# Continuity and general perturbation of the Drazin inverse for closed linear operators

N. Castro González, J. J. Koliha and V. Rakočević

**Abstract**

We study perturbations and the continuity of the Drazin inverse of a closed linear operator $A$ and obtain explicit error estimates in terms of the gap between closed operators and in terms of the gap between ranges and nullspaces of operators. The results are used to derive a theorem on the continuity of the Drazin inverse for closed operators and to describe the asymptotic behaviour of operator semigroups.

## 1 Introduction

In this paper we investigate a perturbation of the Drazin inverse $A^{\mathrm{D}}$ of a closed linear operator $A$; the main tool for obtaining the estimates is the gap between subspaces and operators.

By $\mathcal{C}(X)$ we denote the set of all closed linear operators acting on a linear subspace of $X$ to $X$, where $X$ is a complex Banach space. We write $\mathcal{D}(A)$, $\mathcal{N}(A)$, $\mathcal{R}(A)$, $\rho(A)$, $\sigma(A)$ and $R(\lambda, A)$ for the domain, nullspace, range, resolvent set, spectrum and the resolvent of an operator $A \in \mathcal{C}(X)$. All relevant concepts from the theory of closed linear operators can be found in [10, 24]. The set of all operators $T \in \mathcal{C}(X)$ with $\mathcal{D}(T) = X$ will be denoted by $\mathcal{B}(X)$; we recall that operators in $\mathcal{B}(X)$ are bounded, and the operator norm of $T \in \mathcal{B}(X)$ will be denoted by $\|T\|$. An operator $A \in \mathcal{C}(X)$ is called *quasipolar* if 0 is a resolvent point or an isolated spectral point of

2000 *Mathematics Subject Classification*: 47A10, 47A60

*Key words and phrases*: Drazin inverse, closed linear operator, perturbation, operator semigroups

$A$. The *spectral projection* of a quasipolar operator $A \in \mathcal{C}(X)$ is the unique idempotent operator $A^\pi \in \mathcal{B}(X)$ such that $AA^\pi x = A^\pi Ax$ for all $x \in \mathcal{D}(A)$, $AA^\pi$ is quasinilpotent and $A + A^\pi$ is invertible in $\mathcal{C}(X)$.

With an eye to further development in this paper we choose the following definition of the (generalized) Drazin inverse among the equivalent formulations given in [15].

DEFINITION 1.1. An operator $A \in \mathcal{C}(X)$ is *Drazin invertible* if $A$ is quasipolar. The *Drazin inverse* of $A$ is defined by

$$A^{\mathrm{D}} = (A + A^\pi)^{-1}(I - A^\pi), \tag{1.1}$$

where $A^\pi$ is the spectral projection of $A$ corresponding to 0 (see [24, Theorem V.9.2]); note that $A^{\mathrm{D}}$, $A^\pi \in \mathcal{B}(X)$, and that

$$A^\pi = I - AA^{\mathrm{D}}.$$

The *Drazin index* $i(A)$ of a Drazin invertible operator $A \in \mathcal{C}(X)$ is equal to 0 if $A$ is invertible, to the nilpotency index of the operator $AA^\pi$ if $A$ is singular and $AA^\pi$ nilpotent, and to $\infty$ if $AA^\pi$ is quasinilpotent but not nilpotent.

This definition further generalizes the concept of the Drazin inverse introduced in [11], which in turn is a generalization of the original definition of pseudoinverse given by Drazin [8].

We need another characterization of the Drazin inverse for future use. A proof of the formula is given for completeness.

LEMMA 1.2. *If $A \in \mathcal{C}(X)$ is Drazin invertible, then*

$$A^{\mathrm{D}} = f(A),$$

*where $f(\lambda) = 0$ in $\Omega_0$ and $f(\lambda) = \lambda^{-1}$ in $\Omega$, where $\Omega_0$ and $\Omega$ are disjoint open neighbourhoods of* 0 *and $\sigma(A) \setminus \{0\}$, respectively.*

*Proof.* Since $A$ is quasipolar, we can choose open neighbourhoods $\Omega_0$ and $\Omega$ of 0 and $\sigma(A) \setminus \{0\}$, respectively, such that the complement $\Omega^c$ of $\Omega$ is compact and $\Omega_0 \subset \Omega^c$. Let $\gamma$ be a cycle in $\Omega \setminus \sigma(A)$ such that $\operatorname{ind}(\gamma, \lambda) = 0$ if $\lambda \in \sigma(A) \setminus \{0\}$ and $\operatorname{ind}(\gamma, \lambda) = 1$ if $\lambda \in \Omega^c$. Using the holomorphic calculus for $A$ with $f$ given in the statement of the theorem, we get

$$\begin{aligned} f(A) &= -\frac{1}{2\pi i} \int_\gamma \lambda^{-1} A R(\lambda, A)\, \mathrm{d}\lambda \\ &= \left( \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}\lambda}{\lambda} \right) I - \frac{1}{2\pi i} \int_\gamma R(\lambda, A)\, \mathrm{d}\lambda = I - A^\pi. \end{aligned}$$

