is a useful construct that finds its applications in a number of areas. In the present paper we concentrate on the g-Drazin inverse in the Banach algebra $B(X)$ of bounded linear operators, and continue the investigation of the continuity of the g-Drazin

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inverse [14] by studying its differentiability. For matrices, this was studied by Campbell [1] and Hartwig and Shoaf [9]. Drazin [5] considered differentiation of the conventional Drazin inverse in associative rings, using a general derivation in the ring.

We can briefly describe the contents of this paper as follows: If $A(z)$ is a function of a real or complex variable with values in the space of all bounded linear operators on a Banach space with each $A(z)$ g-Drazin invertible, we study the conditions under which the g-Drazin inverse $A^{\mathsf{D}}(z)$ is differentiable. From our results we recover a theorem due to Campbell on the differentiability of the Drazin inverse of a matrix valued function and a result on differentiation of the Moore–Penrose inverse in Hilbert spaces.

By $B(X)$ we denote the Banach algebra of all bounded linear operators acting on the complex Banach space X with the usual operator norm. By $\rho(T)$, $\sigma(T)$ and $r(T)$ we denote the resolvent set, the spectrum and the spectral radius of $T \in B(X)$, respectively. We also write $\sigma_0(T)$ for $\sigma(T)\backslash\{0\}$. The sets of all isolated and accumulation spectral points of T are denoted by $\operatorname{iso}\sigma(T)$ and $\operatorname{acc}\sigma(T)$. If $\lambda \in \rho(T)$, then $R(\lambda; T) = (\lambda I - T)^{-1}$ is the resolvent of T. We recall [12] that $0 \in \operatorname{iso}\sigma(T)$ if and only if there exists a nonzero projection $P \in B(X)$ such that

$$AP = PA \text{ is quasinilpotent and } A + P \text{ is invertible;}$$

P is the *spectral projection of* T at 0, and is denoted by A^π [12, Theorem 1.2].

Definition 1.1. (Koliha [11, Definition 2.3]) An operator $A \in B(X)$ is *g-Drazin invertible* if there exists $B \in B(X)$ such that

$$AB = BA, \quad BAB = B, \quad ABA = A + U, \text{ where } r(U) = 0. \tag{1.2}$$

The operator B is called the *g-Drazin inverse of* A, denoted by A^{D}. The *Drazin index* $i(A)$ *of* A is 0 if A is invertible, k if A is not invertible and U is nilpotent of index k, and ∞ otherwise. Definition 1.1 with $i(A)$ finite coincides with the definition of the conventional Drazin inverse (see [3, 4, 10]). An operator A has a conventional Drazin inverse if and only if 0 is at most a pole of the resolvent of A; A has the g-Drazin inverse if and only if $0 \notin \operatorname{acc}\sigma(A)$ ([11, Theorem 4.2], [12, Theorem 1.2]).

We need a representation of A^{D} in terms of the holomorphic calculus for A. A *cycle* is a formal linear combination Γ of loops with integral coefficients; Γ is a *Cauchy cycle* relative to the pair (Ω, K), where K is a compact subset of a nonempty open set $\Omega \subset \mathbb{C}$, if $\Gamma \subset \Omega \backslash K$, $\operatorname{ind}(\Gamma, \lambda) = 0$ for all $\lambda \notin \Omega$ and $\operatorname{ind}(\Gamma, \mu) = 1$ for all $\mu \in K$. The existence of a Cauchy cycle relative to any such pair (Ω, K) is proved in [16, Theorem 13.5]. By [11, Theorem 4.4],

$$A^{\mathsf{D}} = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{-1} R(\lambda; A)\, d\lambda, \tag{1.3}$$

where Γ is a Cauchy cycle relative to $(\mathbb{C}\backslash\{0\}, \sigma_0(A))$. (In the case that A is quasinilpotent, the formula is interpreted in the following way: As $\sigma_0(A) = \emptyset$, Γ can be any cycle in $\mathbb{C}\backslash\{0\}$ with $\operatorname{ind}(\Gamma, 0) = 0$. The integral in (1.3) is zero, which agrees with $A^{\mathsf{D}} = 0$.)

