

# Weyl type theorems for bounded linear operators

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## Abstract

For a bounded linear operator  $T$  acting on a Banach space let  $\sigma_{SBF_+^-}(T)$  be the set of all  $\lambda \in \mathbb{C}$  such that  $T - \lambda I$  is upper semi- $B$ -Fredholm and  $\text{ind}(T - \lambda I) \leq 0$ , and let  $E^a(T)$  be the set of all isolated eigenvalues of  $T$  in the approximate point spectrum  $\sigma_a(T)$  of  $T$ . We say that  $T$  satisfies generalized  $a$ -Weyl's theorem if  $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus E^a(T)$ . Among other things, we show in this paper that if  $T$  satisfies generalized  $a$ -Weyl's theorem, then it also satisfies generalized Weyl's theorem  $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$ , where  $\sigma_{BW}(T)$  is the  $B$ -Weyl spectrum of  $T$  and  $E(T)$  is the set of all eigenvalues of  $T$  which are isolated in the spectrum of  $T$ .

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## 1 Introduction

We consider a Banach space  $X$  and the Banach algebra  $L(X)$  of bounded linear operators acting on  $X$ . For  $T \in L(X)$  we will denote by  $N(T)$  the null space of  $T$ , by  $\alpha(T)$  the nullity of  $T$ , by  $R(T)$  the range of  $T$  and by  $\beta(T)$  its defect. If the range  $R(T)$  of  $T$  is closed and  $\alpha(T) < \infty$  (resp.  $\beta(T) < \infty$ ), then  $T$  is called an *upper semi-Fredholm* (resp. a *lower semi-Fredholm*) operator. If  $T \in L(X)$  is either upper or lower semi-Fredholm, then  $T$  is called a *semi-Fredholm operator*, and the *index* of  $T$  is defined by  $\text{ind}(T) = \alpha(T) - \beta(T)$ . If both  $\alpha(T)$  and  $\beta(T)$  are finite, then  $T$  is a *Fredholm operator*. For a bounded linear operator  $T$  and a nonnegative integer  $n$  define  $T_{[n]}$  to be the restriction of  $T$  to  $R(T^n)$  viewed as a map from  $R(T^n)$  into  $R(T^n)$  (in particular  $T_{[0]} = T$ ). If for some integer  $n$  the range space  $R(T^n)$  is closed and  $T_{[n]}$  is an upper (resp. a lower) semi-Fredholm operator, then  $T$  is called an *upper* (resp. *lower*) *semi- $B$ -Fredholm operator*. Moreover if  $T_{[n]}$  is a

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Fredholm operator, then  $T$  is called a *B-Fredholm operator*. A *semi-B-Fredholm operator* is an upper or a lower semi-B-Fredholm operator. In [5, Theorem 2.7] it is shown that if  $H$  is a Hilbert space, then an operator  $T \in L(H)$  is a semi-B-Fredholm operator if and only if  $T = T_0 \oplus T_1$ , where  $T_0$  is a semi-Fredholm operator and  $T_1$  is nilpotent.

Let  $T$  be a semi-B-Fredholm operator and let  $d$  be the degree of the stable iteration of  $T$  (Definition 2.2). It follows from [5, Proposition 2.1] that  $T_{[n]}$  is a semi-Fredholm operator, and  $\text{ind}(T_{[m]}) = \text{ind}(T_{[d]})$  for each  $m \geq d$ . This enables us to define the *index of a semi-B-Fredholm operator*  $T$  as the index of the semi-Fredholm operator  $T_{[d]}$ .

It is shown in [6] that Drazin invertibility is a good tool for the investigation of the class of B-Fredholm operators and of the induced B-Weyl spectrum. Following [12, Definition 4.1] we say that  $T \in L(X)$  is *Drazin invertible* (with a finite index) if there exist  $B, U \in L(X)$  such that  $U$  is nilpotent and

$$TB = BT, \quad BTB = B, \quad TBT = T + U.$$

Recall that the concept of Drazin invertibility was originally introduced by Drazin in [9] where elements of an associative semigroup satisfying an equivalent relation were called pseudo-invertible. It is well known that  $T$  is Drazin invertible if and only if it has a finite ascent and descent (Definition 2.1), which is also equivalent to the fact that  $T = T_0 \oplus T_1$ , where  $T_0$  is invertible and  $T_1$  nilpotent. (See [11, Proposition A] and [16, Corollary 2.2].)

In this paper we introduce the concept of left Drazin invertible operators and of left poles (Definitions 2.5 and 2.6) to deal with the generalized B-Weyl spectrum. In the case that 0 is isolated in the approximate point spectrum of  $T$ , we are able to give a useful characterization of left Drazin invertibility (Theorems 2.8 and 2.11). Generalized  $a$ -Weyl's and  $a$ -Browder's theorems are formulated in terms of left poles.

Recall that B-Weyl operators and the B-Weyl spectrum were defined in [3] as follows:

**Definition 1.1.** An operator  $T \in L(X)$  is called a *B-Weyl operator* if it is a B-Fredholm operator of index 0. The *B-Weyl spectrum*  $\sigma_{BW}(T)$  of  $T$  is defined by

$$(1.1) \quad \sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator}\}.$$

In the case of a normal operator  $T$  acting on a Hilbert space  $H$ , Berkani [3, Theorem 4.5] showed that

$$\sigma_{BW}(T) = \sigma(T) \setminus E(T),$$

where  $\sigma_{BW}(T)$  is the B-Weyl spectrum of  $T$  and  $E(T)$  is the set of all eigenvalues of  $T$  which are isolated in the spectrum of  $T$ . This result gives a generalization of the classical

Weyl's theorem. Recall that classical Weyl's theorem [22] asserts that if  $T$  is a normal operator acting on a Hilbert space  $H$ , then the Weyl spectrum

$$(1.2) \quad \sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Fredholm operator of index } 0\}$$

is the set of all points in  $\sigma(T)$  except for the set  $E_0(T)$  of all eigenvalues  $\lambda$  of finite multiplicity ( $0 < \alpha(T - \lambda I) < \infty$ ) isolated in  $\sigma(T)$ ; that is,

$$\sigma_W(T) = \sigma(T) \setminus E_0(T).$$

Let  $SF_+(X)$  be the class of all upper semi-Fredholm operators,  $SF_+^-(X)$  the class of all  $T \in SF_+(X)$  with  $\text{ind}(T) \leq 0$ , and for any  $T \in L(X)$  let

$$(1.3) \quad \sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SF_+^-(X)\}.$$

Let  $E_0^a(T)$  be the set of all eigenvalues of  $T$  of finite multiplicity which are isolated in the approximate point spectrum  $\sigma_a(T) = \{\lambda \in \mathbb{C} : \inf_{\|x\|=1} \|(T - \lambda I)(x)\| = 0\}$ . Following [19], we say that  $T$  satisfies  $a$ -Weyl's theorem if  $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus E_0^a(T)$ . It follows from [19, Corollary 2.5] that an operator satisfying  $a$ -Weyl's theorem satisfies Weyl's theorem.