Since $A^\pi = g(A)$, where $g(\lambda) = 1$ in $\Omega_0$ and $g(\lambda) = 0$ in $\Omega$, we have $A^\pi f(A) = g(A)f(A) = (gf)(A) = 0$, and $(A + A^\pi)f(A) = I - A^\pi$. Hence $f(A) = (A + A^\pi)^{-1}(I - A^\pi) = A^{\mathrm{D}}$. □

From the last paragraph of the preceding proof we see that the Drazin inverse can also be given by the formula $A^{\mathrm{D}} = (A + \xi A^\pi)^{-1}(I - A^\pi)$ for any $\xi \neq 0$; in particular, we have

$$A^{\mathrm{D}} = (A - A^\pi)^{-1}(I - A^\pi).$$

## 2 Some results on the gap between subspaces

First we quantify results of [14] by obtaining explicit error bounds for the norm distance $\|P - Q\|$ between two idempotent operators $P$, $Q \in \mathcal{B}(X)$ in terms of the gap between their null spaces and ranges.

Let $\mathcal{S}(X)$ be the set of all closed subspaces of the Banach space $X$. The *gap beween subspaces* $M$, $N \in \mathcal{S}(X)$ is defined in [9] and [10, Section IV.2.1]:

$$\operatorname{gap}(M, N) = \max\{\Delta(M, N), \Delta(N, M)\}, \tag{2.1}$$

where $\Delta(M, N) = \sup\{\operatorname{dist}(x, N) : x \in M,\ \|x\| = 1\}$. We summarize the properties of the gap function relevant to our investigation in the following lemma; these results can be found in [9, 10].

Lemma 2.1. *The following are true for any $M$, $N$, $K \in \mathcal{S}(X)$:*

(i) $0 \le \operatorname{gap}(M,N) \le 1$.

(ii) $\operatorname{gap}(M,N) = 0$ *implies* $M = N$.

(iii) $\operatorname{gap}(M,N) = \operatorname{gap}(N,M)$.

(iv) $\operatorname{gap}(M,N) \le 2(\operatorname{gap}(M,K) + \operatorname{gap}(K,N))$.

(v) $\operatorname{gap}(M^\perp, N^\perp) = \operatorname{gap}(M,N)$, *where* $M^\perp$, $N^\perp$ *are the annihilators of* $M$, $N$ *in* $X^*$.

(vi) *If* $\operatorname{gap}(M,N) < 1$, *then* $M$ *and* $N$ *have the same dimension.*

The sets $U(M,\varepsilon) = \{N \in \mathcal{S}(X) : \operatorname{gap}(M,N) < \varepsilon\}$, where $M \in \mathcal{S}(X)$ and $\varepsilon > 0$, form a base for the so-called *gap topology* on $\mathcal{S}(X)$; this is a complete metrizable topology on $\mathcal{S}(X)$ [10, Section IV.2.1]. For a detailed discussion of the gap topology see [18]. We write $M_n \xrightarrow{\mathcal{G}} M$ for $\operatorname{gap}(M_n, M) \to 0$ in $\mathcal{S}(X)$.

Proposition 2.2. *Let* $P$, $Q \in \mathcal{B}(X)$ *be idempotent operators and let*

$$\nu(P,Q) = \operatorname{gap}(\mathcal{N}(P), \mathcal{N}(Q)), \quad \rho(P,Q) = \operatorname{gap}(\mathcal{R}(P), \mathcal{R}(Q)). \tag{2.2}$$

*Then*

$$\max\{\nu(P,Q), \rho(P,Q)\} \le \|P - Q\| \le \frac{\|P\|\|I-P\|(\nu(P,Q)+\rho(P,Q))}{1 - \|P\|\nu(P,Q) - \|I-P\|\rho(P,Q)}, \tag{2.3}$$

*where the second inequality holds if*

$$\|P\|\nu(P,Q) + \|I-P\|\rho(P,Q) < 1. \tag{2.4}$$

*Proof.* The first inequality in (2.3) is well known—see [9, Equation (1.4)].

To prove the second inequality in (2.3), we apply [14, Lemma 3.1] to obtain the following string of inequalities:

$$\begin{aligned} \|P - Q\| &\le \|(I-P)Q\| + \|P(I-Q)\| \\ &\le \|I-P\|\|Q\|\rho(P,Q) + \|P\|\|I-Q\|\nu(P,Q) \end{aligned}$$

$$\begin{aligned}
&\leq \|I-P\|(\|P\|+\|P-Q\|)\rho(P,Q) \\
&\quad + \|P\|(\|I-P\|+\|P-Q\|)\nu(P,Q) \\
&= \|I-P\|\|P\|(\nu(P,Q)+\rho(P,Q)) \\
&\quad + (\|I-P\|\rho(P,Q)+\|P\|\nu(P,Q))\|P-Q\|.
\end{aligned}$$

Then

$$\begin{aligned}
(1-\|I-P\|\rho(P,Q)-&\|P\|\nu(P,Q))\|P-Q\| \\
&\leq \|I-P\|\|P\|(\nu(P,Q)+\rho(P,Q)),
\end{aligned}$$

and the second inequality in (2.3) is proved provided (2.4) holds. □

From the preceding proposition we recover [14, Lemma 3.3].