In the sequel we use the following perturbation result involving operator resolvents which follows from [6, Lemma VII.6.3].

Lemma 1.2. *Let A, $A(z) \in B(X)$ for all z in some neighborhood U of z_0, and let $\|A(z) - A\| \to 0$ as $z \to z_0$. If K is a compact subset of the complex plane contained in the resolvent sets of A and $A(z)$ for all $z \in U$, then*

$$\lim_{z \to z_0} R(\lambda; A(z)) = R(\lambda; A) \text{ uniformly for } \lambda \in K. \tag{1.4}$$

2 Differentiability properties of the g-Drazin inverse

In this section, U denotes an open interval in $\mathbb{R}$ or an open subset of $\mathbb{C}$, z_0 a point in U, and $A : U \to B(X)$ an operator valued function. By $A'(z)$ we denote the derivative of $A(z)$ at z, and by $A^{\mathsf{D}}(z)$ the g-Drazin inverse $A(z)^{\mathsf{D}}$. Our main result on the differentiability of the g-Drazin inverse is given in the following theorem:

Theorem 2.1. *Let A be a $B(X)$-valued function defined on U such that $A(z)$ is g-Drazin invertible for all $z \in U$, and differentiable at $z_0 \in U$. Then*

$A^{\mathsf{D}}(z)$ is differentiable at z_0 if and only if $A^{\mathsf{D}}(z)$ is continuous at z_0. In this case the derivative $(A^{\mathsf{D}})'(z_0)$ is given by

$$(A^{\mathsf{D}})'(z_0) = \frac{1}{2\pi i}\int_\Gamma \lambda^{-1} R(\lambda; A(z_0))A'(z_0)R(\lambda; A(z_0))\,d\lambda, \tag{2.1}$$

where Γ is a Cauchy cycle relative to $(\mathbb{C}\backslash\{0\}, \sigma_0(A(z_0)))$.

Proof. Assume that $A^{\mathsf{D}}(z)$ is continuous at z_0. From [14, Theorem 4.1] (see Equation (2.5) below) it follows that there exists $r > 0$ such that

$$0 < |\lambda| < r \implies \lambda \in \rho(A(z)) \text{ for all } z \in U. \tag{2.2}$$

Let $\Omega = \{\lambda : |\lambda| > r\}$, and let Ω_1 be a bounded open set with $\sigma_0(A(z_0)) \subset \Omega_1 \subset \overline{\Omega}_1 \subset \Omega$. From the upper semicontinuity of the spectrum it follows that there exists $\delta > 0$ such that the sets $\sigma_0(A(z))$ are contained in Ω_1 whenever $|z - z_0| < \delta$. (The cases $\sigma_0(A(z)) = \emptyset$ or $\sigma_0(A(z_0)) = \emptyset$ are not excluded.) There exists a Cauchy cycle Γ relative to $(\Omega, \overline{\Omega}_1)$, and

$$A^{\mathsf{D}}(z) = \frac{1}{2\pi i}\int_\Gamma \lambda^{-1} R(\lambda; A(z))\,d\lambda, \quad |z - z_0| < \delta, \tag{2.3}$$

by (1.3). Consider the existence of the limit

$$\lim_{z\to z_0} \frac{A^{\mathsf{D}}(z) - A^{\mathsf{D}}(z_0)}{z - z_0}.$$

Using the second resolvent equation, we get

$$\begin{aligned}\frac{A^{\mathsf{D}}(z) - A^{\mathsf{D}}(z_0)}{z - z_0} &= \frac{1}{z - z_0}\,\frac{1}{2\pi i}\int_\Gamma \lambda^{-1}[R(\lambda; A(z)) - R(\lambda; A(z_0))]\,d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma \lambda^{-1} R(\lambda; A(z))\,\frac{A(z) - A(z_0)}{z - z_0}\,R(\lambda; A(z_0))\,d\lambda.\end{aligned}$$

In view of Lemma 1.2,

$$\lim_{z\to z_0} R(\lambda; A(z))\,\frac{A(z) - A(z_0)}{z - z_0} = R(\lambda; A(z_0))A'(z_0)$$

uniformly for $\lambda \in \Gamma$. Hence (2.1) follows.