Similarly, let  $SBF_+(X)$  be the class of all upper semi- $B$ -Fredholm operators, and  $SBF_+^-(X)$  the class of all  $T \in SBF_+(X)$  such that  $\text{ind}(T) \leq 0$ , and

$$(1.4) \quad \sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SBF_+^-(X)\}.$$

In this paper we investigate several Weyl type theorems. As a first step, we consider an operator  $T$  satisfying generalized  $a$ -Weyl's theorem  $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus E^a(T)$ , where  $E^a(T)$  is the set of all eigenvalues of  $T$  which are isolated in the approximate point spectrum  $\sigma_a(T)$ . We show that in this case  $T$  satisfies  $a$ -Weyl's theorem  $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus E_0^a(T)$ , and also satisfies generalized Weyl's theorem  $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$ . Moreover we show that if  $T$  satisfies generalized Weyl's theorem  $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$ , then it also satisfies Weyl's theorem  $\sigma_W(T) = \sigma(T) \setminus E_0(T)$ .

We consider also generalized  $a$ -Browder's theorem (see Definition 2.13 and Table 1 in Section 5) and obtain similar results. In particular we show that if  $T$  satisfies generalized Browder's theorem  $\sigma_{BW}(T) = \sigma(T) \setminus \Pi(T)$ , then it satisfies Browder's theorem  $\sigma_W(T) = \sigma(T) \setminus \Pi_0(T)$ , where  $\Pi(T)$  is the set of the poles of  $T$ , and  $\Pi_0(T)$  is the set of all poles of  $T$  of finite rank.

## 2 Preliminaries

We begin by some definitions motivated by [7] and [10].

**Definition 2.1.** For any  $T \in L(X)$  we define the sequences  $(c_n(T))$ ,  $(c'_n(T))$  and  $(k_n(T))$  as follows:

- (i)  $c_n(T) = \dim (R(T^n)/R(T^{n+1}))$ .
- (ii)  $c'_n(T) = \dim (N(T^{n+1})/N(T^n))$ .
- (iii)  $k_n(T) = \dim [(R(T^n) \cap N(T))/R(T^{n+1}) \cap N(T)]$ .

The *descent*  $\delta(T)$  and *ascent*  $a(T)$  of  $T$  are defined by

$$\begin{aligned}\delta(T) &= \inf \{n : c_n(T) = 0\} = \inf \{n : R(T^n) = R(T^{n+1})\}, \\ a(T) &= \inf \{n : c'_n(T) = 0\} = \inf \{n : N(T^n) = N(T^{n+1})\}.\end{aligned}$$

**Definition 2.2.** (See [15].) Let  $T \in L(X)$  and let

$$\Delta(T) = \{n \in \mathbb{N} : \forall m \in \mathbb{N} \ m \geq n \Rightarrow (R(T^n) \cap N(T)) \subset (R(T^m) \cap N(T))\}.$$

Then the *degree of stable iteration*  $\text{dis}(T)$  of  $T$  is defined as  $\text{dis}(T) = \inf \Delta(T)$ .

**Definition 2.3.** (See [10].) Let  $T \in L(X)$  and let  $d \in \mathbb{N}$ . Then  $T$  has a *uniform descent* for  $n \geq d$  if  $R(T) + N(T^n) = R(T) + N(T^d)$  for all  $n \geq d$  (equivalently,  $k_n(T) = 0$  for all  $n \geq d$ ). If, in addition,  $R(T) + N(T^d)$  is closed, then  $T$  is said to have a *topological uniform descent* for  $n \geq d$ .

In [14] Kordula and Müller defined the concept of regularity as follows:

**Definition 2.4.** A non-empty subset  $\mathcal{R} \subset L(X)$  is called a *regularity* if it satisfies the following conditions:

- (i) If  $A \in L(X)$  and  $n \geq 1$ , then  $A \in \mathcal{R}$  if and only if  $A^n \in \mathcal{R}$ .
- (ii) If  $A, B, C, D \in L(X)$  are mutually commuting operators satisfying  $AC + BD = I$ , then  $AB \in \mathcal{R}$  if and only if  $A, B \in \mathcal{R}$ .

A regularity  $\mathcal{R}$  induces the spectrum  $\sigma_{\mathcal{R}}(T) = \{\lambda \in C : T - \lambda I \notin \mathcal{R}\}$  for any  $T \in L(X)$ .

Define a set  $LD(X)$  by

$$LD(X) = \{T \in L(X) : a(T) < \infty \text{ and } R(T^{a(T)+1}) \text{ is closed}\}.$$

In [17],  $LD(X)$  is denoted by  $\mathcal{R}_8$ , and it is shown there that  $\mathcal{R}_8$  is a regularity.

**Definition 2.5.** Let  $X$  be a Banach space. Then  $T \in L(X)$  is called *left Drazin invertible* if  $T \in LD(X)$ ; the *left Drazin spectrum*  $\sigma_{LD}(T)$  of  $T$  is defined as the spectrum induced by the regularity  $\mathcal{R}_8 = LD(X)$ .

**Definition 2.6.** We will say that  $\lambda \in \sigma_a(T)$  is a *left pole of  $T$*  if  $T - \lambda I \in LD(X)$ , and that  $\lambda \in \sigma_a(T)$  is a *left pole of  $T$  of finite rank* if  $\lambda$  is a left pole of  $T$  and  $\alpha(T - \lambda I) < \infty$ . We will denote by  $\Pi^a(T)$  the set of all left poles of  $T$ , and by  $\Pi_0^a(T)$  the set of all left poles of  $T$  of finite rank.

**Remark 2.7.** If  $\lambda \in \Pi^a(T)$ , then it's easily seen that  $T - \lambda I$  is an operator of topological uniform descent. Therefore from [10, Theorem 4.7], it follows that  $\lambda$  is isolated in  $\sigma_a(T)$ .

