Weighted Opial inequalities

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Abstract

This paper presents a class of very general weighted Opial type inequalities. The notivation comes from the monograph of Agarwal and Pang (*Opial Inequalities with Applications in Differential and Difference Equations*, Kluwer Acad., Dordrecht 1995) and the work of Anastassiou and Pečarić (*J. Math. Anal. Appl.* **239** (1999), 402–418). Assuming only a very general inequality, we extend the latter paper in several directions. A new result generalizing the original Opial's inequality is obtained, and applications to fractional derivatives are given.

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1 Introduction and preliminaries

The Opial inequality, which appeared in [7], is of great interest in differential and difference equations and other areas of mathematics, and has attracted a great deal of attention in the recent literature (see, for instance, [1, 2, 3, 4, 5, 8]). Recall that the original inequality [7] (see also [6, p. 114]) states the following:

Theorem 1.1. *Let $a > 0$. If $f \in C^1[0,a]$ with $f(0) = f(a) = 0$ and $f(t) > 0$ on $(0,a)$, then*

$$\int_0^a |f(t)f'(t)|\,dt \le \frac{a}{4}\int_0^a (f'(t))^2\,dt.$$

The constant $a/4$ is the best possible.

Our paper is motivated by the work of Anastassiou and Pečarić [5] on Opial inequalities for linear differential operators. Unlike [5], this paper does not initially assume any relation between the functions y and h except for the inequality (2.1); this leads to a very general type of inequalities in Section 2, extending the results of

[5] in several directions. In Section 3 we derive a new generalization of the original Opial's inequality, and in Section 4 we apply our results to fractional derivatives.

2 Results

The following hypotheses are assumed throughout this section: Let I be a closed interval in $\mathbb{R}$, a a fixed point in I, let Φ be a continuous function nonnegative on $I\times I$, and let y, $h\in C(I)$. We assume that the following condition involving Φ, h and y is satisfied:

$$|y(x)|\le\left|\int_a^x\Phi(x,t)|h(t)|\,dt\right|,\quad x\in I. \tag{2.1}$$

We give some typical examples of the condition (2.1).

Example 2.1. Let K be a continuous function on $I\times I$ and let y be defined by

$$y(s)=\int_a^s K(s,t)h(t)\,dt,\quad s\in I.$$

Then (2.1) holds with $\Phi(s,t)=|K(s,t)|$. A useful modification of this example—easier to attain in practice—is obtained when a function $z\in C(I)$ defined by

$$z(s)=\int_a^s K(s,t)h(t)\,dt$$

satisfies the inequality $|z(t)|\ge|y(t)|$. Again, (2.1) holds with $\Phi(s,t)=|K(s,t)|$.

In general, there need not be any relation between the functions y and h apart from the inequality (2.1). However, the following two examples describe useful applications with y and h closely related.

Example 2.2. Let $f\in C^n(I)$ and let $f^{(j)}(a)=0$ for $j=0,1,\ldots,n-1$. Then, for any $k\in\{0,1,\ldots,n-1\}$ and any $s\in I$,

$$f^{(k)}(s)=\frac{1}{(n-k-1)!}\int_a^s(s-t)^{n-k-1}f^{(n)}(t)\,dt. \tag{2.2}$$

(Observe that for $s<a$, this formula can be written as

$$f^{(k)}(s)=\frac{(-1)^{n-k}}{(n-k-1)!}\int_s^a(t-s)^{n-k-1}f^{(n)}(t)\,dt.)$$

Then (2.1) is satisfied with

$$\Phi(s,t)=\frac{|s-t|^{n-k-1}}{(n-k-1)!},\qquad y(t)=f^{(k)}(t),\qquad h(t)=f^{(n)}(t).$$

Example 2.3. More generally, our results will yield Opial type inequalities for linear differential operators (see [1, 3, 4]). Let

$$L = \sum_{j=0}^{n-1} a_j(t)D^j + D^n, \quad t \in I, \tag{2.3}$$

be a linear differential operator with $a_j \in C(I)$, let $h \in C(I)$, and let $G(x,t)$ be the Green's function for L. It is known that

$$y(x) = \int_a^x G(x,t)h(t)\,dt \tag{2.4}$$

is the unique solution to the initial value problem

$$Ly = h, \qquad y^{(j)}(a) = 0,\ j = 0, 1, \dots, n-1. \tag{2.5}$$

Then (2.1) is satisfied for y and h with $\Phi(s,t) = |G(s,t)|$.