The converse is clear. □

We note that Theorem 4.1 of [14] holds when sequences are replaced by functions of z; that theorem gives twelve conditions equivalent to the continuity of $A^{\mathsf{D}}(z)$ at z_0. For the sake of completeness we restate four of these conditions relevant to the present investigation. Under the hypotheses of Theorem 2.1, $A^{\mathsf{D}}(z) \to A^{\mathsf{D}}(z_0)$ as $z \to z_0$ if and only if any of the following conditions holds:

$$\sup\{\|A^{\mathsf{D}}(z)\| : |z - z_0| < \delta\} < \infty \text{ for some } \delta > 0, \tag{2.4}$$

$$\sup\{r(A^{\mathsf{D}}(z)) : |z - z_0| < \delta\} < \infty \text{ for some } \delta > 0, \tag{2.5}$$

$$A^{\mathsf{D}}(z)A(z) \to A^{\mathsf{D}}(z_0)A(z_0) \text{ as } z \to z_0, \tag{2.6}$$

$$A^{\pi}(z) \to A^{\pi}(z_0) \text{ as } z \to z_0. \tag{2.7}$$

We take this opportunity to correct a mistake in [14, Theorem 4.1]: Conditions (4.14) and (4.15) of that theorem should be

$$C_n \to C \text{ and } \gamma(C_n) \to \gamma(C) \text{ and}$$
$$C_n \to C \text{ and } \inf_n \gamma(C_n) > 0,$$

respectively, where $\gamma(A)$ denotes the reduced minimum modulus of an operator $A \in B(X)$.

Note 2.2. The preceding argument works with appropriate interpretation in the case that $r(A(t_0)) = 0$.

Note 2.3. Hartwig and Shoaf [9, (3.10)] used holomorphic calculus to give a formula for the derivative of the Drazin inverse of a complex matrix in terms of the spectral components of $A(t)$.

In the case that the operators $A(t)$ have the conventional Drazin inverse and the indices of $A(t)$ are uniformly bounded, we are able to obtain a stronger result.

Theorem 2.4. *Let A be a $B(X)$-valued function defined on U such that $A(z)$ is g-Drazin invertible for all $z \in U$ and differentiable at $z_0 \in U$. If the indices $i(A(z))$ are uniformly bounded and the spectral projections $A^{\pi}(z)$ are of finite rank, then $A^{\mathsf{D}}(z)$ is differentiable at z_0 if and only if there exists $\delta > 0$ such that*

$$\operatorname{rank} A^{\pi}(z) = \operatorname{rank} A^{\pi}(z_0) \textit{ whenever } |z - z_0| < \delta.$$

Proof. Follows from Theorem 2.1 and [14, Theorem 5.1]. □

From the preceding theorem we recover the main result of [1] on the differentiability of the matrix Drazin inverse. The *core part* $C(z)$ of $A(z)$ is defined by $C(z) = A(z)(I - A^\pi(z))$; the *core rank* of $A(z)$ is the rank of $C(z)$.

