

Elements of C^* -algebras commuting with their Moore–Penrose inverse

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Abstract. We give new necessary and sufficient conditions for an element of a C^* -algebra to commute with its Moore–Penrose inverse. We then study conditions which ensure that this property is preserved under multiplication. As a special case of our results we recover a recent theorem of Hartwig and Katz on EP matrices.

1 Introduction

The novelty of our approach to the study of Moore–Penrose inverse in C^* -algebras is considering it in terms of the Drazin inverse. For elements of a C^* -algebra that commute with their Moore–Penrose inverse, the Moore–Penrose inverse in fact coincides with the Drazin inverse. Proofs found in the literature may resort to special constructions, often very ingenious. Many of these arguments can be now presented more systematically relying on standard properties of the Drazin inverse and on properties of spectral projections.

We retain the notation of [10]. In particular, \mathfrak{A} is a unital C^* -algebra with unit e ; \mathfrak{A}^{-1} , $QN(\mathfrak{A})$ and \mathfrak{A}^\dagger denote the sets of all invertible, quasinilpotent and regular elements of \mathfrak{A} , respectively. An element $a \in \mathfrak{A}$ is *quasipolar* if 0 is an isolated—possibly removable—singularity of the resolvent of a , and *polar* if 0 is at most a pole of the resolvent. By $\sigma(a)$ we denote the spectrum of $a \in \mathfrak{A}$.

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The set of all quasipolar elements of \mathfrak{A} will be denoted by \mathfrak{A}^D . Observe that $\mathfrak{A}^D \cap \mathfrak{A}^\dagger \supset \mathfrak{A}^{-1}$. We write $L(H)$ for the C^* -algebra of all bounded linear operators on a Hilbert space H .

PROPOSITION 1.1 [9, Theorem 4.2] *Let $a \in \mathfrak{A}$. Then the following condition are equivalent.*

- (i) $a \in \mathfrak{A}^D$.
- (ii) *There exists (a unique) $x \in \mathfrak{A}$ such that*

$$ax = xa, \quad xax = x, \quad axa = a + u, \quad u \in QN(\mathfrak{A}). \quad (1.1)$$

- (iii) *There exists (a unique) $p = p^2 \in \mathfrak{A}$ such that $ap = pa \in QN(\mathfrak{A})$ and $a + p \in \mathfrak{A}^{-1}$.*

An element x in (ii) is the *Drazin inverse* of a , written $x = a^D$. (The original definition of this inverse [4] required that u be nilpotent.) The idempotent p satisfying (iii) is the *spectral idempotent* of $a \in \mathfrak{A}^D$ at 0, written $p = a^\pi$. We recall that a is polar if and only if $a^k a^\pi = 0$ for some nonnegative integer k ; if $aa^\pi = 0$, a is *simply polar*. It is known [9] that

$$a^\pi = e - a^D a \quad \text{and} \quad a^D = (a + a^\pi)^{-1}(e - a^\pi). \quad (1.2)$$

We observe that if a is quasipolar, then so is a^* , and $(a^*)^\pi = (a^\pi)^*$.

PROPOSITION 1.2 [5, Theorem 6], [10, Theorem 2.8] *Let $a \in \mathfrak{A}$. Then $a \in \mathfrak{A}^\dagger$ if and only if there exists (a unique) element $x \in \mathfrak{A}$ satisfying the equations*

$$xax = x, \quad axa = a, \quad (ax)^* = ax, \quad (xa)^* = xa; \quad (1.3)$$

x is called the *Moore–Penrose inverse* of a , written $x = a^\dagger$. (The original definition of this inverse was given in [13] for matrices.)

The two inverses are related by the following result.