Corollary 2.3. *Let $P_n$, $P \in \mathcal{B}(X)$ be idempotent operators. Then*

$$\|P_n - P\| \to 0 \Leftrightarrow \mathcal{N}(P_n) \xrightarrow{\mathcal{G}} \mathcal{N}(P) \text{ and } \mathcal{R}(P_n) \xrightarrow{\mathcal{G}} \mathcal{R}(P).$$

Let $H$ be a Hilbert space. Then the gap between closed subspaces $M$, $N$ is equal to

$$\operatorname{gap}(M,N) = \|P_M - P_N\|,$$

with $P_M$, $P_N$ the orthogonal projections onto $M$, $N$, respectively [10, Theorem I.6.34]. Rakočević recently proved [21] that, for an idempotent operator $P$, $\operatorname{gap}(\mathcal{N}(P)^\perp, \mathcal{R}(P)) = (1-\|P\|^{-2})^{1/2}$, that is,

$$\|P\| = (1-\operatorname{gap}^2(\mathcal{N}(P)^\perp, \mathcal{R}(P)))^{-1/2}.$$

This yields a simple proof of the well known equality

$$\|I-P\| = \|P\|.$$

## 3 Perturbations of the Drazin inverse

Let $A$ and $B$ be closed linear operators on $X$. In the case that both operators are Drazin invertible, we want to derive an error bound for the norm $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|$. From [14, 20] we recall that for Drazin invertible bounded linear operators satisfying $A_n \to A$ (in the operator norm) we have

$$A_n^{\mathrm{D}} \to A^{\mathrm{D}} \Leftrightarrow A_n^{\pi} \to A^{\pi}.$$

This suggests that we should consider perturbations $B$ of $A$ with $B$ close to $A$ in some sense and also with $\|B^{\pi} - A^{\pi}\|$ small.

If $A \in \mathcal{C}(X)$, we write $\mathrm{G}\,(A) = \{(u, Au) \in X \times X : u \in \mathcal{D}(A)\}$ for the graph of the operator $A$. For any $A \in \mathcal{C}(X)$, the set $\mathrm{G}\,(A)$ is a closed subspace of the product space $X \times Y$ equipped with the norm $\|(x, y)\| = (\|x\|^2 + \|y\|^2)^{1/2}$. We can then define the *gap between operators* $A$, $B \in \mathcal{C}(X)$ [10, Section IV.4.4] by

$$\operatorname{gap}(A, B) = \operatorname{gap}(\mathrm{G}\,(A), \mathrm{G}\,(B)). \tag{3.1}$$

The sets $\mathcal{U}(A, \varepsilon) = \{B \in \mathcal{C}(X) : \operatorname{gap}(A, B) < \varepsilon\}$, where $A \in \mathcal{C}(X)$ and $\varepsilon > 0$, form a base for the so-called *operator gap topology* in $\mathcal{C}(X)$; this is (in general an incomplete) metrizable topology on $\mathcal{C}(X)$. For a detailed discussion of this topology see [19]. We write $A_n \xrightarrow{\mathcal{G}} A$ for $\operatorname{gap}(A_n, A) \to 0$ in $\mathcal{C}(X)$.

In the sequel we need the following result on holomorphic calculus for closed operators that follows from [10, Theorem IV.3.15].

Lemma 3.1. *Let $A$, $A_n \in \mathcal{C}(X)$ $(n = 1, 2, \dots)$. Let $G$ be an open set containing $\sigma(A)$ and $\sigma(A_n)$ $(n = 1, 2, \dots)$ and $\Omega$ an open neighbourhood of $\overline{G}$ with $\Omega^c$ compact. If $f$ is a function holomorphic on $\Omega \cup \{\infty\}$, then*

$$A_n \xrightarrow{\mathcal{G}} A \Rightarrow \|f(A_n) - f(A)\| \to 0.$$

For the rest of the section we assume that $A,\, B \in \mathcal{C}(X)$ are Drazin invertible. Then

$$\begin{aligned} B^{\mathrm{D}} - A^{\mathrm{D}} &= (B+B^\pi)^{-1}(I-B^\pi) - (A+A^\pi)^{-1}(I-B^\pi) \\ &\quad + (A+A^\pi)^{-1}(I-B^\pi) - (A+A^\pi)^{-1}(I-A^\pi) \\ &= \left[(B+B^\pi)^{-1} - (A+A^\pi)^{-1}\right](I-B^\pi) \\ &\quad + (A+A^\pi)^{-1}(A^\pi - B^\pi). \end{aligned} \tag{3.2}$$

In order to obtain an error bound for $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|$ we estimate

$$\|(B+B^\pi)^{-1} - (A+A^\pi)^{-1}\| \text{ and } \|I - B^\pi\|$$

in terms of the quantities $\operatorname{gap}(A,B)$ and $\|B^\pi - A^\pi\|$.