**Theorem 2.8.** *Let  $T \in L(X)$  and  $\lambda \in \mathbb{C}$  be isolated in  $\sigma_a(T)$ . Then  $\lambda \in \Pi^a(T)$  if and only if  $\lambda \notin \sigma_{SBF_+^-}(T)$ , and  $\lambda \in \Pi_0^a(T)$  if and only if  $\lambda \notin \sigma_{SF_+^-}(T)$ .*

*Proof.* Without loss of generality we can assume  $\lambda = 0$ . If  $0 \in \Pi^a(T)$ , then  $d = a(T) < \infty$  and  $R(T^{d+1})$  is closed. From [17, Lemma 12] it follows that  $R(T^d)$  is closed and  $T_{[d]}$  is an upper semi-Fredholm operator such that  $\text{ind}(T_{[d]}) \leq 0$ . Therefore  $0 \notin \sigma_{SBF_+^-}(T)$ .

Conversely suppose that  $0 \notin \sigma_{SBF_+^-}(T)$ . Then  $T$  is upper semi- $B$ -Fredholm, and, in particular, an operator of topological uniform descent. Since  $0$  is isolated in  $\sigma_a(T)$ , from [10, Theorem 4.7] it follows that for  $n$  large enough we have  $c_n(T) = 0$ . So  $a(T) < \infty$ . Since  $T$  is semi- $B$ -Fredholm, it is also an operator of topological uniform descent, and by [17, Lemma 12],  $R(T^{a(T)+1})$  is closed. Now it is easily seen that an upper semi- $B$ -Fredholm operator is upper semi-Fredholm if and only if  $\alpha(T) < \infty$ . Using this fact we can show analogously that  $0 \in \Pi_0^a(T)$  if and only if  $0 \notin \sigma_{SF_+^-}(T)$ .  $\square$

**Proposition 2.9.** *Let  $H$  be a Hilbert space. Then  $T \in L(H)$  is an upper semi- $B$ -Fredholm operator if and only if  $T = T_1 \oplus T_2$ , where  $T_1$  is upper semi-Fredholm and  $T_2$  nilpotent. Moreover, in this case we have  $\text{ind}(T) = \text{ind}(T_1)$ .*

*Proof.* We know from [5, Theorem 2.6] that  $T$  is an upper semi- $B$ -Fredholm operator if and only if  $T = T_1 \oplus T_2$ , where  $T_1$  is upper semi-Fredholm and  $T_2$  nilpotent. We only have to show that if such a decomposition exists for  $T$ , then  $\text{ind}(T) = \text{ind}(T_1)$ .

If  $T_1$  is a  $B$ -Fredholm operator then  $T$  is also  $B$ -Fredholm, and from [2, Proposition 2.1] it follows that  $\text{ind}(T) = \text{ind}(T_1)$ . If  $T_1$  is not  $B$ -Fredholm, let  $m$  be such that  $T_2^m = 0$ , and let  $d \geq \max(\text{dis}(T), m)$ . Then  $R(T^d) = R(T_1^{(d+1)})$ . Therefore  $\dim(R(T^d)/R(T^{(d+1)})) = \dim(R(T_1^d)/R(T_1^{(d+1)}))$ . Since  $T_1$  is upper semi-Fredholm operator which is not a  $B$ -Fredholm operator, then  $\dim(R(T_1^d)/R(T_1^{(d+1)})) = \infty = \dim(R(T^d)/R(T^{(d+1)}))$ . Consequently  $\text{ind}(T_1) = -\infty = \text{ind}(T)$ .  $\square$

We have immediately the following corollary:

**Corollary 2.10.** *Let  $H$  be a Hilbert space and  $T \in L(H)$ . Then  $T \in SBF_+^-(H)$  if and only if  $T = T_1 \oplus T_2$ , where  $T_1$  is an upper semi-Fredholm operator with  $\text{ind}(T_1) \leq 0$ , and  $T_2$  is nilpotent.*

In the case of a Hilbert space  $H$  we can give yet another characterization of isolated left poles of an operator.

**Theorem 2.11.** *Let  $H$  be a Hilbert space,  $T \in L(H)$ , and let  $\lambda$  be an isolated point in  $\sigma_a(T)$ . Then the following properties are equivalent:*

- (i)  $\lambda$  is a left pole of  $T$ .
- (ii) *There exist  $T$ -invariant subspaces  $M$  and  $N$  of  $H$  such that  $H = M \oplus N$ ,  $(T - \lambda I)|_M$  is bounded below and  $(T - \lambda I)|_N$  is nilpotent.*

*Proof.* Let  $\lambda$  be a left pole of  $T$  and let  $S = T - \lambda I$ . Then  $d = a(S) < \infty$  and both  $R(S^{d+1})$  and  $R(S^d)$  are closed, which implies that  $N(S_{[d]}) = \{0\}$  and  $R(S_{[d]}) = R(S^{d+1})$  is closed in  $R(S^d)$ ; hence  $S_{[d]}$  is bounded below. From [7, Corollary 3.13] it follows that  $S = S_1 \oplus S_2$ , where  $S_1$  is bounded below and  $S_2$  nilpotent.

Conversely, suppose there exist  $T$ -invariant subspaces  $M$  and  $N$  of  $H$  such that  $H = M \oplus N$ ,  $(T - \lambda I)|_M$  is bounded below and  $(T - \lambda I)|_N$  is nilpotent. Then from the previous corollary it follows that  $T - \lambda I \in SBF_+^-(H)$ . By Theorem 2.8,  $\lambda$  is a left pole of  $T$ .  $\square$

**Lemma 2.12.** Let  $X$  be a Banach space and let  $T \in L(X)$ . Then

$$\sigma_{SBF_+^-}(T) \subset \sigma_{LD}(T) \subset \sigma_a(T).$$

*Proof.* If  $\lambda \notin \sigma_a(T)$ , then  $\inf_{\|x\|=1} \|(T - \lambda I)x\| > 0$ . So there exists  $\varepsilon > 0$  such that  $\|(T - \lambda I)x\| \geq \varepsilon\|x\|$  for all  $x \in X$ . Hence  $N(T - \lambda I) = \{0\}$ , and  $a(T - \lambda I) = 0$ . Moreover since  $\|(T - \lambda I)x\| \geq \varepsilon\|x\|$ ,  $R(T - \lambda I)$  is closed. Consequently  $\lambda \notin \sigma_{LD}(T)$  and  $\sigma_{LD}(T) \subset \sigma_a(T)$ .