Assuming the conditions stated at the beginning of this section we derive our first result which extends Theorems 1 and 2 and Corollary 1 of [5].

Theorem 2.4. *Assume that* (2.1) *holds. Let* $x \in I$, *let* α, $\beta > 0$, $r > \max(1, \alpha)$, *and let* U, $V \in C(I)$ *be such that* $U(s) \ge 0$ *and* $V(s) > 0$ *for all* $s \in I$. *Then*

$$\left| \int_a^x U(s)|y(s)|^\beta |h(s)|^\alpha\,ds \right| \le C(x) \left| \int_a^x V(s)|h(s)|^r\,ds \right|^{(\alpha+\beta)/r}, \tag{2.6}$$

where

$$C(x) := \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha/r} \left| \int_a^x \left(U^r(s)V^{-\alpha}(s)\right)^{1/(r-\alpha)} P(s)^{\beta(r-1)/(r-\alpha)}\,ds \right|^{(r-\alpha)/r}, \tag{2.7}$$

$$P(s) := \left| \int_a^s V(t)^{-1/(r-1)} \Phi(s,t)^{r/(r-1)}\,dt \right|. \tag{2.8}$$

Proof. Assume that $x \ge a$. Then, using (2.1) and Hölder's inequality with the conjugate indices r and $u = r/(r-1)$, we obtain

$$\begin{aligned} |y(s)| &\le \int_a^s \Phi(s,t)|h(t)|\,dt \\ &= \int_a^s V(t)^{-1/r}\Phi(s,t) \cdot V(t)^{1/r}|h(t)|\,dt \\ &\le \left(\int_a^s V(t)^{-1/(r-1)}\Phi(s,t)^u\,dt\right)^{1/u} \left(\int_a^s V(t)|h(t)|^r\,dt\right)^{1/r} \\ &\le P(s)^{1/u}\varphi(s)^{1/r}, \end{aligned}$$

where $\varphi'(s) = V(s)|h(s)|^r$ and $\varphi(a) = 0$. For any $\alpha > 0$,

$$|h(s)|^\alpha = V(s)^{-\alpha/r}\big(\varphi'(s)\big)^{\alpha/r}.$$

Then, for $\beta > 0$,

$$U(s)|y(s)|^\beta |h(s)|^\alpha \le U(s)P(s)^{\beta/u}V(s)^{-\alpha/r}\varphi(s)^{\beta/r}(\varphi'(s))^{\alpha/r}. \tag{2.9}$$

Integrate (2.9) over $[a, x]$ and apply Hölder's inequality with the conjugate indices r/α and $v = r/(r-\alpha)$ to obtain

$$\begin{aligned}
&\int_a^x U(s)|y(s)|^\beta |h(s)|^\alpha\, ds \\
&\le \left(\int_a^x U(s)^v V(s)^{-\alpha v/r} P(s)^{\beta v/u}\, ds\right)^{1/v} \left(\int_a^x \varphi(s)^{\beta/\alpha}\varphi'(s)\, ds\right)^{\alpha/r} \\
&= \left(\int_a^x U(s)^{r/(r-\alpha)} V(s)^{-\alpha/(r-\alpha)} P(s)^{\beta(r-1)/(r-\alpha)}\, ds\right)^{(r-\alpha)/\alpha} \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha/r} \varphi(s)^{(\alpha+\beta)/r} \\
&= C(x)\left(\int_a^x V(t)|h(t)|^r\, dt\right)^{(\alpha+\beta)/r}.
\end{aligned}$$

This proves (2.6).

The case $x < a$ follows from the preceding proof by using the relation $\int_x^a (\cdot)\, ds = -\int_a^x (\cdot)\, ds$. □

We remark that [5, Corollary 1]—proved for linear differential operators—is recovered from the theorem when y, h and G satisfy conditions of Example 2.3, that is,

$$y(s) = \int_a^s G(s,t)h(t)\, dt, \quad s \in I.$$

In this case $\Phi(s,t) = |G(s,t)|$.

In particular, if $r = 2$ in Theorem 2.4, we have the following specialization (see also [5, Corollary 2]).