First we derive an upper estimate for $\|(B+B^\pi)^{-1} - (A+A^\pi)^{-1}\|$ writing

$$\alpha = 1 + \|(A+A^\pi)^{-1}\|^2, \quad \beta = 1 + \|A^\pi\|^2 \tag{3.3}$$

and assuming that

$$\operatorname{gap}(A+A^\pi, B+B^\pi) < \alpha^{-1/2}. \tag{3.4}$$

By [10, Theorem IV.2.20],

$$\operatorname{gap}((A+A^\pi)^{-1}, (B+B^\pi)^{-1}) = \operatorname{gap}(A+A^\pi, B+B^\pi).$$

When we apply [10, Theorem IV.4.13], we obtain the inequality

$$\|(B+B^\pi)^{-1} - (A+A^\pi)^{-1}\| \le \frac{\alpha \operatorname{gap}(A+A^\pi, B+B^\pi)}{1 - \alpha^{1/2}\operatorname{gap}(A+A^\pi, B+B^\pi)}. \tag{3.5}$$

We estimate $\operatorname{gap}(A+A^\pi, B+B^\pi)$ with the help of [10, Theorem IV.2.17]:

$$\begin{aligned} \operatorname{gap}(A+A^\pi, B+B^\pi) &= \operatorname{gap}(A+A^\pi, (B+B^\pi-A^\pi)+A^\pi) \\ &\le 2(1+\|A^\pi\|^2)\operatorname{gap}(A, B+B^\pi-A^\pi). \end{aligned}$$

To estimate $\operatorname{gap}(A, B + B^\pi - A^\pi)$ we use Lemma 2.1 (iv) and [10, Theorem IV.2.14]:

$$\begin{aligned}\operatorname{gap}(A, B + B^\pi - A^\pi) &\le 2(\operatorname{gap}(A, B) + \operatorname{gap}(B, B + B^\pi - A^\pi)) \\ &\le 2(\operatorname{gap}(A, B) + \|B^\pi - A^\pi\|).\end{aligned}$$

Then

$$\operatorname{gap}(A + A^\pi, B + B^\pi) \le 4(1 + \|A^\pi\|^2)(\operatorname{gap}(A, B) + \|B^\pi - A^\pi\|).$$

If, in addition, we assume that

$$\operatorname{gap}(A, B) + \|B^\pi - A^\pi\| < (4\alpha^{1/2}\beta)^{-1},$$

then inequality (3.4) is satisfied, and

$$\|(B + B^\pi)^{-1} - (A + A^\pi)^{-1}\| \le \frac{4\alpha\beta\,(\operatorname{gap}(A, B) + \|B^\pi - A^\pi\|)}{1 - 4\alpha^{1/2}\beta\,(\operatorname{gap}(A, B) + \|B^\pi - A^\pi\|)} \tag{3.6}$$

since the function $\varphi(t) = \alpha t/(1 - \alpha^{1/2} t)$ is increasing for $0 \le t < \alpha^{-1/2}$.

The estimate for $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|$ based on (3.2) is obtained when we observe that

$$\|I - B^\pi\| \le \|I - A^\pi\| + \|B^\pi - A^\pi\|.$$

We can then summarize our results in the following theorem.

Theorem 3.2. *Let $A$, $B \in \mathcal{C}(X)$ be Drazin invertible. If*

$$\operatorname{gap}(A, B) + \|B^\pi - A^\pi\| < (4\alpha^{1/2}\beta)^{-1}, \tag{3.7}$$

*where $\alpha$ and $\beta$ are defined by* (3.3), *then*

$$\begin{aligned}\|B^{\mathrm{D}} - A^{\mathrm{D}}\| \le{}& \frac{4\alpha\beta\,(\operatorname{gap}(A, B) + \|B^\pi - A^\pi\|)}{1 - 4\alpha^{1/2}\beta\,(\operatorname{gap}(A, B) + \|B^\pi - A^\pi\|)} \times \\ &\times (\|I - A^\pi\| + \|B^\pi - A^\pi\|) + \|(A + A^\pi)^{-1}\|\|B^\pi - A^\pi\|.\end{aligned} \tag{3.8}$$

# 4 The continuity of the Drazin inverse

Theorem 3.2 will enable us to obtain the following result on the continuity of the Drazin inverse of closed linear operators.

THEOREM 4.1. *Let $A$, $A_n \in \mathcal{C}(X)$ be Drazin invertible operators ($n = 1, 2, \dots$) such that $A_n \xrightarrow{\mathcal{G}} A$. Then the following conditions are equivalent.*

(i) $\|A_n^{\mathrm{D}} - A^{\mathrm{D}}\| \to 0$.

(ii) $\sup_n \|A_n^{\mathrm{D}}\| < \infty$.

(iii) *There exists $r > 0$ such that $0 < |\lambda| < r$ implies $\lambda \in \rho(A_n)$ for all $n$.*

(iv) $\|A_n A_n^{\mathrm{D}} - AA^{\mathrm{D}}\| \to 0$.

(v) $\|A_n^{\pi} - A^{\pi}\| \to 0$.

*Proof.* The implication (i) ⇒ (ii) is clear.

(ii) ⇒ (iii) Suppose that $\|A_n^{\mathrm{D}}\| \le M$ for all $n$ and some $M > 0$, and let $r = M^{-1}$. In accordance with Lemma 1.2 we write $A_n^{\mathrm{D}} = f_n(A_n)$, where $f_n(\lambda) = 0$ in a neighbourhood $\Delta_n$ of 0 and $f(\lambda) = \lambda^{-1}$ in a neighbourhood $\Omega_n$ of $\sigma(A_n) \setminus \{0\}$, where $\Delta_n \cap \Omega_n = \emptyset$. If $\lambda \in \sigma(A_n) \setminus \{0\}$, then

$$|\lambda^{-1}| = |f_n(\lambda)| \le \|f_n(A_n)\| = \|A_n^{\mathrm{D}}\| \le M,$$

which gives $|\lambda| \ge r$. This implies (iii).