If  $\lambda \notin \sigma_{LD(X)}(T)$ , then  $d = a(T - \lambda I) < \infty$  and  $R(T^{d+1})$  is closed. From [17, Lemma 12] we deduce that  $R(T^d)$  is closed and  $T_{[d]}$  is an upper semi-Fredholm operator of negative index. So  $\lambda \notin \sigma_{SBF_+^-}(T)$  and  $\sigma_{SBF_+^-}(T) \subset \sigma_{LD}(T) \subset \sigma_a(T)$ .  $\square$

**Definition 2.13.** Let  $T \in L(X)$ . We will say that:

- (i)  $T$  satisfies Weyl's theorem if  $\sigma_W(T) = \sigma(T) \setminus E_0(T)$ .

- (ii)  $T$  satisfies generalized Weyl's theorem if  $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$ .
- (iii)  $T$  satisfies Browder's theorem if  $\sigma_W(T) = \sigma(T) \setminus \Pi_0(T)$ .
- (iv)  $T$  satisfies generalized Browder's theorem if  $\sigma_{BW}(T) = \sigma(T) \setminus \Pi(T)$ .
- (v)  $T$  satisfies  $a$ -Weyl's theorem if  $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus E_0^a(T)$ .
- (vi)  $T$  satisfies generalized  $a$ -Weyl's theorem if  $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus E^a(T)$ .
- (vii)  $T$  satisfies  $a$ -Browder's theorem if  $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \Pi_0^a(T)$ .
- (viii)  $T$  satisfies generalized  $a$ -Browder's theorem if  $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \Pi^a(T)$ .

(For a convenient summary of these definitions see Table 1 in Section 5.)

### 3 Weyl type theorems for Banach space operators

At the beginning of this section we give a characterization of operators obeying generalized  $a$ -Weyl's theorem or  $a$ -Weyl's theorem. This characterization extends results of Barnes [1, Theorem 4], proved in the case of Weyl's theorem, and Berkani [4, Theorem 2.5], proved in the case of generalized Weyl's theorem.

**Theorem 3.1.** *Let  $X$  be a Banach space and let  $T \in L(X)$ . Then the following are true:*

- (i)  $\sigma_{SBF_+^-}(T) \subseteq \sigma_a(T) \setminus E^a(T)$  if and only if  $E^a(T) = \Pi^a(T)$ .
- (ii)  $\sigma_{SBF_+^-}(T) \supseteq \sigma_a(T) \setminus E^a(T)$  if and only if  $\sigma_{SBF_+^-}(T) = \sigma_{LD}(T)$ .

*Proof.* (i) Suppose that  $\sigma_{SBF_+^-}(T) \subseteq \sigma_a(T) \setminus E^a(T)$  and let  $\lambda \in E^a(T)$ . Then  $\lambda$  is isolated  $\sigma_a(T)$ , and  $\lambda \notin \sigma_{SBF_+^-}(T)$ . So  $T - \lambda I$  is in  $SBF_+^-(X)$ . From Theorem 2.8 it follows that  $\lambda$  is a left pole of  $T$ , and so  $\lambda \in \Pi^a(T)$ . As we have always  $\Pi^a(T) \subset E^a(T)$ , then  $E^a(T) = \Pi^a(T)$ .

Conversely if  $E^a(T) = \Pi^a(T)$  and  $\lambda \in E^a(T)$ , then  $(T - \lambda I) \in SBF_+^-(X)$ . Therefore  $\lambda \notin \sigma_{SBF_+^-}(T)$  and so  $\sigma_{SBF_+^-}(T) \subseteq \sigma_a(T) \setminus E^a(T)$ .

(ii) Suppose that  $\sigma_{SBF_+^-}(T) \supseteq \sigma_a(T) \setminus E^a(T)$ , and let  $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$ . Then  $\lambda \in E^a(T)$ , in particular  $\lambda$  is isolated in  $\sigma_a(T)$ . Moreover  $T - \lambda I$  is an upper semi-B-Fredholm operator and  $\text{ind}(T - \lambda I) \leq 0$ . From Theorem 2.8 it follows that  $T - \lambda I$  is left Drazin invertible and  $\sigma_{LD}(T) \subset \sigma_{SBF_+^-}(T)$ . As it is always true that  $\sigma_{SBF_+^-}(T) \subset \sigma_{LD}(T)$ , then  $\sigma_{SBF_+^-}(T) = \sigma_{LD}(T)$ .

Conversely suppose that  $\sigma_{SBF_+^-}(T) = \sigma_{LD}(T)$ . Let  $\lambda \notin \sigma_{SBF_+^-}(T)$ , then  $\lambda \notin \sigma_{LD}(T)$ . So  $T - \lambda I$  is left Drazin invertible and  $\lambda \in E^a(T)$ . Hence  $\sigma_{SBF_+^-}(T) \supseteq \sigma_a(T) \setminus E^a(T)$ .  $\square$

From this theorem we obtain immediately the following corollaries:

**Corollary 3.2.** *Let  $X$  be a Banach space and let  $T \in L(X)$ . Then  $T$  satisfies generalized  $a$ -Weyl's theorem if and only if  $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \Pi^a(T)$  and  $E^a(T) = \Pi^a(T)$ .*

**Corollary 3.3.** *Let  $X$  be a Banach space and let  $T \in L(X)$ . If  $T$  satisfies generalized  $a$ -Weyl's theorem, then it satisfies generalized  $a$ -Browder's theorem.*

Similarly we have the following characterization of operators satisfying  $a$ -Weyl's theorem, which we give without proof.

**Theorem 3.4.** *Let  $X$  be a Banach space and let  $T \in L(X)$ . Then  $T$  satisfies  $a$ -Weyl's theorem if and only if  $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \Pi_0^a(T)$  and  $E_0^a(T) = \Pi_0^a(T)$ .*

As a consequence of this theorem we have the following result:

**Corollary 3.5.** *If  $T \in L(X)$  satisfies  $a$ -Weyl's theorem, then it satisfies  $a$ -Browder's theorem.*

**Remark 3.6.** In [19, Theorem 1.1] Rakočević proved that  $T \in L(X)$  satisfies  $a$ -Weyl's theorem if and only if the following conditions are satisfied:

- (i) If  $\lambda \in \Pi_0^a(T)$ , then  $R(T - \lambda I)$  is closed.
- (ii) If  $T - \lambda I \in SF_+^-(X)$ , then  $\lambda$  is not an interior point of  $\sigma_a(T)$ .

