INTEGRAL REPRESENTATION OF THE DRAZIN INVERSE *

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Abstract. In this note we present an integral representation for the Drazin inverse A^{D} of a complex square matrix A which does not require any restriction on its eigenvalues.

Key words. Drazin inverse, Integral representation

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1. Introduction. It is a well-known fact that if the eigenvalues of $A \in \mathbb{C}^{n\times n}$ lie in the open right halfplane, then the inverse of A can be represented by

$$A^{-1} = \int_0^\infty \exp(-tA)\,dt. \tag{1.1}$$

This representation was extended to the Drazin inverse by Koliha and Straškraba (see [2, Theorem 6.3]) in the form

$$A^{\mathrm{D}} = \int_0^\infty \exp(-tA)(I - A^\pi)\,dt \tag{1.2}$$

for those nonsingular matrices whose nonzero eigenvalues lie in the open right halfplane and for which $\mathrm{ind}(A) = 1$; here A^π is the eigenprojection of A corresponding to the eigenvalue 0. (Recall that $\mathrm{ind}(A)$, the index of A, is the least nonnegative k for which the nullspace of of A^k coincides with the nullspace of A^{k+1}.)

Recently, Castro, Koliha and Wei [1, Corollary 2.5] obtained a simple integral representation of the Drazin inverse A^{D} for matrices $A \in \mathbb{C}^{n\times n}$ (and more generally elements of a Banach algebra) for which the nonzero eigenvalues of A^{m+1} lie in the open right halfplane for some $m \geq \mathrm{ind}(A)$:

$$A^{\mathrm{D}} = \int_0^\infty \exp(-tA^{m+1})A^m\,dt. \tag{1.3}$$

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It is natural to ask whether we can drop the restriction on the spectrum of A^{m+1}. In this note we will establish an integral representation for the Drazin inverse A^{D} which holds without any restriction on the eigenvalues of A.

2. Integral representation for the Drazin inverse A^{D}. We mention that for the Moore–Penrose inverse $A^{\dagger}$ of a matrix $A \in \mathbb{C}^{n\times n}$ (and more generally of a bounded Hilbert space operator A with closed range) there is a well known integral representation due to Showalter [3],

$$A^{\dagger} = \int_0^{\infty} \exp(-tA^*A)A^*\,dt, \tag{2.1}$$

generalized recently by Wei and Wu to the weighted Moore–Penrose inverse [4].

Our main result which follows bears a certain resemblance to this representation.

THEOREM 2.1. *Suppose that* $A \in \mathbb{C}^{n\times n}$ *and* $k = \operatorname{ind}(A)$. *Then*

$$A^{\mathrm{D}} = \int_0^{\infty} \exp[-tA^k(A^{2k+1})^*A^{k+1}]A^k(A^{2k+1})^*A^k\,dt. \tag{2.2}$$

Proof. For each matrix $A \in \mathbb{C}^{n\times n}$ there exists a nonsingular matrix P such that

$$A = P\begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} P^{-1},$$

where C is a nonsingular matrix and N is a nilpotent matrix of index k; either block C or block N may be empty.

The Drazin inverse of A can be then expressed by

$$A^{\mathrm{D}} = P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

We partition the Hermitian matrices P^*P and $(P^*P)^{-1}$ into block matrices compatible with the above partitioning of A (and A^{D}):

$$P^*P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \qquad (P^*P)^{-1} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}.$$

Since P^*P and $(P^*P)^{-1}$ are positive definite Hermitian matrices, so are the submatrices P_{11} and Q_{11}. By a direct computation we obtain

$$\begin{aligned} A^k(A^{2k+1})^*A^k &= P\begin{bmatrix} C^k & 0 \\ 0 & 0 \end{bmatrix} P^{-1}P^{-*}\begin{bmatrix} (C^{2k+1})^* & 0 \\ 0 & 0 \end{bmatrix} P^*P\begin{bmatrix} C^k & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \\ &= P\begin{bmatrix} C^k & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}\begin{bmatrix} (C^{2k+1})^* & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} C^k & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \end{aligned}$$

$$= P \begin{bmatrix} C^k Q_{11}(C^{2k+1})^* P_{11} C^k & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

Similarly, we get

$$A^k (A^{2k+1})^* A^{k+1} = P \begin{bmatrix} C^k Q_{11}(C^{2k+1})^* P_{11} C^{k+1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

Write $\sigma(A)$ for the spectrum of A (that is, the set of all eigenvalues of A). Then

$$\begin{aligned} \sigma[C^k Q_{11}(C^{2k+1})^* P_{11} C^{k+1}] &= \sigma[Q_{11}(C^{2k+1})^* P_{11} C^{2k+1}] \\ &= \sigma[Q_{11}^{1/2}(C^{2k+1})^* P_{11}^{1/2} P_{11}^{1/2} C^{2k+1} Q_{11}^{1/2}] \\ &= \sigma[(P_{11}^{1/2} C^{2k+1} Q_{11}^{1/2})^* (P_{11}^{1/2} C^{2k+1} Q_{11}^{1/2})], \end{aligned}$$

where the last spectrum is positive being the spectrum of a positive definite Hermitian matrix. Thus

$$\begin{aligned} &\int_0^\infty \exp[-tA^k (A^{2k+1})^* A^{k+1}] A^k (A^{2k+1})^* A^k \, dt \\ &= P \begin{bmatrix} \int_0^\infty \exp[-tC^k Q_{11}(C^{2k+1})^* P_{11} C^{k+1}] \, dt & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C^k Q_{11}(C^{2k+1})^* P_{11} C^k & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \\ &= P \begin{bmatrix} [C^k Q_{11}(C^{2k+1})^* P_{11} C^{k+1}]^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C^k Q_{11}(C^{2k+1})^* P_{11} C^k & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \\ &= P \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \\ &= A^{\mathrm{D}}. \end{aligned}$$

This completes the proof. □

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