Corollary 2.5. *Assume that* (2.1) *holds. Let* $x \in I$, $0 < \alpha < 2$ *and* $\beta > 0$. *Let* $U, V \in C(I)$ *be such that* $U(s) \ge 0$ *and* $V(s) > 0$ *for all* $s \in I$. *Then*

$$\left|\int_a^x U(s)|y(s)|^\beta |h(s)|^\alpha\, ds\right| \le \widetilde{C}(x)\left|\int_a^x V(s)|h(s)|^2\, ds\right|^{(\alpha+\beta)/2}, \tag{2.10}$$

where

$$\widetilde{C}(x) := \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha/2} \left|\int_a^x \big(U^2(s)V^{-\alpha}(s)\big)^{1/(2-\alpha)} \widetilde{P}(s)^{\beta/(2-\alpha)}\, ds\right|^{(2-\alpha)/2}, \tag{2.11}$$

$$\widetilde{P}(s) := \left|\int_a^s V(t)^{-1/(2-1)}\Phi(s,t)^2\, dt\right|. \tag{2.12}$$

The following extreme case analogous to [5, Proposition 1] is proved similarly as Theorem 2.4.

Theorem 2.6. *Assume that* (2.1) *holds. Let* $x \in I$*, let* α*,* $\beta > 0$*,* $r > \max(1, \alpha)$*, and let* V*,* $V \in C(I)$ *be such that* $U(s) \geq 0$ *and* $V(s) > 0$ *for all* $s \in I$*. Then*

$$\left| \int_a^x U(s)|y(s)|^\beta |h(s)|^\alpha \, ds \right| \leq \int_a^x U(w) \left| \int_a^x V(t)\Phi(w,t)\, dt \right|^{(r-\alpha)/r} \|V\|_\infty^\beta \|h\|_\infty^{\alpha+\beta}, \tag{2.13}$$

where $\|f\|_\infty = \sup\{|f(t)| : t \in [a,x] \cup [x,a]\}$ *for any* $f \in C(I)$*.*

Following [5], we consider a situation when the exponents α, β and r in Theorem 2.4 are not necessarily positive. In this case the inequality (2.1) must be strengthened to equality

$$|y(s)| = \left| \int_a^s \Phi(s,t)|h(t)|\, dt \right|, \tag{2.14}$$

where Φ is again a nonnegative continuous function on $I \times I$, and y, $h \in C(I)$. As before, a is a fixed point in the interval I.

The proof of the following theorem is omitted as it is similar to the proofs of Theorems 3–6 in [5]. Let us remark that our result applies to completely general functions y and h as long as they satisfy (2.14) for some Φ, while [5] treats the case of linear differential operators with y and h related as in Example 2.3.

Theorem 2.7. *Assume that* (2.14) *holds. Let* $x \in I$*, and let* $U, V \in C(I)$ *be such that* $U(s) \geq 0$ *and* $V(s) > 0$ *for all* $s \in I$*. Let* $C(x)$ *be defined by* (2.7) *and* (2.8)*. Consider real numbers* α*,* β, r *and the following relations*:

(i) $r > 1$, $\beta > 0$, $0 < \alpha < r$;

(ii) $r < \alpha < 0$, $\beta < 0$;

(iii) $-\alpha < \beta < 0$, $0 < \alpha < r < 1$;

(iv) $\beta > 0$, $0 < r < \min(\alpha, 1)$;

(v) $\alpha < 0 < r < 1$, $0 < \beta < -\alpha$;

(vi) $\beta < 0$, $\alpha < 0$, $r > 1$;

(vii) $1 < r < \alpha$, $-\alpha < \beta < 0$;

(viii) $\beta > 0$, $r < 0 < \alpha$;

(ix) $\alpha < r < 0$, $0 < \beta < -\alpha$.