Our characterization of operators obeying  $a$ -Weyl's theorem appears to have a simpler formulation than Rakočević's theorem.

**Theorem 3.7.** *If  $T \in L(X)$  satisfies generalized  $a$ -Weyl's theorem, then it satisfies generalized Weyl's theorem.*

*Proof.* Suppose that  $T$  satisfies generalized  $a$ -Weyl's theorem and  $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$ . Then  $T - \lambda I$  is a  $B$ -Fredholm operator of index 0 and hence also upper semi- $B$ -Fredholm with index 0. Therefore  $\lambda \notin \sigma_{SBF_+^-}(T)$ . Let  $\lambda \notin \sigma_a(T)$ . Since  $T - \lambda I$  is an operator of topological uniform descent, then there exist  $\varepsilon > 0$  such that if  $0 < |\lambda - \mu| < \varepsilon$ , then for  $n$  large enough we have  $c_n(T - \lambda I) = c_0(T - \mu I)$  and  $c'_n(T - \lambda I) = c'_0(T - \mu I)$ . Since  $T - \lambda I$  is a  $B$ -Weyl operator, then for  $n$  large enough we have  $c_n(T - \lambda I) = c'_n(T - \lambda I)$ . Since  $\lambda \notin \sigma_a(T)$  we have  $c'_0(T - \lambda I) = 0$ . It follows then that  $c_0(T - \lambda I) = c'_0(T - \lambda I) = 0$ .

Consequently  $\lambda \notin \sigma(T)$ , which is a contradiction. Hence  $\lambda \in \sigma_a(T)$ . As  $T$  satisfies generalized  $a$ -Weyl's theorem, then  $\lambda \in E^a(T)$ . If  $\mu \neq \lambda$  and  $|\lambda - \mu|$  is small enough, then  $\mu \notin \sigma_a(T)$ . So  $c'_0(T - \mu I) = 0$ . Hence  $c_n(T - \lambda I) = 0$  for  $n$  large enough. Since  $\text{ind}(T - \lambda I) = 0$ , then  $T - \mu I$  is invertible. Consequently  $\lambda \in E(T)$ .

Conversely, if  $\lambda \in E(T) \subset E^a(T)$ , then  $\lambda \notin \sigma_{SBF_+^-}(T)$ . Hence  $T - \lambda I$  is an operator of topological uniform descent. So there is  $\varepsilon > 0$ , such that if  $0 < |\lambda - \mu| < \varepsilon$ , then  $c_n(T - \lambda I) = c_n(T - \mu I)$  and  $c'_n(T - \lambda I) = c'_n(T - \mu I)$  for  $n$  large enough. Since  $\lambda$  is isolated in  $\sigma(T)$ , if  $\varepsilon$  is chosen small enough, then  $c_n(T - \mu I) = c'_n(T - \mu I) = 0$ . So  $T - \lambda I$  is Drazin invertible. Therefore  $\lambda \notin \sigma_{BW}(T)$  and  $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$ .  $\square$

**Theorem 3.8.** *If  $T \in L(X)$  satisfies generalized  $a$ -Browder's theorem, then it also satisfies the generalized Browder's theorem.*

*Proof.* Obtained by an argument similar to the one in the preceding proof.  $\square$

Further we have:

**Theorem 3.9.** *If  $T \in L(X)$  satisfies generalized Weyl's theorem, then it also satisfies Weyl's theorem.*

*Proof.* Suppose that  $T$  satisfies generalized Weyl's theorem, and let  $\lambda \in \sigma(T)$  and  $\lambda \notin \sigma(T) \setminus E_0(T)$ . Then  $\lambda \in E_0(T) \subset E(T)$ . Since  $T$  satisfies generalized Weyl's Theorem,  $T - \lambda I$  is a  $B$ -Fredholm operator of index 0. As  $\lambda$  is isolated in the spectrum of  $T$ , it follows from [7, Theorem 4.5] that  $\lambda$  is a pole of  $T$ . Hence there is an integer  $n$  such that  $X = N((T - \lambda I)^n) \oplus R((T - \lambda I)^n)$ . Since  $\alpha(T - \lambda I) < \infty$ , then from [21, Lemma 3.3 (a)] we have  $\alpha((T - \lambda I)^n) < \infty$ . Therefore  $(T - \lambda I)^n$  is a Fredholm operator of index 0, and so is  $T - \lambda I$ . Hence  $\lambda \notin \sigma_W(T)$ .

Conversely, if  $\lambda \in \sigma(T)$  and  $\lambda \notin \sigma_W(T)$ , then  $T - \lambda I$  is Fredholm operator of index 0, and hence also a  $B$ -Fredholm operator of index 0. Therefore  $\lambda \notin \sigma_{BW}(T)$ . Since  $T$  satisfies the generalized Weyl theorem, we have  $\lambda \in E(T)$ , while  $\alpha(T - \lambda I) < \infty$ . So  $\lambda \in E_0(T)$  and  $\sigma_W(T) = \sigma(T) \setminus E_0(T)$ .  $\square$

**Corollary 3.10.** *If  $T \in L(X)$  satisfies generalized  $a$ -Weyl's theorem, then it satisfies Weyl's theorem.*

**Theorem 3.11.** *If  $T \in L(X)$  satisfies generalized  $a$ -Weyl's theorem, then it satisfies  $a$ -Weyl's theorem.*

*Proof.* Let  $T \in L(X)$ . Suppose that  $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus E^a(T)$  and let us show that  $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus E_0^a(T)$ . If  $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$ , then  $T - \lambda I$  is an upper semi-Fredholm operator and  $\text{ind}(T - \lambda I) \leq 0$ . Therefore  $\lambda \notin \sigma_{SBF_+^-}(T)$ . Since  $T$  satisfies the generalized  $a$ -Weyl theorem, then  $\lambda \in E^a(T)$ . As  $\alpha(T - \lambda I) < \infty$ , then  $\lambda \in E_0^a(T)$ .

Conversely suppose that  $\lambda \in E_0^a(T)$ . Then  $\lambda \notin \sigma_{SBF_+^-}(T)$ , and  $T - \lambda I$  is an upper semi- $B$ -Fredholm operator with  $\text{ind}(T - \lambda I) \leq 0$ . Since  $\alpha(T - \lambda I) < \infty$ , then  $T - \lambda I$  is an upper semi-Fredholm operator with  $\text{ind}(T - \lambda I) \leq 0$ . So  $\lambda \notin \sigma_{SF_+^-}(T)$ .  $\square$

The converse of the preceding theorem is not true as shown by the following example:

**Example 3.12.** There is an operator  $T \in L(X)$  which obeys  $a$ -Weyl's theorem and Weyl's theorem, but does not obey generalized Weyl's theorem or generalized  $a$ -Weyl's theorem.