PROPOSITION 1.3 [10, Theorem 2.5] *Let $a \in \mathfrak{A}$. Then $a \in \mathfrak{A}^\dagger$ if and only if a^*a (respectively aa^*) is simply polar, in which case*

$$a^\dagger = (a^*a)^D a^* = a^*(aa^*)^D. \quad (1.4)$$

If a is regular, then so is a^ , and $(a^*)^\dagger = (a^\dagger)^*$.*

For future use we need the following two lemmas and some notation. Following [6], for any $a \in \mathfrak{A}$ we define the nullspace ideals

$$a^{-1}(0) = \{x \in \mathfrak{A} : ax = 0\}, \quad a_{-1}(0) = \{x \in \mathfrak{A} : xa = 0\}.$$

LEMMA 1.4 (See also [5, Theorem 9]) *Let $a \in \mathfrak{A}$ be simply polar with the spectral idempotent a^π at 0. Then*

$$\begin{aligned} a^{-1}(0) &= a^\pi \mathfrak{A}, & a\mathfrak{A} &= (a^\pi)^{-1}(0), \\ a_{-1}(0) &= \mathfrak{A}a^\pi, & \mathfrak{A}a &= a_{-1}^\pi(0), \end{aligned}$$

and $\mathfrak{A} = a^{-1}(0) \oplus a\mathfrak{A} = a_{-1}(0) \oplus \mathfrak{A}a$ with $a\mathfrak{A}$ and $\mathfrak{A}a$ closed.

Proof. We prove only the results for $a^{-1}(0)$ and $a\mathfrak{A}$, the rest follows by symmetry.

Let $ax = 0$. Then $a^\pi x = (e - a^D a)x = x$, and $a^{-1}(0) \subset a^\pi \mathfrak{A}$. If $x = a^\pi x$, then $ax = aa^\pi x = 0$, and $a^\pi \mathfrak{A} \subset a^{-1}(0)$.

Let $x = au$ for some $u \in \mathfrak{A}$. Then $a^\pi x = a^\pi au = 0$, and $a\mathfrak{A} \subset (a^\pi)^{-1}(0)$. Let $a^\pi x = 0$. Then $(e - aa^D)x = 0$, and $x = aa^D x$; hence $(a^\pi)^{-1}(0) \subset a\mathfrak{A}$.

Since a^π is an idempotent, $\mathfrak{A} = a^\pi \mathfrak{A} \oplus (a^\pi)^{-1}(0)$.

LEMMA 1.5 *Let $a \in \mathfrak{A}^\dagger$. Then*

$$a^\dagger = (a^*a + (a^*a)^\pi)^{-1} a^* = a^*(aa^* + (aa^*)^\pi)^{-1}, \quad (1.5)$$

$$a^* \mathfrak{A}^{-1} = a^\dagger \mathfrak{A}^{-1} \quad \text{and} \quad \mathfrak{A}^{-1} a^* = \mathfrak{A}^{-1} a^\dagger, \quad (1.6)$$

$$a^{*-1}(0) = (a^\dagger)^{-1}(0) \quad \text{and} \quad a_{-1}^*(0) = a_{-1}^\dagger(0). \quad (1.7)$$

Proof. First we observe that, for any $a \in \mathfrak{A}^\dagger$, $a(a^*a)^\pi = 0$ and $(a^*a)^\pi a^* = 0$. The first equation follows from $a(a^*a)^\pi = a(e - a^\dagger a) = 0$, the second is obtained by taking adjoints. By (1.4) and (1.2),

$$a^\dagger = (a^*a)^D a^* = (a^*a + (a^*a)^\pi)^{-1} (e - (a^*a)^\pi) a^* = (a^*a + (a^*a)^\pi)^{-1} a^*.$$

The second part of (1.5) is proved similarly. This proves (1.6), and (1.7) follows.

Condition (1.6) appears in the proof of [6, Theorem 10] as equation (10.6).

2 Elements commuting with their Moore–Penrose inverse

In this paper we are concerned with the elements $a \in \mathfrak{A}^\dagger$ satisfying $a^\dagger a = aa^\dagger$. Matrices with this property are called EP_r or EP matrices [1, 7]. We give a new characterization—in terms of spectral idempotents—of the elements of a C^* -algebra which commute with their Moore–Penrose inverse.