Corollary 2.5. (Campbell [1, Theorem 4]) *Let A be a $p \times p$ matrix valued function defined on U and differentiable at $z_0 \in U$. Then $A^{\mathsf{D}}(z)$ is differentiable at z_0 if and only if the core rank of $A(z)$ is constant in some neighborhood of z_0.*

Proof. Follows from Theorem 2.4 and the result for the core rank of $A(z)$, which states that $\operatorname{rank} C(z) = p - \operatorname{rank} A^\pi(z)$. □

Let us remark that our approach differs from the one adopted by Campbell in [1], who derived his theorem from the known differentiation result for the Moore–Penrose inverse and from the relation between the Drazin inverse A^{D} of a $p \times p$ matrix A and the Moore–Penrose inverse $A^\dagger$ of A:

$$A^{\mathsf{D}} = A^p (A^{2p+1})^\dagger A^p.$$

3 Series expansion for $(A^{\mathsf{D}})'$

Let U be an open interval in $\mathbb{R}$ or an open set in $\mathbb{C}$, and $A(z)$ an operator valued function on U satisfying the hypotheses of Theorem 2.1 such that $A^{\mathsf{D}}(z)$ is continuous at z_0. To simplify notation, we write A, A^{D}, A', A^π for $A(z_0)$, $A^{\mathsf{D}}(z_0)$, $A'(z_0)$, $A^\pi(z_0)$. Then (2.2) holds, and we pick $R > \max\{r, r(A)\}$. In formula (2.1) we choose $\Gamma = \omega_R - \omega_r$, where $\omega_\rho(s) = \rho \exp(is)$ for any $\rho > 0$, $s \in [0, 2\pi]$. It can be verified that Γ is a Cauchy cycle relative to the pair $(\mathbb{C}\backslash\{0\}, \sigma_0(A))$.

According to (2.1),

(3.1)
$$(A^{\mathsf{D}})' = \frac{1}{2\pi i} \int_{\omega_R} \lambda^{-1} R(\lambda; A) A' R(\lambda; A)\, d\lambda - \frac{1}{2\pi i} \int_{\omega_r} \lambda^{-1} R(\lambda; A) A' R(\lambda; A)\, d\lambda.$$

Since $R(\lambda;A) = O(|\lambda|^{-1})$ as $|\lambda| \to \infty$, $\| \int_{\omega_R} \lambda^{-1} R(\lambda;A)A'R(\lambda;A)\,d\lambda\| = O(R^{-2})$ as $R \to \infty$. This shows that

$$\frac{1}{2\pi i}\int_{\omega_R} \lambda^{-1}R(\lambda;A)A'R(\lambda;A)\,d\lambda = 0.$$

By assumption, $0 \notin \operatorname{acc}\sigma(A)$; in view of [11, Theorem 5.1] there exists $r_0 > 0$ such that

$$R(\lambda;A) = \sum_{n=0}^{\infty} \lambda^{-n-1}A^n A^\pi - \sum_{n=0}^{\infty} \lambda^n (A^{\mathsf{D}})^{n+1} =: U_\lambda - V_\lambda$$

for $0 < |\lambda| < r_0$. If $0 < \rho < \min(r, r_0)$, then

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\omega_r} \lambda^{-1}R(\lambda;A)A'R(\lambda;A)\,d\lambda &= \frac{1}{2\pi i}\int_{\omega_\rho} \lambda^{-1}(U_\lambda - V_\lambda)A'(U_\lambda - V_\lambda)\,d\lambda \\
= \quad & \frac{1}{2\pi i}\int_{\omega_\rho} \lambda^{-1}U_\lambda A'U_\lambda\,d\lambda + \frac{1}{2\pi i}\int_{\omega_\rho} \lambda^{-1}V_\lambda A'V_\lambda\,d\lambda \\
& - \frac{1}{2\pi i}\int_{\omega_\rho} \lambda^{-1}U_\lambda A'V_\lambda\,d\lambda - \frac{1}{2\pi i}\int_{\omega_\rho} \lambda^{-1}V_\lambda A'U_\lambda\,d\lambda \\
= \quad & \sum_{m,n=0}^{\infty} A^\pi A^m A' A^n A^\pi \,\frac{1}{2\pi i}\int_{\omega_\rho} \lambda^{-m-n-3}\,d\lambda \\
& + \sum_{m,n=0}^{\infty} (A^{\mathsf{D}})^{m+1}A'(A^{\mathsf{D}})^{n+1}\,\frac{1}{2\pi i}\int_{\omega_r} \lambda^{m+n-1}\,d\lambda \\
& - \sum_{m,n=0}^{\infty} A^\pi A^m A'(A^{\mathsf{D}})^{n+1}\,\frac{1}{2\pi i}\int_{\omega_r} \lambda^{-m+n-2}\,d\lambda \\
& - \sum_{m,n=0}^{\infty} (A^{\mathsf{D}})^{n+1}A'A^m A^\pi\,\frac{1}{2\pi i}\int_{\omega_r} \lambda^{-m+n-2}\,d\lambda \\
= \quad & A^{\mathsf{D}}A'A^{\mathsf{D}} - \sum_{n=0}^{\infty} A^\pi A^n A'(A^{\mathsf{D}})^{n+2} - \sum_{n=0}^{\infty}(A^{\mathsf{D}})^{n+2}A'A^nA^\pi
\end{aligned}$$