Let  $Q$  be defined for each  $x = (\xi_i) \in \ell^1$  by

$$Q(\xi_1, \xi_2, \xi_3, \dots, \xi_k, \dots) = (0, \alpha_1 \xi_1, \alpha_2 \xi_2, \dots, \alpha_{k-1} \xi_{k-1}, \dots),$$

where  $(\alpha_i)$  is a sequence of complex numbers such that  $0 < |\alpha_i| \leq 1$  and  $\sum_{i=1}^{\infty} |\alpha_i| < \infty$ . We observe that

$$\overline{R(Q^n)} \neq R(Q^n), \quad n = 1, 2, \dots$$

Indeed, for a given  $n \in \mathbb{N}$  let  $x_k^{(n)} = (1, \dots, 1, 0, 0, \dots)$  (with  $n+k$  zeros). Then the limit  $y^{(n)} = \lim_{k \rightarrow \infty} x_k^{(n)}$  exists and lies in  $\overline{R(Q^n)}$ . However, there is no element  $x^{(n)} \in \ell^1$  satisfying the equation  $Q^n x^{(n)} = y^{(n)}$  as the algebraic solution to this equation is  $(1, 1, 1, \dots) \notin \ell^1$ .

Define  $T$  on  $X = \ell^1 \oplus \ell^1$  by  $T = Q \oplus 0$ . Then  $N(T) = \{0\} \oplus \ell^1$ ,  $\sigma(T) = \sigma_a(T) = \{0\}$ ,  $E(T) = \{0\}$ ,  $E_0^a(T) = \emptyset$ . Since  $R(T^n) = R(Q^n) \oplus \{0\}$ ,  $R(T^n)$  is not closed for any  $n \in \mathbb{N}$ ; so  $T$  is not a  $B$ -Weyl operator, and  $\sigma_{BW}(T) = \{0\}$ . Further,  $T \notin SF_+^-(X)$ , and  $\sigma_{SF_+^-}(T) = \{0\}$ . We then have

$$\sigma_{BW}(T) \neq \sigma(T) \setminus E(T), \quad \sigma_{SF_+^-}(T) = \sigma_a(T) \setminus E_0^a(T).$$

Hence  $T$  satisfies  $a$ -Weyl's theorem but not generalized Weyl's theorem. Then  $T$  does not obey generalized  $a$ -Weyl's theorem (Theorem 3.7), but obeys Weyl's theorem [19, Corollary 2.5].

**Theorem 3.13.** *If  $T \in L(X)$  satisfies generalized  $a$ -Browder's theorem, then it satisfies  $a$ -Browder's theorem.*

*Proof.* Suppose that  $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \Pi^a(T)$ , and let us show that  $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \Pi_0^a(T)$ . If  $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$ , then  $T - \lambda I$  is an upper semi-Fredholm operator and  $\text{ind}(T - \lambda I) \leq 0$ . Then  $\lambda \notin \sigma_{SBF_+^-}(T)$ . Since  $T$  satisfies generalized  $a$ -Browder's theorem, then  $\lambda \in \Pi^a(T)$ . As  $\alpha(T - \lambda I) < \infty$ , we have  $\lambda \in \Pi_0^a(T)$ .

Conversely suppose that  $\lambda \in \Pi_0^a(T)$ . Then  $\lambda \notin \sigma_{SBF_+^-}(T)$ . Therefore  $T - \lambda I$  is an upper semi- $B$ -Fredholm operator with  $\text{ind}(T - \lambda I) \leq 0$ . Since  $\alpha(T - \lambda I) < \infty$ ,  $T - \lambda I$  is an upper semi-Fredholm operator with  $\text{ind}(T - \lambda I) \leq 0$ . So  $\lambda \notin \sigma_{SF_+^-}(T)$ .  $\square$

Combining Corollary 3.3 with the preceding theorem we obtain the following result (which also follows from Theorem 3.11 and Corollary 3.5).

**Corollary 3.14.** *If  $T \in L(X)$  satisfies generalized  $a$ -Weyl's theorem, then it satisfies  $a$ -Browder's theorem.*

**Theorem 3.15.** *If  $T \in L(X)$  satisfies generalized Browder's theorem, then it also satisfies Browder's theorem.*

*Proof.* If  $\lambda \in \sigma(T) \setminus \sigma_W(T)$ , then  $T - \lambda I$  is a Fredholm operator of index 0 and also a  $B$ -Fredholm operator of index 0. So  $\lambda$  is a pole of  $T$ , and by [16, Theorem 1.2] there is an integer  $n$  such that  $X = N((T - \lambda I)^n) \oplus R((T - \lambda I)^n)$ . Since  $\alpha(T - \lambda I) < \infty$ , then from [21, Lemma 3.3, (a)] we have  $\alpha((T - \lambda I)^n) < \infty$ . From [1, Proposition 2] it follows that  $\lambda$  is a pole of finite rank of  $T$ .

Conversely, if  $\lambda$  is a pole of finite rank of  $T$ , then  $T - \lambda I$  is a  $B$ -Fredholm operator of index 0 since  $T$  satisfies generalized Browder's theorem. So there is an integer  $n$  such that  $X = N((T - \lambda I)^n) \oplus R((T - \lambda I)^n)$  and  $\alpha((T - \lambda I)^n) < \infty$ . Therefore  $T - \lambda I$  is also a Fredholm operator of index 0. Hence we have  $\sigma_W(T) = \sigma(T) \setminus \Pi_0(T)$ .  $\square$

## 4 Finite rank perturbations for Hilbert space operators

In this section  $H$  denotes a Hilbert space and  $F_0(H)$  the set of finite rank operators in  $L(H)$ . Classical characterizations of Weyl's and Browder's spectrum state that, for every  $T \in L(H)$ ,

$$\sigma_W(T) = \bigcap_{F \in F_0(H)} \sigma(T + F), \quad \sigma_B(T) = \bigcap_{F \in F_0(H), TF=FT} \sigma(T + F).$$

Many extensions of these results have been obtained (for instance, [1, 13, 18, 20]). In the following we present some results analogous to the above characterizations of Weyl's and Browder's spectra, and also show that the validity of generalized  $a$ -Browder's theorem and of  $a$ -Browder's theorem is preserved under commuting finite rank perturbations.