THEOREM 2.1 *Let $a \in \mathfrak{A}$. Then $a^\dagger a = aa^\dagger$ if and only if a is simply polar with a selfadjoint spectral idempotent at 0. In this case*

$$a^\pi = (a^*)^\pi = (a^*a)^\pi = (aa^*)^\pi. \quad (2.1)$$

Proof. From Proposition 1.1 we deduce that if a is simply polar, then a^* is also simply polar with $(a^*)^\pi = (a^\pi)^*$. If $p = a^\pi$ is selfadjoint, then $(a^*a)p = p(a^*a) = 0$ and

$$a^*a + p = (a^* + p)(a + p) \in \mathfrak{A}^{-1}.$$

By Proposition 1.1 again, a^*a is simply polar and $(a^*a)^\pi = a^\pi$. Equality $(aa^*)^\pi = a^\pi$ follows by symmetry. By Proposition 1.3, (1.4) and the first part of (1.2), we have $a \in \mathfrak{A}^\dagger$ and

$$a^\dagger a = (a^*a)^D a^*a = e - (a^*a)^\pi = e - (aa^*)^\pi = aa^*(aa^*)^D = aa^\dagger.$$

Conversely, let $a \in \mathfrak{A}^\dagger$ and let $a^\dagger a = aa^\dagger$. Since $a^\dagger aa^\dagger = a^\dagger$ and $aa^\dagger a = a + u$ with $u = 0$, a^D exists and $a^D = a^\dagger$. Further, $aa^\pi = a(e - a^D a) = a(e - a^\dagger a) = 0$, and a is simply polar. Finally,

$$(a^\pi)^* = (e - a^D a)^* = (e - a^\dagger a)^* = e - (a^\dagger a)^* = e - a^\dagger a = a^\pi.$$

From the theorem and its proof we can deduce the following alternative necessary and sufficient conditions for a^\dagger to commute with a . The equivalence of (i) and (ii) is [10, Proposition 2.2].

COROLLARY 2.2 *Let $a \in \mathfrak{A}$. Then the following conditions are equivalent.*

- (i) $a \in \mathfrak{A}^\dagger$ and $a^\dagger a = aa^\dagger$;
- (ii) $a \in \mathfrak{A}^\dagger \cap \mathfrak{A}^D$ and $a^\dagger = a^D$;
- (iii) a is simply polar and $(a^*)^\pi = a^\pi$;
- (iv) a is simply polar and $a^\pi = (a^*a)^\pi$ (respectively $a^\pi = (aa^*)^\pi$);
- (v) $a \in \mathfrak{A}^\dagger$ and $(a^*a)^\pi = (aa^*)^\pi$.

We now show that the commuting Moore–Penrose inverse can be expressed in terms of the holomorphic calculus.

COROLLARY 2.3 *Let $a \in \mathfrak{A}^\dagger$. Then $a^\dagger a = aa^\dagger$ if and only if*

$$a^\dagger = f(a) \tag{2.2}$$

for some function f holomorphic in a neighbourhood of $\sigma(a)$.

Proof. Let $a^\dagger a = aa^\dagger$. By the preceding corollary, $a^\dagger = a^D$. According to [9, Theorem 4.4], $a^D = f(a)$, where f is holomorphic in a neighbourhood of $\sigma(a)$, and $f(\lambda) = 0$ in a neighbourhood of 0, $f(\lambda) = \lambda^{-1}$ in a neighbourhood of $\sigma(a) \setminus \{0\}$.

Conversely, if $a^\dagger = f(a)$ for some function holomorphic in a neighbourhood of $\sigma(a)$, then by a property of the holomorphic calculus, a^\dagger commutes with a .

Corollary 2.3 yields a result of Wong [15], who showed that, for an operator A on a finite dimensional space, $AA^\dagger = A^\dagger A$ if and only if $A^\dagger = f(A)$ for some polynomial f .

It is interesting to observe that if a^\dagger commutes with $a \in \mathfrak{A}$, then it double commutes with a , that is,

$$ax = xa \Rightarrow a^\dagger x = xa^\dagger, \quad x \in \mathfrak{A}.$$

This follows from the holomorphic calculus representation (2.2) of a^\dagger .