(iii) ⇒ (v) Condition (iii) and the quasipolarity of $A$ imply that we can choose disjoint open sets $\Delta$ and $\Omega$ such that $0 \in \Delta$, $\Omega$ contains the nonzero spectrum of $A$ and of all the $A_n$, and $\Omega^c$ is compact. Let $f$ be defined on $\Delta \cup \Omega$ by setting $f(\lambda) = 1$ on $\Delta$ and $f(\lambda) = 0$ on $\Omega$. According to Lemma 3.1, $\|f(A_n) - f(A)\| \to 0$, that is,

$$\|A_n^{\pi} - A^{\pi}\| \to 0.$$

(iv) and (v) are equivalent since $T^{\pi} = I - TT^{\mathrm{D}}$ for any Drazin invertible operator $T \in \mathcal{C}(X)$.

Finally, the implication (v) $\Rightarrow$ (i) follows from Theorem 3.2. $\square$

# 5 An error bound for $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|$ using an upper estimate for $\|B^\pi - A^\pi\|$ and $\|(A + A^\pi)^{-1}\|$

If $\Delta$ is an upper estimate for $\|B^\pi - A^\pi\|$ and $\operatorname{gap}(A, B) + \Delta < (4\sqrt{\alpha}\beta)^{-1}$, where $\alpha$, $\beta$ are defined by (3.3), then according to Theorem 3.2,

$$\begin{aligned}\|B^{\mathrm{D}} - A^{\mathrm{D}}\| \le{}& \frac{4\alpha\beta\,(\operatorname{gap}(A,B)+\Delta)}{1-4\sqrt{\alpha}\beta\,(\operatorname{gap}(A,B)+\Delta)}(\|I - A^\pi\| + \Delta)\\ &+ \|(A + A^\pi)^{-1}\|\Delta.\end{aligned} \tag{5.1}$$

In this section we find a value of $\Delta$ satisfying (5.1) that can be calculated from the gaps between spaces associated with the operators $A$, $B$.

The *spectral projection* $A^\pi$ of a Drazin invertible operator $A \in \mathcal{C}(X)$ is the bounded linear operator with $\mathcal{R}(A^\pi) = \mathcal{H}(A)$ and $\mathcal{N}(A^\pi) = \mathcal{K}(A)$, where the spaces $\mathcal{H}(A)$ and $\mathcal{K}(A)$ are defined by

$$\begin{aligned}\mathcal{H}(A) &= \{x \in \mathcal{D}_\infty(A) : \limsup_{n\to\infty} \|A^n x\|^{1/n} = 0\},\\ \mathcal{K}(A) &= \{x \in X : \exists x_n \in \mathcal{D}_n(A) \text{ such that}\\ &\qquad Ax_1 = x,\ Ax_{n+1} = x_n \text{ for } n = 1, 2, \dots\\ &\qquad \text{and } \limsup_{n\to\infty} \|x_n\|^{1/n} < \infty\}\end{aligned}$$

(see Mbekhta [16]). It is known [13] that $A$ is Drazin invertible if and only if $X$ is the direct sum $X = \mathcal{H}(A) \oplus \mathcal{K}(A)$ with at least one of the component spaces closed. If 0 is a pole of $A$ of order $p$, then

$$\mathcal{K}(A) = \mathcal{N}(A^p), \qquad \mathcal{H}(A) = \mathcal{R}(A^p).$$

Theorem 5.1. *Let $A$, $B \in \mathcal{C}(X)$ be Drazin invertible operators and let*

$$\Delta_H = \operatorname{gap}(\mathcal{H}(A), \mathcal{H}(B)), \qquad \Delta_K = \operatorname{gap}(\mathcal{K}(A), \mathcal{K}(B)). \tag{5.2}$$

*If $\|I - A^\pi\|\Delta_H + \|A^\pi\|\Delta_K < 1$ , then (5.1) holds with $\Delta$ defined by*

$$\Delta = \frac{\|I - A^\pi\|\|A^\pi\|(\Delta_H + \Delta_K)}{1 - \|I - A^\pi\|\Delta_H - \|A^\pi\|\Delta_K} \tag{5.3}$$

*provided* $\operatorname{gap}(A,B)+\Delta<(4\sqrt{\alpha}\beta)^{-1}$.