If one of the conditions (i)–(iii) *is satisfied, then*

$$\left|\int_a^x U(s)|y(s)|^\beta |h(s)|^\alpha\, ds\right| \le C(x) \left|\int_a^x V(s)|h(s)|^r\, ds\right|^{(\alpha+\beta)/r}.$$

If one of the conditions (iv)–(ix) *is satisfied, then*

$$\left|\int_a^x U(s)|y(s)|^\beta |h(s)|^\alpha\, ds\right| \ge C(x) \left|\int_a^x V(s)|h(s)|^r\, ds\right|^{(\alpha+\beta)/r}.$$

3 Further results

In this section we assume that I is a closed interval in $\mathbb{R}$ and a, b are two fixed points in I such that $a < b$. Further we assume that Φ_1 and Φ_2 are two nonnegative continuous functions on $I \times I$, and that y, $h \in C(I)$. In place of (2.1) we assume that

$$|y(s)| \le \begin{cases} \displaystyle\int_a^x \Phi_1(x,t)|h(t)|\, dt & \text{if } x \ge a, \\ \displaystyle\int_x^b \Phi_2(x,t)|h(t)|\, dt & \text{if } x \le b. \end{cases} \tag{3.1}$$

A typical example of this condition:

Example 3.1. Let $f \in C^n(I)$ and let $f^{(j)}(s) = 0$ for $s = a, b$, $j = 0, 1, \ldots, n-1$. Then, for any $k \in \{0, 1, \ldots, n-1\}$ and any $s \in I$,

$$f^{(k)}(s) = \frac{1}{(n-k-1)!} \int_a^s (s-t)^{n-k-1} f^{(n)}(t)\, dt, \quad s \ge a, \tag{3.2}$$

$$f^{(k)}(s) = \frac{(-1)^{n-k}}{(n-k-1)!} \int_s^b (t-s)^{n-k-1} f^{(n)}(t)\, dt, \quad s \le b. \tag{3.3}$$

In this case (3.1) holds with

$$\Phi_i(s,t) = \frac{|s-t|^{n-k-1}}{(n-k-1)!},\ i = 1, 2, \qquad y(t) = f^{(k)}(t), \qquad h(t) = f^{(n)}(t).$$

As in previous examples concerning (2.1), this can be extended to linear differential operators.

In the next proposition it is assumed that $r = \alpha + \beta$.

Proposition 3.2. *Assume that condition* (3.1) *is satisfied. Let* α, $\beta > 0$, $\alpha + \beta > 1$ *and let* U, $V \in C(I)$ *be such that* $U(s) \ge 0$ *and* $V(s) > 0$ *for all* $s \in I$.

(i) *If $x \geq a$, then*

$$\int_a^x U(s)|y(s)|^\beta |h(s)|^\alpha\, ds \leq A(x) \int_a^x V(s)|h(s)|^{\alpha+\beta}\, ds, \tag{3.4}$$

where

$$A(x) := \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha/(\alpha+\beta)} \left(\int_a^x \left(U^{\alpha+\beta}(s)V^{-\alpha}(s)\right)^{1/\beta} Q_1(s)^{\alpha+\beta-1}\, ds\right)^{\beta/(\alpha+\beta)}, \tag{3.5}$$

$$Q_1(s) := \int_a^s V(t)^{-1/(\alpha+\beta-1)} \Phi_1(s,t)^{(\alpha+\beta)/(\alpha+\beta-1)}\, dt. \tag{3.6}$$

(ii) *If $x \leq b$, then*

$$\int_x^b U(s)|y(s)|^\beta |h(s)|^\alpha\, ds \leq B(x) \int_x^b V(s)|h(s)|^{\alpha+\beta}\, ds, \tag{3.7}$$

where

$$B(x) := \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha/(\alpha+\beta)} \left(\int_x^b \left(U^{\alpha+\beta}(s)V^{-\alpha}(s)\right)^{1/\beta} Q_2(s)^{\alpha+\beta-1}\, ds\right)^{\beta/(\alpha+\beta)}, \tag{3.8}$$

$$Q_2(s) := \int_s^b V(t)^{-1/(\alpha+\beta-1)} \Phi_2(s,t)^{(\alpha+\beta)/(\alpha+\beta-1)}\, dt. \tag{3.9}$$

Proof. The result follows from Theorem 2.4 for the special case $r = \alpha + \beta$. □

The following result generalizes Opial's inequality.