3 Further conditions

Previously Brock [2] characterized the bounded linear operators A on a Hilbert space satisfying $A^\dagger A = AA^\dagger$, and Harte and Mbekhta [6] generalized this characterization to C^* -algebras. We extend their results in the following theorem.

THEOREM 3.1 *If $a \in \mathfrak{A}^\dagger$, then the following are equivalent.*

- (i) $aa^\dagger = a^\dagger a$;
- (ii) $a^2 a^\dagger = a = a^\dagger a^2$;
- (iii) $(a^* a)^\pi a = 0 = a(aa^*)^\pi$;
- (iv) $a^{-1}(0) = a^{*-1}(0)$;
- (v) $a_{-1}(0) = a_{-1}^*(0)$;
- (vi) $a\mathfrak{A} = a^*\mathfrak{A}$;
- (vii) $\mathfrak{A}a = \mathfrak{A}a^*$;
- (viii) $a\mathfrak{A}^{-1} = a^*\mathfrak{A}^{-1}$;
- (ix) $\mathfrak{A}^{-1}a = \mathfrak{A}^{-1}a^*$;
- (x) $a \in a^\dagger\mathfrak{A} \cap \mathfrak{A}a^\dagger$;
- (xi) $a \in a^\dagger\mathfrak{A}^{-1} \cap \mathfrak{A}^{-1}a^\dagger$.

Proof. Observe that (i), (ii) and (iii) are equivalent. (Recall that $(a^*a)^\pi = e - a^\dagger a$ and $(aa^*)^\pi = e - aa^\dagger$.)

We prove that (i) implies (iv)–(xi). Suppose that (i) holds. According to Theorem 2.1, a is simply polar and the spectral idempotent a^π is selfadjoint. By Lemma 1.4,

$$\begin{aligned} a^{-1}(0) &= a^\pi \mathfrak{A} = (a^\pi)^* \mathfrak{A} = (a^*)^\pi \mathfrak{A} = a^{*-1}(0), \\ a\mathfrak{A} &= (a^\pi)^{-1}(0) = (a^\pi)^{*^{-1}}(0) = ((a^*)^\pi)^{-1}(0) = a^*\mathfrak{A}. \end{aligned}$$

Equations $a_{-1}(0) = a_{-1}^*(0)$ and $\mathfrak{A}a = \mathfrak{A}a^*$ follow by taking adjoints. This proves (iv), (v), (vi) and (vii). To prove (xi) we write $a = a^2 a^\dagger = (a + a^\pi)^2 a^\dagger$ and observe that $a + a^\pi \in \mathfrak{A}^{-1}$ by Proposition 1.1; similarly $a = a^\dagger (a + a^\pi)^2$. Condition (x) follows from (xi), and (viii) and (ix) follow from (xi) and (1.6).

Conversely, we show that any of the conditions (iv)–(xi) implies (ii). We note that (viii) and (ix) are equivalent (take adjoints), and together they imply (xi); (xi) in turn implies (x). Conditions (vi) and (vii) are equivalent (adjoints), and together they also imply (x). From (x) we deduce (ii): Indeed, if $a = ua^\dagger$, then $a - a^2 a^\dagger = u(a^\dagger - a^\dagger a a^\dagger) = 0$; if $a = a^\dagger v$, then $a - a^\dagger a^2 = (a^\dagger - a^\dagger a a^\dagger)v = 0$. Conditions (iv) and (v) are equivalent (take adjoints) and together they imply (ii): $e - aa^\dagger \in (a^\dagger)^{-1}(0) = (a^*)^{-1}(0) = a^{-1}(0)$ by (1.7) and (iv), and similarly $e - a^\dagger a \in a_{-1}(0)$ by (1.7) and (v).

We have recovered [6, Theorem 10] which gives the equivalence of (i), (iv), (v), (viii) and (ix) of the preceding theorem.