**Theorem 4.1.** *Let  $T \in L(H)$ . Then*

$$\sigma_{SBF_+^-}(T) = \bigcap_{F \in F_0(H)} \sigma_{LD}(T + F).$$

*Proof.* If  $\lambda \notin \sigma_{SBF_+^-}(T)$ , then  $T - \lambda I$  is a semi- $B$ -Fredholm operator and  $\text{ind}(T - \lambda I) \leq 0$ . Therefore  $H = H_1 \oplus H_2$ , and  $T - \lambda I = T_1 \oplus T_2$  relatively to this decomposition, with  $T_1$  in  $SF_+^-(H)$  and  $T_2$  a nilpotent operator. By [18, Corollary 2.7] there exists a finite rank operator  $F_0$  such that  $T_1 \oplus F_0$  is bounded below. Let  $F = F_0 \oplus 0$ . Then  $F$  is a finite rank operator,  $T - \lambda I + F = T_1 + F \oplus T_2$  is in  $LD(X)$  and  $\lambda \notin \bigcap \{\sigma_{LD}(T + F) : F \in F_0(X)\}$ .

Conversely suppose that  $\lambda \notin \bigcap \{\sigma_{LD}(T + F) : F \in F_0(X)\}$ . Then there exists a finite rank operator  $F$  such that  $T - \lambda I + F$  is in  $LD(X)$ . In particular  $T - \lambda I + F$  is semi- $B$ -Fredholm with  $\text{ind}(T - \lambda I + F) \leq 0$ . As  $F$  is of finite rank, then by [5, Proposition 2.7],  $T - \lambda I$  is also an upper semi- $B$ -Fredholm operator. Moreover there exists an integer  $n_0$  such that for any  $n \geq n_0$ ,  $T - \lambda I - \frac{1}{n}I$  and  $T - \lambda I + F - \frac{1}{n}I$  are semi-Fredholm operators,  $\text{ind}(T - \lambda I - \frac{1}{n}I) = \text{ind}(T - \lambda I)$  and  $\text{ind}(T - \lambda I + F - \frac{1}{n}I) = \text{ind}(T - \lambda I + F)$ . Since  $F$  is a finite rank operator and  $T - \lambda I - \frac{1}{n}I$  is semi-Fredholm, by the known properties of the index we have  $\text{ind}(T - \lambda I + F - \frac{1}{n}I) = \text{ind}(T - \lambda I - \frac{1}{n}I)$ . So  $\text{ind}(T - \lambda I) = \text{ind}(T - \lambda I + F) \leq 0$ .  $\square$

**Theorem 4.2.** *Let  $T \in L(H)$ . Then*

$$\sigma_{LD}(T) = \bigcap_{F \in F_0(X), FT=TF} \sigma_{LD}(T + F).$$

*Proof.* Let  $\lambda \notin \sigma_{LD}(T)$ . Then  $T + 0 - \lambda I$  is left Drazin invertible. Since 0 is a finite rank operator, it follows that  $\lambda \notin \bigcap \{\sigma_{LD}(T + F) : F \in F_0(X), FT = TF\}$ .

Conversely suppose that  $\lambda \notin \bigcap \{\sigma_{LD}(T + F) : F \in F_0(X), FT = TF\}$ . Then there exists a finite rank operator  $F$  commuting with  $T$  such that  $T + F - \lambda I$  is left Drazin invertible. Hence from [5, Proposition 2.7],  $T - \lambda I = (T - \lambda I + F) - F$  is a semi- $B$ -Fredholm operator. In particular the two operators  $T - \lambda I$  and  $T - \lambda I + F$  are of topological uniform descent. By [10, Theorem 5.8], for  $n$  large enough we have  $c_n(T - \lambda I) = c_n(T - \lambda I + F)$  and  $c'_n(T - \lambda I) = c'_n(T - \lambda I + F)$ . Since  $T - \lambda I + F$  is left Drazin invertible, then for  $n$  large enough we have  $c'_n(T - \lambda I + F) = 0$ . So for  $n$  large enough we have  $c'_n(T - \lambda I) = 0$ . Hence  $T - \lambda I$  is left Drazin invertible.  $\square$

**Theorem 4.3.** *If  $T \in L(H)$  satisfies generalized  $a$ -Browder's theorem and if  $F$  is a finite rank operator such that  $TF = FT$ , then  $T + F$  satisfies generalized  $a$ -Browder's theorem.*

*Proof.* From the characterization of  $\sigma_{SBF_+^-}(T)$  it follows that if  $F$  is a finite rank operator, then  $\sigma_{SBF_+^-}(T + F) = \sigma_{SBF_+^-}(T)$ . Moreover, if  $F$  commutes with  $T$ , it follows from Theorem 4.2 that  $\sigma_{LD}(T + F) = \sigma_{LD}(T)$ . If  $T$  satisfies generalized  $a$ -Browder's theorem, then  $\sigma_{SBF_+^-}(T) = \sigma_{LD}(T)$ . Hence  $\sigma_{SBF_+^-}(T + F) = \sigma_{LD}(T + F)$ , and so  $T + F$  satisfies generalized  $a$ -Browder's theorem.  $\square$

Similarly, using properties of  $\sigma_{SF_+^-}(T)$  for an operator  $T$  acting on a Banach space  $X$ , we have the following:

**Theorem 4.4.** *Let  $X$  be a Banach space. If  $T \in L(X)$  satisfies  $a$ -Browder's theorem and  $F$  is a finite rank operator such that  $TF = FT$ , then  $T + F$  satisfies  $a$ -Browder's theorem.*

We give a generalization of a result involving poles of finite rank [1, Fact 10], and arbitrary poles [4, Theorem 2.9], which in general holds for Banach space operators.