Theorem 3.3. *Let the hypotheses of Proposition* 3.2 *be satisfied with $A(b) \neq 0$ and $B(a) \neq 0$, where A and B are defined by* (3.5) *and* (3.8)*, respectively. Then there exists $x_0 \in (a,b)$ such that $A(x_0) = B(x_0) =: D$, and*

$$\int_a^b U(t)|y(t)|^\beta |h(t)|^\alpha\, dt \leq D \int_a^b V(t)|h(s)|^{\alpha+\beta}\, ds. \tag{3.10}$$

Proof. The function $S(x) := A(x) - B(x)$ is continuous for $x \in [a,b]$, and $S(a) = -B(a) < 0$, $S(b) = A(b) > 0$. By the intermediate value theorem there exists $x_0 \in (a,b)$ such that $S(x_0) = 0$, that is, $A(x_0) = B(x_0) := D$. According to Proposition 3.2,

$$\begin{aligned}\int_a^b U(t)|y(t)|^\beta |h(t)|^\alpha\, dt &= \int_a^{x_0} U(t)|y(t)|^\beta |h(t)|^\alpha\, dt + \int_{x_0}^b U(t)|y(t)|^\beta |h(t)|^\alpha\, dt \\ &\leq A(x_0) \int_a^{x_0} V(t)|h(s)|^{\alpha+\beta}\, ds + B(x_0) \int_{x_0}^b V(t)|h(s)|^{\alpha+\beta}\, ds \\ &= D \int_a^b V(t)|h(s)|^{\alpha+\beta}\, ds.\end{aligned}$$

□

Remark 3.4. The original Opial's inequality is recovered from Theorem 3.3 when $y(t) = f(t)$, $h(t) = f'(t)$, $U(t) = V(t) = 1$ and $\alpha = \beta = 1$, where $f \in C^1(I)$ and $f(a) = f(b) = 0$. The condition (3.1) holds with $\Phi_i(s,t) = 1$, $i = 1, 2$, as gleaned from the representations

$$f(s) = \int_a^s f'(t)\,dt = -\int_s^b f'(t)\,dt, \quad a \le s \le b.$$

We calculate $A(x_0) = (x_0 - a)/2$ and $B(x_0) = (b - x_0)/2$. From $A(x_0) = B(x_0)$ we obtain $x_0 = (a+b)/2$ and $D = (b-a)/4$ in agreement with Theorem 1.1.

Remark 3.5. The constant D depends on the choice of Φ_1 and Φ_2 in Theorem 3.3. If we make a non-optimal choice in the preceding remark, say $\Phi_1(s,t) = 1$ and $\Phi_2(s,t) = 2$, a calculation yields $D = (b-a)/(2+\sqrt{2}) > (b-a)/4$.

4 Applications to fractional derivatives

First we review basic facts about fractional derivatives needed below following essentially Chapter 1 of the monograph [9] by Samko, Kilbas and Marichev. Let $x > 0$. By $C^m[0,x]$ we denote the space of all functions on $[0,x]$ which have continuous derivatives up to order m, and $AC[0,x]$ is the space of all absolutely continuous functions on $[0,x]$. By $AC^m[0,x]$ we denote the space of all functions $g \in C^m[0,x]$ with $g^{(m-1)} \in AC[0,x]$. For any $\alpha \in \mathbb{R}$ we denote by $[\alpha]$ the integral part of α (the integer k satisfying $k \le \alpha < k+1$). By $L(0,x)$ we denote the space of all Lebesgue integrable functions on the interval $(0,x)$ and by $L^\infty(0,x)$ the set of all Lebesgue measurable functions essentially bounded on $[0,x]$.

Let $\alpha > 0$. For any $f \in L(0,x)$ the *Riemann–Liouville fractional integral of* f of order α is defined by

$$I^\alpha f(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} f(t)\,dt, \quad s \in [0,x]. \tag{4.1}$$

The integral on the right side of (4.1) exists for almost all $s \in [0,x]$ (see [9]), and $I^\alpha f \in L(0,x)$. The *Riemann–Liouville fractional derivative of* $f \in L(0,x)$ of order α is defined by

$$D^\alpha f(s) = \left(\frac{d}{ds}\right)^m I^{m-\alpha} f(s) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{ds}\right)^m \int_0^s (s-t)^{m-\alpha-1} f(t)\,dt \tag{4.2}$$

where $m = [\alpha] + 1$, provided that the derivative exists. In addition, we stipulate

$$D^0 f := f =: I^0 f, \quad I^{-\alpha} f := D^\alpha f \text{ if } \alpha > 0, \quad D^{-\alpha} f := I^\alpha f \text{ if } 0 < \alpha \le 1. \tag{4.3}$$

If α is a positive integer, then $D^\alpha f = (d/ds)^\alpha f$.