In [2], Brock proved the equivalence of the following conditions for operators on a Hilbert space. We write $N(A)$ for the nullspace of $A \in L(H)$, and $R(A)$ for the range of A .

COROLLARY 3.2 (Brock [2]) *Let $A \in L(H)$ be a closed range operator on a Hilbert space H . Then the following conditions are equivalent:*

- (i) $A^\dagger A = AA^\dagger$;
- (ii) $H = N(A) \oplus^\perp R(A)$;
- (iii) $N(A) = N(A^*)$;
- (iv) $A^* = PA$ for some $P \in L(H)^{-1}$.

Proof. Let $\mathfrak{A} = L(H)$, the full algebra of bounded linear operators on H . Brock's condition (i) \Leftrightarrow (ii) follows from Theorem 2.1 (A simply polar with a selfadjoint spectral projection). Equivalence (i) \Leftrightarrow (iv) follows from Theorem 3.1 (ix). To recover (i) \Leftrightarrow (iii), we observe that, for any pair of bounded linear operators A, B on H ,

$$[N(L_A) \subset N(L_B)] \Leftrightarrow [N(A) \subset N(B)] \quad (3.1)$$

(where L_T is the linear mapping $U \mapsto TU$ on $L(H)$), and apply Theorem 3.1 (iv).

COROLLARY 3.3 *Let $A \in L(H)$ be an upper semi-Fredholm operator on a Hilbert space H . Then the following conditions are equivalent:*

- (i) $A^\dagger A = AA^\dagger$;
- (ii) $(A^*A)^\pi A = 0$;
- (iii) $A(AA^*)^\pi = 0$;

Proof. Recall that an operator $A \in L(H)$ is upper semi-Fredholm if $R(A)$ is closed and $N(A)$ finite dimensional. Closed range operators on a Hilbert space are Moore–Penrose invertible.

If $A^\dagger A = AA^\dagger$, then (ii) and (iii) hold by Theorem 3.1 (iii).

Conversely, if (ii) (which is the first half of condition (iii) in Theorem 3.1) is satisfied, then $A^*(A^*A)^\pi = 0$, which means that

$$N(A) = N(A^*A) = R((A^*A)^\pi) \subset N(A^*).$$

Since $N(A)$ has a finite dimension equal to that of $N(A^*)$, we have $N(A) = N(A^*)$. This implies that $A^\dagger A = AA^\dagger$ by Corollary 3.2.

Finally, condition (iii) is condition (ii) with A^* in place of A , and hence, by the preceding argument, (iii) implies that $(A^*)^\dagger$ commutes with A^* , which then implies (i).

From this Corollary we recover the result of Marek and Žitný [12, p. 143], who proved the foregoing criterion under the assumption that A is a linear operator on a finite dimensional Hilbert space. Their result follows since linear operators on a finite dimensional space are Fredholm.

4 The product of elements commuting with their Moore–Penrose inverse

By $\mathfrak{A}_{\text{com}}^\dagger$ we denote the class of all elements $a \in \mathfrak{A}^\dagger$ such that $a^\dagger a = aa^\dagger$. It is well known that $\mathfrak{A}_{\text{com}}^\dagger$ is not closed under multiplication of elements in \mathfrak{A} ; this is evident even for finite matrices.

EXAMPLE 4.1 In the C^* -algebra $\mathfrak{A} = \mathbb{C}^{3 \times 3}$ let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad AB = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $A, B \in \mathfrak{A}_{\text{com}}^\dagger$ as $A^\dagger = A$ and $B^\dagger = B^{-1}$. But the matrices $(AB)^\dagger$, $(AB)(AB)^\dagger$ and $(AB)^\dagger(AB)$ are equal to

$$\frac{1}{10} \begin{bmatrix} 0 & 2 & 0 \\ 5 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{1}{5} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 4 \end{bmatrix},$$

respectively, and $(AB)(AB)^\dagger \neq (AB)^\dagger(AB)$.

Adding commutativity makes a difference. The following result is known for matrices [8].