*Proof.* Follows from Theorem 3.2 and Proposition 2.2. $\square$

Next we find an upper estimate for the quantity $\|(A+A^\pi)^{-1}\|$. We observe that from $(A+A^\pi)A^\pi = A^\pi(I+AA^\pi)$ it follows that

$$(A+A^\pi)^{-1}A^\pi = (I+AA^\pi)^{-1}A^\pi.$$

Since the operator $AA^\pi$ is bounded and quasinilpotent, we have

$$\begin{aligned}(A+A^\pi)^{-1}A^\pi &= (I+AA^\pi)^{-1}A^\pi = \left(\sum_{i=0}^{\infty}(-1)^i(AA^\pi)^i\right)A^\pi\\ &= \left(I+\sum_{i=1}^{\infty}(-1)^iA^iA^\pi\right)A^\pi.\end{aligned}$$

Consequently,

$$\begin{aligned}\|(A+A^\pi)^{-1}\| &\le \|(A+A^\pi)^{-1}(I-A^\pi)\| + \|(A+A^\pi)^{-1}A^\pi\|\\ &\le \|A^{\mathrm{D}}\| + \|A^\pi\|\bigl(1+\bigl\|\sum_{i=1}^{\infty}(-1)^iA^iA^\pi\bigr\|\bigr),\end{aligned}$$

which gives

$$\|(A+A^\pi)^{-1}\| \le \|A^{\mathrm{D}}\| + \|A^\pi\|\bigl(1+\bigl\|\sum_{i=1}^{\infty}(-1)^iA^iA^\pi\bigr\|\bigr) =: \Theta. \tag{5.4}$$

Theorem 5.2. *Let $A$, $B\in\mathcal{C}(X)$ be Drazin invertible operators and let $\Delta$ be an upper estimate for $\|B^\pi-A^\pi\|$ such that* $\operatorname{gap}(A,B)+\Delta<(4\sqrt{\alpha}\beta)^{-1}$, *where $\alpha$, $\beta$ are defined by* (3.3). *Then*

$$\|B^{\mathrm{D}}-A^{\mathrm{D}}\| \le \frac{4\alpha\beta\,(\operatorname{gap}(A,B)+\Delta)}{1-4\sqrt{\alpha}\beta\,(\operatorname{gap}(A,B)+\Delta)}(\|I-A^\pi\|+\Delta)+\Theta\Delta, \tag{5.5}$$

*where $\Theta$ is given by* (5.4).

## 6 The case of bounded operators

In this section we assume that the operators in question are in $\mathcal{B}(X)$. In this case we are able to use the operator norm rather than the operator gap to deduce the following explicit error estimate. Specializing the theorems of this section to matrices, we recover recent results of the second author [12].

THEOREM 6.1. *Let $A$, $B \in \mathcal{B}(X)$ be Drazin invertible operators. If*

$$(\|B - A\| + \|B^\pi - A^\pi\|)\|(A + A^\pi)^{-1}\| < 1, \tag{6.1}$$

*then*

$$\|B^{\mathrm{D}} - A^{\mathrm{D}}\| \le \frac{\|(A + A^\pi)^{-1}\|^2(\|B - A\| + \|B^\pi - A^\pi\|)}{1 - \|(A + A^\pi)^{-1}\|(\|B - A\| + \|B^\pi - A^\pi\|)} \times$$
$$\times (\|I - A^\pi\| + \|B^\pi - A^\pi\|) + \|(A + A^\pi)^{-1}\|\|B^\pi - A^\pi\|. \tag{6.2}$$

*If $\|I - A^\pi\|\Delta_H + \|A^\pi\|\Delta_K < 1$ and $\Theta(\|B - A\| + \Delta) < 1$, then*

$$\|B^{\mathrm{D}} - A^{\mathrm{D}}\| \le \frac{\Theta^2(\|B - A\| + \Delta)}{1 - \Theta(\|B - A\| + \Delta)}(\|I - A^\pi\| + \Delta) + \Theta\Delta, \tag{6.3}$$

*where $\Delta$ and $\Theta$ are defined by* (5.3) *and* (5.4).

*Proof.* According to (3.2),

$$B^{\mathrm{D}} - A^{\mathrm{D}} = [(B + B^\pi)^{-1} - (A + A^\pi)^{-1}](I - B^\pi) + (A + A^\pi)^{-1}(A^\pi - B^\pi).$$

The standard error estimate for the perturbation of the ordinary inverse of operators $T$, $S \in \mathcal{B}(X)$ gives

$$\|S^{-1} - T^{-1}\| \le \frac{\|T^{-1}\|^2\|S - T\|}{1 - \|T^{-1}\|\|S - T\|}$$

under the assumption $\|T^{-1}\|\|S - T\| < 1$. Setting $T = A + A^\pi$, $S = B + B^\pi$ we obtain (6.2). □

Finally, we apply the results obtained in this section to operators on Hilbert spaces. If $A \in \mathcal{B}(H)$, then

$$\|A^\pi\| = \|I - A^\pi\| \le \kappa_{\mathrm{D}}(A), \tag{6.4}$$

where $\kappa_{\mathrm{D}}(A) = \|A\|\|A^\pi\|$ is the Drazin condition number. The estimate (5.4) yields

$$\|(A+A^\pi)^{-1}\| \le \|A^{\mathrm{D}}\| + \kappa_{\mathrm{D}}(A)\big(1 + \|\sum_{i=1}^{\infty}(-1)^i A^i A^\pi\|\big) =: \Theta_1. \tag{6.5}$$

Suppose that $\kappa_{\mathrm{D}}(A)(\Delta_H + \Delta_K) < 1$. By (2.3),

$$\|B^\pi - A^\pi\| \le \frac{\kappa_{\mathrm{D}}(A)^2(\Delta_H + \Delta_K)}{1 - \kappa_{\mathrm{D}}(A)(\Delta_H + \Delta_K)} =: \Delta_1. \tag{6.6}$$

Then (6.2) combined with Theorem 3.2 yields the following result.