as $\int_{\omega_\rho} \lambda^k\,d\lambda$ is equal to $2\pi i$ if $k = -1$ and to 0 otherwise. Substituting this into (3.1) we get the following result.

Theorem 3.1. *Let A be a $B(X)$-valued function defined on U such that $A(z)$ is g-Drazin invertible for all $z \in U$ and differentiable at $z_0 \in U$. If A^{D} is continuous at z_0, then*

$$(A^{\mathsf{D}})' = -A^{\mathsf{D}}A'A^{\mathsf{D}} + \sum_{n=0}^{\infty} A^{\pi}A^{n}A'(A^{\mathsf{D}})^{n+2} + \sum_{n=0}^{\infty}(A^{\mathsf{D}})^{n+2}A'A^{n}A^{\pi}, \tag{3.2}$$

where A, A^{D}, A', A^{π} stand for $A(z_0)$, $A^{\mathsf{D}}(z_0)$, $A'(z_0)$, $A^{\pi}(z_0)$, respectively.

In the case that the Drazin indices $i(A(z))$ are finite and uniformly bounded, the preceding theorem subsumes the differentiation formula of Campbell [1, Theorem 2]; the summation then becomes finite. Let us observe that Campbell's proof is based on the differentiation of the defining equations in the case that A has the Drazin index 1, that is, on the differentiation of the equations

$$AA^{\mathsf{D}}A = A, \quad A^{\mathsf{D}}AA^{\mathsf{D}} = A^{\mathsf{D}}, \quad AA^{\mathsf{D}} = A^{\mathsf{D}}A.$$

Hartwig and Shoaf obtained Campbell's formula from a difference relation [9, (4.16)]. Under the assumption of finite and uniformly bounded indices, formula (3.2) formally agrees with Drazin's result [5, Theorem 2], which is derived for the conventional Drazin inverse in associative rings.

We note that if $i(A) \le 1$, formula (3.2) reduces to

$$(A^{\mathsf{D}})' = -A^{\mathsf{D}}A'A^{\mathsf{D}} + A^{\pi}A'(A^{\mathsf{D}})^2 + (A^{\mathsf{D}})^2A'A^{\pi}. \tag{3.3}$$

For matrices this yields [1, Theorem 1].

If A satisfies the hypotheses of Theorem 2.1 and A^{D} is continuous at z_0, Equation (3.3) can be used to describe $(A^{\mathsf{D}})'$ in terms of the derivative C' of the core part of A bearing in mind that C has Drazin index not exceeding one:

$$\begin{aligned}(A^{\mathsf{D}})' = (C^{\mathsf{D}})' &= -C^{\mathsf{D}}C'C^{\mathsf{D}} + C^{\pi}C'(C^{\mathsf{D}})^2 + (C^{\mathsf{D}})^2C'C^{\pi} \\ &= -A^{\mathsf{D}}C'A^{\mathsf{D}} + A^{\pi}C'(A^{\mathsf{D}})^2 + (A^{\mathsf{D}})^2C'A^{\pi};\end{aligned}$$

it is known that $A^{\mathsf{D}} = C^{\mathsf{D}}$ and $A^{\pi} = C^{\pi}$.