THEOREM 4.2 *Suppose that $a, b \in \mathfrak{A}_{\text{com}}^\dagger$ and $ab = ba$. Then $ab \in \mathfrak{A}_{\text{com}}^\dagger$ and $(ab)^\dagger = a^\dagger b^\dagger = b^\dagger a^\dagger$.*

Proof. By Corollary 2.2, $a^\dagger = a^D$ and $b^\dagger = b^D$. The set $\{a, b, a^D, b^D\}$ is commutative. By [9, Theorem 5.5], ab is Drazin invertible with $(ab)^D = a^D b^D = b^D a^D$. We verify that ab is simply polar with a selfadjoint spectral idempotent.

We have

$$\begin{aligned} (ab)(ab)^\pi &= ab(e - (ab)(ab)^D) = ab(e - aa^D b b^D) \\ &= ab - (a^2 a^D)(b^2 b^D) \\ &= ab - ab = 0, \end{aligned}$$

which shows that ab is simply polar. By Theorem 2.1, the spectral idempotents $a^\pi = e - a^D a$ and $b^\pi = e - b^D b$ are selfadjoint; hence $a^D a$ and $b^D b$ are selfadjoint, and

$$((ab)(ab)^D)^* = (aa^D bb^D)^* = (bb^D)^*(aa^D)^* = bb^D aa^D = (ab)(ab)^D.$$

Therefore $(ab)^\pi = e - (ab)(ab)^D$ is selfadjoint, $ab \in \mathfrak{A}_{\text{com}}^\dagger$ by Theorem 2.1, and

$$(ab)^\dagger = (ab)^D = a^D b^D = a^\dagger b^\dagger.$$

Hartwig and Katz [7, Theorem 1] recently gave necessary and sufficient conditions for the product of two EP matrices to be an EP matrix. The following theorem, the main result of this section, generalizes their result to C^* -algebras.

THEOREM 4.3 *Let $a, b \in \mathfrak{A}_{\text{com}}^\dagger$ and let $a^{-1}(0)$, $b^{-1}(0)$ be finite dimensional vector subspaces of \mathfrak{A} . Then the following conditions are equivalent.*

- (i) $ab \in \mathfrak{A}_{\text{com}}^\dagger$;
- (ii) $(ab)a^\pi = 0$ and $b^\pi(ab) = 0$;
- (iii) $a^{-1}(0) \subset (ab)^{-1}(0)$ and $b_{-1}(0) \subset (ab)_{-1}(0)$;
- (iv) $(ab)^{-1}(0) = a^{-1}(0) + b^{-1}(0)$ and $(ab)_{-1}(0) = a_{-1}(0) + b_{-1}(0)$.

Proof. (i) \Rightarrow (ii) By Theorem 3.1 (viii), there is $c \in \mathfrak{A}$ such that $ab = c(b^* a^*)$. Since a^π is selfadjoint, $(ab)a^\pi = cb^* a^* a^\pi = cb^*(a^\pi a)^* = 0$. From $ab \in \mathfrak{A}_{\text{com}}^\dagger$ it follows that $b^* a^* \in \mathfrak{A}_{\text{com}}^\dagger$; hence $(b^* a^*)b^\pi = 0$ by the foregoing argument. Hence $b^\pi(ab) = 0$.

(ii) \Rightarrow (iii) Since $(ab)a^\pi = 0$ by assumption and since $a^\pi \mathfrak{A} = a^{-1}(0)$ by Lemma 1.4, we have $a^{-1}(0) \subset (ab)^{-1}(0)$. Similarly, $b^\pi(ab) = 0$ and $\mathfrak{A}b^\pi = b_{-1}(0)$ imply $b_{-1}(0) \subset (ab)_{-1}(0)$.