THEOREM 6.2. *Let* $\Theta_1$ *and* $\Delta_1$ *be defined by* (6.5) *and* (6.6), *and let* $A, B \in \mathcal{B}(H)$ *be operators such that* $\Theta_1(\|B-A\|+\Delta_1) < 1$ *and* $\kappa_{\mathrm{D}}(A)(\Delta_H + \Delta_K) < 1$. *Then*

$$\|B^{\mathrm{D}} - A^{\mathrm{D}}\| \le \frac{\Theta_1^2(\|B-A\| + \Delta_1)}{1 - \Theta_1(\|B-A\| + \Delta_1)}(\kappa_{\mathrm{D}}(A) + \Delta_1) + \Theta_1\Delta_1. \tag{6.7}$$

## 7 Applications to operator semigroups

In this section we give a representation of the Drazin inverse of the infinitesimal generator of a $C_0$-semigroup with a special asymptotic behaviour.

THEOREM 7.1. *Let* $T(t)$ *be a bounded* $C_0$*-semigroup with the infinitesimal generator* $A$ *and let* $P \in \mathcal{B}(X)$ *be a nonzero idempotent operator such that*

(a) $T(t)P = PT(t)$ *for all* $t \ge 0$;

(b) $\mathcal{R}(P) \subset \mathcal{D}(A)$;

(c) $\|T(t)(I-P)\| \to 0$ *as* $t \to \infty$;

(d) $\sigma(AP) = \{0\}$.

*Then* $A$ *is Drazin invertible with* $A^\pi = P$, *there are positive constants* $M$, $\mu$ *such that*

$$\|T(t)(I-P)\| \le Me^{-\mu t} \quad \textit{for all } t \ge 0, \tag{7.1}$$

*and*

$$A^{\mathrm{D}} = -\int_0^\infty T(t)(I-P)\,\mathrm{d}t. \tag{7.2}$$

*Proof.* We observe that $APx = PAx$ for all $x \in \mathcal{D}(A)$. Write

$$S(t) = T(t)\exp(-tP), \quad t \geq 0.$$

A direct verification shows that $S(t)$ is a $C_0$-semigroup whose generator $C$ is calculated from

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\bigg|_0 S(t)x = \frac{\mathrm{d}^+}{\mathrm{d}t}\bigg|_0 T(t)\,\exp(0P)x + T(0)\,\frac{\mathrm{d}^+}{\mathrm{d}t}\bigg|_0 \exp(-tP)x = Ax - Px$$

for all $x \in \mathcal{D}(A)$; hence, $C = A - P$. Observing that $\exp(-tP) = I - P + e^{-t}P$, we obtain that

$$\|S(t)\| \leq \|T(t)(I-P)\| + \|T(t)\|\|P\|e^{-t} \to 0 \text{ as } t \to \infty.$$

Applying results from [17, Chapter 3], we conclude that there exist positive constants $K$, $\mu$ such that

$$\|S(t)\| \leq Ke^{-\mu t}, \quad t \geq 0.$$

In addition, the spectrum of the generator $C$ of $S(t)$ is contained in the open left half plane, which means that $C$ is invertible.

To prove that $A$ is Drazin invertible with the spectral projection $P$, we need to show that 0 is an isolated singularity of the resolvent $R(\lambda, A)$ with $P$ the residue of the resolvent at 0. For each $x \in \mathcal{D}(A)$,

$$(\lambda I - A)x = (\lambda I - AP)Px + (\lambda I - C)(I-P)x.$$

There exists $\rho > 0$ such that $\lambda I - C$ is invertible if $|\lambda| < \rho$. Then, for each $\lambda$ satisfying $0 < |\lambda| < \rho$,

$$R(\lambda, A) = (\lambda I - AP)^{-1}P + (\lambda I - C)^{-1}(I-P).$$

Integrating the resolvent along a sufficiently small circular loop $\omega$ centred at 0, we get

$$\begin{aligned} A^\pi &= \frac{1}{2\pi i}\int_\omega R(\lambda, A)\,\mathrm{d}\lambda \\ &= \frac{1}{2\pi i}\int_\omega \sum_{n=0}^{\infty} \lambda^{-n-1}A^n P\,\mathrm{d}\lambda + \frac{1}{2\pi i}\int_\omega R(\lambda, C)(I-P)\,\mathrm{d}\lambda \\ &= \Big(\frac{1}{2\pi i}\int_\omega \frac{\mathrm{d}\lambda}{\lambda}\Big)P = P, \end{aligned}$$

which completes the argument.