**Theorem 4.5.** *Let  $\Gamma$  be a nonempty connected subset of  $\mathbb{C}$  such that  $T - \alpha I$  is an upper semi- $B$ -Fredholm operator for all  $\alpha \in \Gamma$ . If there is  $\alpha \in \Gamma$  such that  $T - \alpha I$  is left Drazin invertible, then every point of  $\sigma(T) \cap \Gamma$  is a left pole of  $T$  and  $\sigma_a(T) \cap \Gamma$  is a countable discrete set.*

*Proof.* Since  $T - \alpha I$  is left Drazin invertible, for  $n$  large enough we have  $c'_n(T - \alpha I) = 0$ . Let  $\Omega = \{\mu \in \Gamma : T - \mu I \text{ is left Drazin invertible}\}$ . Then  $\alpha \in \Omega$  and  $\Omega \neq \emptyset$ . If  $\lambda \in \Omega$ , since  $T - \lambda I$  is left Drazin invertible, then there is an open neighborhood  $B(\lambda, \varepsilon)$  such that  $B(\lambda, \varepsilon) \setminus \{\lambda\} \subset \rho_a(T)$ , where  $\rho_a(T)$  is the complement set of  $\sigma_a(T)$ . Therefore  $B(\lambda, \varepsilon) \cap \Gamma \subset \Omega$ , and  $\Omega$  is open in  $\Gamma$ . Now let  $\lambda \in \overline{\Omega} \cap \Gamma$ , where  $\overline{\Omega}$  is the closure of  $\Omega$ . In particular,  $T - \lambda I$  is an upper semi- $B$ -Fredholm operator. Hence there is  $\varepsilon > 0$  such that if  $|\lambda - \mu| < \varepsilon$  then for  $n$  large enough we have  $c_n(T - \lambda I) = c_n(T - \mu I)$ ,  $c'_n(T - \lambda I) = c'_n(T - \mu I)$ . Since  $\lambda \in \overline{\Omega}$ , then  $B(\lambda, \varepsilon) \cap \Omega \neq \emptyset$ . So there is  $\mu \in B(\lambda, \varepsilon) \cap \Omega$ . Hence  $c'_n(T - \lambda I) = 0$ , and so  $\lambda \in \Omega$ . Therefore  $\Omega$  is closed in  $\Gamma$ . Since  $\Gamma$  is connected, then  $\Omega = \Gamma$ . Moreover, if  $\lambda \in \sigma_a(T) \cap \Gamma$ , then  $\lambda$  is a pole of  $T$ , and an isolated point of  $\sigma_a(T)$ . Since  $\sigma_a(T)$  is a compact set,  $\sigma_a(T) \cap \Gamma$  is a discrete set.  $\square$

**Theorem 4.6.** *Let  $T \in L(H)$ . Suppose that  $\sigma_{SBF_+^-}(T)$  is simply connected. Then  $T + F$  satisfies generalized  $a$ -Browder's theorem for every  $F \in F_0(X)$ .*

*Proof.* Suppose that  $\lambda \in \sigma_a(T)$  and  $(T - \lambda I) \in SBF_+^-(X)$ . Let  $\Gamma = \{\alpha \in \mathbb{C} : (T - \alpha I) \in SBF_+^-(X)\}$ . Then  $\Gamma$  is connected. Since  $\Gamma \cap \rho_a(T)$  is non-empty, from the previous theorem it follows that  $\Gamma \cap \sigma_a(T)$  consists entirely of left poles of  $T$ . So  $\lambda \in \Pi^a(T)$  and  $T$  satisfies generalized  $a$ -Browder's theorem.

If  $F \in F_0(X)$ , is a finite rank operator, then  $\sigma_{SBF_+^-}(T + F) = \sigma_{SBF_+^-}(T)$ . Thus  $\sigma_{SBF_+^-}(T + F)$  is simply connected, and so  $T + F$  satisfies generalized Browder's theorem.  $\square$

## 5 Summary of results and open problems

For the sake of simplicity of notation we introduce the abbreviations  $gaW$ ,  $aW$ ,  $gW$  and  $W$  to signify that an operator  $T \in L(X)$  (which is usually understood) obeys generalized  $a$ -Weyl's theorem,  $a$ -Weyl's theorem, generalized Weyl's theorem and Weyl's theorem, respectively.

Analogous meaning is attached to the abbreviations  $gaB$ ,  $aB$ ,  $gB$  and  $B$  with respect to Browder's theorem. The following table summarizes the meaning of various theorems.

$gaW$	$\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus E^a(T)$	$gaB$	$\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \Pi^a(T)$
$aW$	$\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus E_0^a(T)$	$aB$	$\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \Pi_0^a(T)$
$gW$	$\sigma_{BW}(T) = \sigma(T) \setminus E(T)$	$gB$	$\sigma_{BW}(T) = \sigma(T) \setminus \Pi(T)$
$W$	$\sigma_W(T) = \sigma(T) \setminus E_0(T)$	$B$	$\sigma_W(T) = \sigma(T) \setminus \Pi_0(T)$

Table 1

The spectra  $\sigma_W(T)$ ,  $\sigma_{BW}(T)$ ,  $\sigma_{SF_+^-}(T)$  and  $\sigma_{SBF_+^-}(T)$  are defined by (1.2), (1.1), (1.3) and (1.4), respectively. In addition,

- $E(T)$  – eigenvalues of  $T$  isolated in  $\sigma(T)$
- $E_0(T)$  – eigenvalues of  $T$  of finite multiplicity isolated in  $\sigma(T)$
- $E^a(T)$  – eigenvalues of  $T$  isolated in  $\sigma_a(T)$
- $E_0^a(T)$  – eigenvalues of  $T$  of finite multiplicity isolated in  $\sigma_a(T)$
- $\Pi(T)$  – poles of  $T$
- $\Pi_0(T)$  – poles of  $T$  of finite rank
- $\Pi^a(T)$  – left poles of  $T$
- $\Pi_0^a(T)$  – left poles of  $T$  of finite rank

In the following diagram, arrows signify implications between various Weyl and Browder type theorems, and the numbers near the arrows are references to the results in the present paper (numbers without brackets) or to the bibliography therein (the numbers in

square brackets).

$$\begin{array}{ccccccccc}
 gB & \xleftarrow{[4]} & gW & \xleftarrow{3.7} & gaW & \xrightarrow{3.3} & gaB & \xrightarrow{3.8} & gB \\
 \downarrow 3.15 & & \downarrow 3.9 & & \downarrow 3.11 & & \downarrow 3.13 & & \downarrow 3.15 \\
 B & \xleftarrow{[1]} & W & \xleftarrow{[19]} & aW & \xrightarrow{3.5} & aB & \xrightarrow{[8]} & B
 \end{array}$$

Table 2

We finish this paper by posing the following two problems arising from Table 2.

**Open problem 1.** Does there exist an operator satisfying Browder's theorem but not generalized Browder's theorem?

**Open problem 2.** Does there exist an operator satisfying  $a$ -Browder's theorem but not generalized  $a$ -Browder's theorem?

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