(iii) \Rightarrow (iv) We note that $b^{-1}(0) \subset (ab)^{-1}(0)$. Write

$$X = (ab)^{-1}(0), \quad U = a^{-1}(0), \quad V = b^{-1}(0), \quad Y = b\mathfrak{A}.$$

Define $f : X/V \rightarrow \mathfrak{A}$ by $f(x + V) = bx$. The map f is linear and injective with $f(X/V) = U \cap Y$. Therefore X/V and $U \cap Y$ are isomorphic as vector spaces. Since $U \cap Y$ and V are finite dimensional, so is X . By Lemma 1.4, $\mathfrak{A} = b\mathfrak{A} \oplus b^{-1}(0) = Y \oplus V$. Then

$$\begin{aligned} (U \cap Y) \oplus (U \cap V) &\subset U, \quad \text{and} \\ (U \cap Y) \oplus V &\subset U + V \subset X, \end{aligned}$$

taking into account the hypothesis $U = a^{-1}(0) \subset (ab)^{-1}(0) = X$. Then

$$\dim((U \cap Y) \oplus V) = \dim(U \cap Y) + \dim V = \dim(X/V) + \dim V = \dim X.$$

We conclude that $(U \cap Y) \oplus V = U + V = X$, that is, $(ab)^{-1}(0) = a^{-1}(0) + b^{-1}(0)$. We note that $b_{-1}(0) \subset (ab)_{-1}(0)$ implies $b^{*-1}(0) \subset (ab)^{*-1}(0)$. Applying the foregoing result to b^* , a^* , b^*a^* in place of a , b , ab , we obtain $(b^*a^*)^{-1}(0) = a^{*-1}(0) + b^{*-1}(0)$. This, in turn, implies $(ab)_{-1}(0) = a_{-1}(0) + b_{-1}(0)$.

(iv) \Rightarrow (i) By Theorem 3.1, $a^{-1}(0) = a^{*-1}(0)$ and $b^{-1}(0) = b^{*-1}(0)$. Further, $(ab)_{-1}(0) = a_{-1}(0) + b_{-1}(0)$ is equivalent to $(ab)^{*-1}(0) = a^{*-1}(0) + b^{*-1}(0)$. Hence

$$(ab)^{*-1}(0) = a^{-1}(0) + b^{-1}(0) = (ab)^{-1}(0),$$

and $ab \in \mathfrak{A}_{\text{com}}^\dagger$ follows on another application of Theorem 3.1.

The preceding theorem can be modified for the C^* -algebra $\mathfrak{A} = L(H)$ to obtain a result for bounded linear operators on a Hilbert space.

COROLLARY 4.4 *Let A, B be bounded linear upper semi-Fredholm operators on a Hilbert space H satisfying $A^\dagger A = AA^\dagger$ and $B^\dagger B = BB^\dagger$. Then the following conditions are equivalent.*

- (i) $(AB)^\dagger AB = AB(AB)^\dagger$;
- (ii) $(AB)A^\pi = 0$ and $B^\pi(AB) = 0$;
- (iii) $N(A) \subset N(AB)$ and $R(AB) \subset R(B)$;
- (iv) $N(AB) = N(A) + N(B)$ and $R(AB) = R(A) \cap R(B)$.

Proof. Theorem 4.3 cannot be directly applied in the setting of Hilbert space operators since the spaces $N(L_A)$, $N(L_B)$ (where $L_T : U \mapsto TU$) may be infinite dimensional even though $N(A)$ and $N(B)$ are finite dimensional. Instead we retrace the proof of that theorem replacing formally

$$\begin{aligned} b\mathfrak{A} & \text{ by } R(B), \\ a^{-1}(0), b^{-1}(0), (ab)^{-1}(0) & \text{ by } N(A), N(B), N(AB), \\ a_{-1}(0), b_{-1}(0), (ab)_{-1}(0) & \text{ by } N(A^*), N(B^*), N(B^*A^*), \end{aligned}$$

respectively. To this end we note that

$$\begin{aligned} b_{-1}(0) \subset (ab)_{-1}(0) & \Leftrightarrow b^{*-1}(0) \subset (b^*a^*)^{-1}(0), \\ (ab)_{-1}(0) = a_{-1}(0) + b_{-1}(0) & \Leftrightarrow (b^*a^*)^{-1}(0) = a^{*-1}(0) + b^{*-1}(0). \end{aligned}$$

Also, for any bounded linear operators A, B on H we have

$$\begin{aligned} N(B^*) \subset N(B^*A^*) & \Leftrightarrow R(AB) \subset R(B), \\ N(B^*A^*) = N(A^*) + N(B^*) & \Leftrightarrow R(AB) = R(A) \cap R(B). \end{aligned}$$

We can then check that the proof of Theorem 4.3 appropriately modified yields the required result.