Let $\alpha > 0$ and $m = [\alpha] + 1$. A function $f \in L(0, x)$ is said to have an *integrable fractional derivative* $D^\alpha f$ (see the definition and discussion in [9, pp. 43–44]) if

$$D^{\alpha-k} f \in C[0, x],\ k = 1, \ldots, m, \quad \text{and} \quad D^{\alpha-1} f \in AC[0, x]. \tag{4.4}$$

The following theorem is a strong analogue of Taylor's formula with vanishing fractional derivatives of lower orders. An interesting aspect of this formula is that ν and μ can be arbitrarily close.

Theorem 4.1. *Let $\nu > \mu \geq 0$, let $f \in L(0, x)$ have an integrable fractional derivative $D^\nu f$, and let $D^{\nu-k} f(0) = 0$ for $k = 1, \ldots, [\nu] + 1$. Then*

$$D^\mu f(s) = \frac{1}{\Gamma(\nu - \mu)} \int_0^s (s - t)^{\nu-\mu-1} D^\nu f(t)\, dt, \quad s \in [0, x]. \tag{4.5}$$

Proof. Set $\alpha = \nu - \mu > 0$ and $\beta = -\nu < 0$. According to the index law for fractional derivatives (Theorem 2.5 in [9, p. 45]),

$$I^{\nu-\mu} D^\nu f = I^\beta I^\alpha f = I^{\beta+\alpha} f = I^{-\mu} f = D^\mu f.$$

This proves the result. □

We can now give an application of Theorem 2.4 to fractional derivatives.

Theorem 4.2. *Let $x > 0$, let α, $\beta > 0$, $r > \max\{1, \alpha, (\nu-\mu)^{-1}\}$, and let U, $V \in C(I)$ be such that $U(s) \geq 0$ and $V(s) > 0$ for all $s \in I$. Let $f \in L(0, x)$ have an integrable fractional derivative $D^\nu f \in L^\infty(0, x)$ such that $D^{\nu-j} f(0) = 0$ for $j = 1, \ldots, [\nu] + 1$. Then*

$$\int_0^x U(s) |D^\mu f(s)|^\beta |D^\nu f(s)|^\alpha\, ds \leq \Omega(x) \left(\int_0^x V(s) |D^\nu f(s)|^r\, ds \right)^{(\alpha+\beta)/r}, \tag{4.6}$$

where

$$\Omega(x) := \left(\frac{\alpha}{\alpha + \beta} \right)^{\alpha/r} \left(\int_0^x \big(U^r(s) V^{-\alpha}(s) \big)^{1/(r-\alpha)} \Delta(s)^{\beta(r-1)/(r-\alpha)}\, ds \right)^{(r-\alpha)/r}, \tag{4.7}$$

$$\Delta(s) := \int_0^s V(t)^{-1/(r-1)} \left[\frac{1}{\Gamma(\nu - \mu)} (s - t)^{\nu-\mu-1} \right]^{r/(r-1)} dt. \tag{4.8}$$

Proof. According to Theorem 4.1,

$$D^\mu f(s) = \frac{1}{\Gamma(\nu - \mu)} \int_0^s (s - t)^{\nu-\mu-1} D^\nu f(t)\, dt, \quad s \in [0, x]. \tag{4.9}$$

Setting

$$y(s) = D^\mu f(s), \quad h(s) = D^\nu f(s), \quad \Phi(s,t) = \frac{(s-t)_+^{\nu-\mu-1}}{\Gamma(\nu-\mu)},$$

we observe that condition (2.1) is satisfied with $a = 0$ and $I = [0, x]$:

$$|y(s)| \le \int_0^s \Phi(s,t)|h(t)|\,dt, \quad 0 \le s \le x.$$

Write $\gamma = \nu - \mu - 1$. If $\gamma < 0$, a slight modification of the proof of Theorem 2.4 is required as Φ is not continuous on $[0,x] \times [0,x]$. By hypothesis, $\gamma > -1$. For the integral in $\Delta(x)$ to exist, the function

$$t \mapsto V(t)^{-1/(r-1)}(s-t)^{\gamma r/(r-1)}$$

must be integrable on $[0,s]$. As $V(t)$ is continuous and positive on $[0,s]$, we must have $\gamma r/(r-1) > -1$. This is ensured by the condition $r > (\nu-\mu)^{-1}$. The assumption $D^\nu f \in L^\infty(0,x)$ is needed to ensure that the function $t \mapsto V(t)|D^\nu f(t)|^r$ is integrable. The proof of Theorem 2.4 then goes through and the result follows. □

An interesting special case follows.