4 The Moore–Penrose inverse of Hilbert space operators

For H a complex Hilbert space and $A \in B(H)$ it is well known that

$$\begin{aligned} A \text{ has closed range} &\iff A^*A \text{ has closed range} \iff AA^* \text{ has closed range} \\ &\iff 0 \notin \operatorname{acc}\sigma(A^*A) \iff 0 \notin \operatorname{acc}\sigma(AA^*). \end{aligned}$$

For a closed range operator $A \in B(H)$ we can give a definition of the Moore–Penrose inverse $A^\dagger$ of A in terms of the Drazin inverse (see [13, Theorem 2.5]):

$$A^\dagger = (A^*A)^{\mathsf{D}}A^* = A^*(AA^*)^{\mathsf{D}}. \tag{4.1}$$

This equation enables us to obtain results on the continuity and differentiability of the Moore–Penrose inverse using our results on the g-Drazin inverse. (For the continuity of the Moore–Penrose inverse see, for instance, [15].)

Theorem 4.1. *Let A be a $B(X)$-valued function defined on a real interval J differentiable at $t_0 \in J$ with $A(t)$ closed range operators for all $t \in J$. Write $B(t) = A^*(t)A(t)$ and $E(t) = A(t)A^*(t)$ for all $t \in J$. Then the following conditions are equivalent.*

(i) *$B^{\mathsf{D}}(t)$ is continuous at t_0.*

(ii) *$E^{\mathsf{D}}(t)$ is continuous at t_0.*

(iii) *$B^{\mathsf{D}}(t)$ is differentiable at t_0.*

(iv) *$E^{\mathsf{D}}(t)$ is differentiable at t_0.*

(v) *$A^\dagger(t)$ is differentiable at t_0.*

(vi) *$A^\dagger(t)$ is continuous at t_0.*

(vii) *$A^\dagger(t)A(t)$ is continuous at t_0.*

(viii) *$A(t)A^\dagger(t)$ is continuous at t_0.*

(ix) *$\|A^\dagger(t)\|$ is bounded in some neighborhood of t_0.*

Proof. (i) $\implies$ (iii) $\implies$ (v) $\implies$ (vi) $\implies$ (vii) $\implies$ (i): The first four implications by Theorem 2.1, by the product rule for differentiation applied to $A^\dagger(t) = B^{\mathsf{D}}(t)A^*(t)$, by the relation between differentiability and continuity, and by the continuity of the multiplication in $B(X)$, respectively. The last implication follows when we observe that if (vii) holds, then $A^\dagger(t)A(t) = (A^*(t)A(t))^{\mathsf{D}}(A^*(t)A(t)) = I - A^\pi(t)$ is continuous at t_0. Then (i) is true by Theorem 2.1.

(ii) $\implies$ (iv) $\implies$ (v) $\implies$ (vi) $\implies$ (viii) $\implies$ (ii) is proved by a symmetrical argument.

Condition (ix) is equivalent to (vi) when we use the inequality

$$\|A^\dagger(t) - A^\dagger(t_0)\| \le 3\max\{\|A^\dagger(t)\|^2, \|A^\dagger(t_0)\|^2\}\|A(t) - A(t_0)\|$$

(see [2, Theorem 10.4.5]). □

Note 4.2. We note that in the proof of the implication (iii) $\implies$ (v) the differentiability of $A^*(t)$ follows from the differentiability of $A(t)$ via the identity

$$\frac{dA^*(t)}{dt} = \Big(\frac{dA(t)}{dt}\Big)^*,$$

which holds only when t is real. The preceding theorem, unlike Theorem 2.1, does not hold for complex differentiation.

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