When the preceding corollary is specialized to matrices we get the product theorem of Hartwig and Katz [7], which answered a problem that was open for over 25 years (see [1]). For a proof of the Hartwig–Katz theorem it is enough to observe that matrices are Fredholm operators on a finite dimensional Hilbert space, and to verify that

$$\begin{aligned} N(A) \subset N(AB) & \Leftrightarrow \text{RS}(AB) \subset \text{RS}(A), \\ N(AB) = N(A) + N(B) & \Leftrightarrow \text{RS}(AB) = \text{RS}(A) \cap \text{RS}(B), \end{aligned}$$

where $\text{RS}(A)$ is the row space of the matrix A .

COROLLARY 4.5 [7, Theorem 1] *Let A, B be EP matrices. Then the following are equivalent.*

- (i) $R(AB) = R(A) \cap R(B)$ and $\text{RS}(AB) = \text{RS}(A) \cap \text{RS}(B)$.

- (ii) $R(AB) \subset R(B)$ and $RS(AB) \subset RS(A)$.
- (iii) AB is EP.

A simple direct proof of the Hartwig–Katz theorem for matrices was given in [11].

References

- [1] T. S. B a s k e t t and I. J. K a t z, Theorems on products of EP_r matrices, *Linear Algebra Appl.* **2** (1969), 87–103.
- [2] K. G. B r o c k, A note on commutativity of a linear operator and its Moore–Penrose inverse, *Numer. Funct. Anal. Optim.* **11** (1990), 673–678.
- [3] S. L. C a m p b e l l and C. D. M e y e r, *Generalized Inverses of Linear Transformations*, Pitman, London, 1979.
- [4] M. P. D r a z i n, Pseudo-inverse in associative rings and semigroups, *Amer. Math. Monthly* **65** (1958), 506–514.
- [5] R. E. H a r t e and M. M b e k h t a, On generalized inverses in C^* -algebras, *Studia Math.* **103** (1992), 71–77.
- [6] R. E. H a r t e and M. M b e k h t a, On generalized inverses in C^* -algebras II, *Studia Math.* **106** (1993), 129–138.
- [7] R. H a r t w i g and I. J. K a t z, On products of EP matrices, *Linear Algebra Appl.* **252** (1997), 339–345.
- [8] I. J. K a t z, Weigman type theorems for EP_r matrices, *Duke Math. J.* **32** (1965), 423–428.
- [9] J. J. K o l i h a, A generalized Drazin inverse, *Glasgow Math. J.* **38** (1996), 367–381.
- [10] J. J. K o l i h a, The Drazin and Moore–Penrose inverse in C^* -algebras, *Proc. Roy. Irish Acad. Sect. A* **99** (1999), 17–27.
- [11] J. J. K o l i h a, A simple proof of the product theorem for EP matrices, *Lin. Algebra Appl.* **294** (1999), 213–215.

- [12] I. M a r e k and K. Ž i t n ý, *Matrix Analysis for Applied Sciences* vol. 2, Teubner, Leipzig, 1986.
- [13] R. P e n r o s e, A generalized inverse for matrices, *Proc. Cambridge Philos. Soc.* **51** (1955), 406–413.
- [14] J. H. M. Wedderburn, *Lectures on Matrices*, AMS Colloq. Publ. 17, Amer. Math. Soc., Providence, 1934.
- [15] E. T. W o n g, Does the generalized inverse of A commute with A ?, *Math. Mag.* **59** (1986), 230–232.

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