Theorem 4.3. *Let $x > 0$, $\nu > \mu \ge 0$, let α, $\beta > 0$ and let $r > \max\{1, \alpha, (\nu-\mu)^{-1}\}$. Let $f \in L(0,x)$ have an integrable fractional derivative $D^\nu f \in L^\infty(0,x)$ such that $D^{\nu-j}f(0) = 0$ for $j = 1, \ldots, [\nu]+1$. Then*

$$\int_0^x |D^\mu f(s)|^\beta |D^\nu f(s)|^\alpha\,ds \le \Omega_1\, x^{(\beta\sigma+r-\alpha)/(r-\alpha)} \left(\int_0^x |D^\nu f(s)|^r\,ds\right)^{(\alpha+\beta)/r}, \qquad (4.10)$$

where

$$\Omega_1 := \left[\left(\frac{\alpha}{\alpha+\beta}\right)^\alpha \left(\frac{r-1}{\sigma}\right)^{\beta(r-1)} \left(\frac{r-\alpha}{\beta\sigma+r-\alpha}\right)^{r-\alpha}\right]^{1/r} \Gamma(\nu-\mu)^{-\beta} \qquad (4.11)$$

with $\sigma := r\nu - r\mu - 1$.

Proof. By Theorem 4.2,

$$\int_0^x |D^\mu f(s)|^\beta |D^\nu f(s)|^\alpha\,ds \le \Omega(x) \left(\int_0^x |D^\nu f(s)|^r\,ds\right)^{(\alpha+\beta)/r}, \qquad (4.12)$$

where

$$\Omega(x) = \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha/r}.$$

$$\cdot \left(\int_0^x \left(\int_0^s \left[\frac{1}{\Gamma(\nu-\mu)} (s-t)^{\nu-\mu-1} \right]^{r/(r-1)} dt \right)^{\beta(r-1)/(r-\alpha)} ds \right)^{(r-\alpha)/r}$$

$$= \left(\frac{\alpha}{\alpha+\beta} \right)^{\alpha/r} \Gamma(\nu-\mu)^{-\beta} \left(\frac{r-1}{\sigma} \right)^{\beta(r-1)/r} \left(\int_0^x s^{\beta\sigma/(r-\alpha)} ds \right)^{(r-\alpha)/r}$$

$$= \left(\frac{\alpha}{\alpha+\beta} \right)^{\alpha/r} \Gamma(\nu-\mu)^{-\beta} \left(\frac{r-1}{\sigma} \right)^{\beta(r-1)/r} \left(\frac{r-\alpha}{\beta\sigma+r-\alpha} \right)^{(r-\alpha)/r} x^{1+\beta\sigma/(r-\alpha)}.$$

This completes the proof. □

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References

[1] R. P. Agarwal, Sharp Opial-type inequalities involving r-derivatives and their applications, *Tôhoku Math. J.* **47** (1995), 567–593.

[2] R. P. Agarwal and P. Y. H. Pang, Remarks on the generalization of Opial's inequality, *J. Math. Anal. Appl.* **190** (1995), 559–577.

[3] R. P. Agarwal and P. Y. H. Pang, *Opial Inequalities with Applications in Differential and Difference Equations*, Kluwer Acad. Publ., Dordrecht, 1995.

[4] G. A. Anastassiou, Opial type inequalities for linear differential operators, *Math. Inequalities Appl.* **1** (1998), 193–200.

[5] G. A. Anastassiou and J. Pečarić, General weighted Opial inequalities for linear differential operators, *J. Math. Anal. Appl.* **239** (1999), 402–418.

[6] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Acad. Publ., Dordrecht, 1993.

[7] Z. Opial, Sur une inégalité, *Ann. Polon. Math.* **8** (1960), 29–32.

[8] B. G. Pachpatte, On Opial-type inequalities, *J. Math. Anal. Appl.* **120** (1986), 547–556.

[9] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Reading, 1993.