Since $C$ is an invertible generator of a $C_0$-semigroup satisfying $\|S(t)\| \le Ke^{-\mu t}$, we have

$$\Big(\int_0^\infty S(t)\,\mathrm{d}t\Big)Cx = C\Big(\int_0^\infty S(t)\,\mathrm{d}t\Big)x = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}S(t)x\,\mathrm{d}t = -x$$

for all $x \in \mathcal{D}(A)$; hence

$$C^{-1} = -\int_0^\infty S(t)\,\mathrm{d}t.$$

Inequality (7.1) holds with $M = K\|I-P\|$ when we use the equation $T(t)(I-P) = S(T)(I-P)$. Hence $T(t)(I-P)$ is integrable over $[0,\infty)$ with

$$\int_0^\infty T(t)(I-P)\,\mathrm{d}t = \int_0^\infty S(t)(I-P)\,\mathrm{d}t = -(A-P)^{-1}(I-P) = -A^{\mathrm{D}}.$$

This completes the proof. □

A special case of the preceding theorem occurs when $T(t)$ converges in the operator norm as $t \to \infty$; in this case $P$ is the limit of $T(t)$.

## References

[1] S. L. Campbell, Continuity of the Drazin inverse, *Linear and Multilinear Algebra* **8** (1980), 265–268.

[2] S. L. Campbell and C. D. Meyer, Continuity properties of the Drazin pseudoinverse, *Linear Algebra Appl.* **10** (1975), 77–83.

[3] S. L. Campbell and C. D. Meyer, *Generalized Inverses of Linear Transformations*, Pitman, London, 1979.

[4] N. Castro González and J. J. Koliha, Perturbation of the Drazin inverse for closed linear operators, *Integral Equations Operator Theory* **36** (2000), 92–106.

[5] N. Castro González, J. J. Koliha and I. Straškraba, Perturbation of the Drazin inverse, *Soochow J. Math.* **27** (2001), 201–211.

[6] N. Castro González, J. J. Koliha and Yimin Wei, Perturbation of the Drazin inverse for matrices with equal eigenprojections, *Linear Algebra Appl.* **312** (2000), 181–189.

[7] N. Castro González, J. J. Koliha and Yimin Wei, Error bounds for the perturbation of the Drazin inverse of closed operators with equal spectral projections, *Applicable Analysis*, to appear.

[8] M. P. Drazin, Pseudo-inverse in associative rings and semigroups, *Amer. Math. Monthly* **65** (1958), 506–514.

[9] I. C. Gohberg and M. G. Krein, The basic propositions on defect numbers, root numbers and indices of linear operators, *Uspehi Mat. Nauk* **12** (1960), 43–118; *Amer. Math. Soc. Translations Series 2* **13** (1960), 185–264.

[10] T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed., Springer, Berlin, 1980.

[11] J. J. Koliha, A generalized Drazin inverse, *Glasgow Math. J.* **38** (1996), 367–381.

[12] J. J. Koliha, Error bounds for a general perturbation of the Drazin inverse, *Appl. Math. Comput.* **126** (2002), 181–185.

[13] J. J. Koliha and P. W. Poon, Spectral sets II, *Rend. Circ. Mat.* (*Palermo*) **47** (1998), 293–310.

[14] J. J. Koliha and V. Rakočević, Continuity of the Drazin inverse II, *Studia Math.* **131** (1998), 167–177.

[15] J. J. Koliha and T. D. Tran, The Drazin inverse for closed linear operators and the asymptotic behaviour of $C_0$-semigroups, *J. Operator Theory* **46** (2001), 323–336.

[16] M. Mbekhta, Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux, *Glasgow Math. J.* **29** (1987), 159–175.

[17] J. van Neerven, *Asymptotic Behaviour of Operator Semigroups*, Birkhäuser, Basel, 1996.

[18] M. I. Ostrovskii, Topologies on the set of all subspace of a Banach space and related questions of Banach space geometry, *Quaest. Math.* **17** (1994), 259–319.

[19] V. I. Paraska, On a certain metric in the set of closed operators and its application in the theory of perturbations, *Izv. Akad. Nauk Moldav. SSSR* **6** (1964), 91–96.

[20] V. Rakočević, Continuity of the Drazin inverse, *J. Operator Theory* **41** (1999), 55–68.

[21] V. Rakočević, On the norm of idempotent operators in a Hilbert space, *Amer. Math. Monthly* **107** (2000), 748–750.

[22] V. Rakočević and Y. Wei, The perturbation theory for the Drazin inverse and its applications, *J. Aust. Math. Soc.* **70** (2001), 189–197.

[23] G. Rong, The error bounds for the perturbation of the Drazin inverse, *Linear Algebra Appl.* **47** (1982). 159–168.

[24] A. E. Taylor and D. C. Lay, *Introduction to Functional Analysis*, 2nd ed., Wiley, New York, 1980.

[25] Y. Wei and G. Wang, The perturbation theory for the Drazin inverse and its applications, *Linear Algebra Appl.* **258** (1997), 179–186.

N. Castro González
Departmento de Matemática Aplicada
Facultad de Informática
Universidad Politécnica de Madrid
Boadilla del Monte
28660 Madrid
SPAIN
nieves@fi.upm.es

J. J. Koliha
Department of Mathematics and Statistics
The University of Melbourne
VIC 3010
AUSTRALIA
j.koliha@ms.unimelb.edu.au

Vladimir Rakočević
Department of Mathematics
Faculty of Philosophy
University of Niš
1800 Niš
YUGOSLAVIA-SERBIA
vrakoc@bankerinter.net