THE MAPPING $\delta_{x,y}$ IN NORMED LINEAR SPACES AND REFINEMENTS OF THE CAUCHY SCHWARZ INEQUALITY

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Abstract

This paper introduces the mapping $\delta_{x,y}(t) = 2\|x+ty\| - \|x+2ty\|$ in a normed linear space, closely related to the lower and upper semi-inner product, and investigates its monotonicity, boundedness and convexity. This mapping is used as a tool to refine measurements in normed linear spaces obtained through the norm. As a result we obtain refinements of the abstract Cauchy-Schwarz inequality and their applications in concrete spaces.

Key words and phrases: Normed linear spaces, the lower and upper semi-inner product, inner product spaces, the Cauchy-Schwarz inequality.

1 Introduction

In this paper we introduce the mapping $\delta_{x,y}(t) = 2\|x+ty\| - \|x+2ty\|$, which arises naturally in geometric arguments on normed spaces which use the lower and upper semi-inner product defined below. Continuing the investigation of [6, 7, 8], we study properties of monotonicity, boundedness and convexity of this mapping. We give applications to inequalities in analysis, including refinements of the Cauchy-Schwarz inequality, both in normed linear and inner product spaces.

Let $(X, \|\cdot\|)$ be a real normed linear space. We define the *lower* and *upper semi-inner product* by

$$(y, x)_i = \lim_{t\to 0-} \frac{\|x+ty\|^2 - \|x\|^2}{2t},$$

and

$$(y, x)_s = \lim_{t\to 0+} \frac{\|x+ty\|^2 - \|x\|^2}{2t},$$

respectively. These limits are well defined for every pair $x, y \in X$ (see for example [3, 9]); the subscripts i and s stand for inferior and superior, respectively. We mention that $(\cdot,\cdot)_i$ and $(\cdot,\cdot)_s$ are not semi-inner products in the sense of Lumer since they are not additive in the first variable (see (VII) below).

For the sake of completeness we list here some of the main properties of these products that will be used in the sequel (see [3, 4, 5]), assuming that $p, q \in \{s, i\}$ and $p \neq q$:

(I) $(x, x)_p = \|x\|^2$ for all $x \in X$;

(II) $(\alpha x, \beta y)_p = \alpha\beta(x, y)_p$ if $\alpha\beta \geq 0$ and $x, y \in X$;

(III) The Cauchy-Schwarz inequality: $|(x, y)_p| \leq \|x\| \, \|y\|$ for all $x, y \in X$;

(IV) $(\alpha x + y, x)_p = \alpha(x, x)_p + (y, x)_p$ if x, y belong to X and α is a real number;

(V) $(-x, y)_p = -(x, y)_q$ for all $x, y \in X$;

(VI) $(x + y, z)_p \leq \|x\| \, \|z\| + (y, z)_p$ for all $x, y, z \in X$;

(VII) The mapping $(\cdot,\cdot)_p$ is continuous and subadditive (superadditive) in the first variable for $p = s$ (or $p = i$);

(VIII) We have the inequality

$$(y, x)_i \leq (y, x)_s \text{ for all } x, y \in X;$$

(IX) If the norm $\|\cdot\|$ is induced by an inner product $(\cdot,\cdot)$, then

$$(y, x)_i = (y, x) = (y, x)_s \text{ for all } x, y \in X.$$

For other properties of the lower and upper semi-inner product see [3, 4, 5, 10]. See [12] for applications to iterative methods for nonlinear operators, and [2] for generation of nonlinear semi-groups. Good sources of further information on the lower and upper semi-inner product and the related duality mappings are the recent monographs [9] and [1]; the latter discusses many applications to nonlinear problems (see also the review [13]).

The terminology throughout the paper is standard. We mention that for functions we use the terms 'increasing' (and 'strictly increasing'), 'decreasing' (and 'strictly decreasing'), thus avoiding 'nondecreasing' and 'nonincreasing'.

2 Properties of the mapping $\delta_{x,y}$

Let $(X, \|\cdot\|)$ be a real normed linear space, and x, y two fixed elements of X. We introduce and study the mapping $\delta_{x,y} : \mathbb{R} \to \mathbb{R}$ which we define by

$$\delta_{x,y}(t) = 2\|x + ty\| - \|x + 2ty\| .$$

The following theorem describes the main properties of this mapping which is related to the lower and upper semi-inner products.

Theorem 2.1 *Let $(X, \|\cdot\|)$ be a real normed linear space, and x, y two fixed vectors in X, and let*

$$\Phi^p_{x,y}(t) = \frac{(y, x + ty)_p}{\|x + ty\|}, \qquad \Psi^p_{x,y}(t) = \frac{(x, x + ty)_p}{\|x + ty\|}$$

(if x, y are linearly independent). Then we have:

(i) *$\delta_{x,y}$ is bounded and*

$$|\delta_{x,y}(t)| \leq \|x\| \quad \text{for all } t \in \mathbb{R}; \tag{2.1}$$

(ii) *If x, y are linearly independent, then we have the inequalities*

$$\delta_{x,y}(t) \leq \Psi^i_{x,y}(t) \leq \Psi^s_{x,y}(t) \leq \|x\| \quad \text{for all } t \in \mathbb{R} \tag{2.2}$$

and

$$\begin{aligned} \delta_{x,y}(t) &\geq \Psi^s_{x,2y}(t) \geq \Psi^i_{x,2y}(t) \geq \|x + 2ty\| - 2|t|\,\|y\| \\ &\geq \begin{cases} \dfrac{(x,y)_s}{\|y\|} & \text{if } t \geq 0 \\ -\dfrac{(x,y)_i}{\|y\|} & \text{if } t < 0; \end{cases} \end{aligned} \tag{2.3}$$

(iii) *The mapping* $\delta_{x,y}$ *is continuous on* $\mathbb{R}$ *and we have the limits*

$$\lim_{t\to+\infty}\delta_{x,y}(t)=\frac{(x,y)_s}{\|y\|},\qquad \lim_{t\to-\infty}\delta_{x,y}(t)=-\frac{(x,y)_i}{\|y\|}\tag{2.4}$$

provided that x,y *are linearly independent;*

(iv) *The mapping* $\delta_{x,y}$ *has one sided derivatives at each point of* $\mathbb{R}$ *and, if* x,y *are linearly independent, we have*

$$\frac{d^+\delta_{x,y}(t)}{dt}=2\left(\Phi^s_{x,y}(t)-\Phi^s_{x,y}(2t)\right)\tag{2.5}$$

and

$$\frac{d^-\delta_{x,y}(t)}{dt}=2\left(\Phi^i_{x,y}(t)-\Phi^i_{x,y}(2t)\right)\tag{2.6}$$

for all $t\in\mathbb{R}$*;*

(v) *The mapping* $\delta_{x,y}$ *is increasing on* $(-\infty,0]$ *and decreasing on* $(0,+\infty)$*.*

Proof (i) By the triangle inequality for the norm,

$$|\delta_{x,y}(t)|=|\,\|2x+2ty\|-\|x+2ty\|\,|\le\|2x+2ty-x-2ty\|=\|x\|$$

for all $t\in\mathbb{R}$, and (2.1) obtains.

(ii) Using the Cauchy-Schwarz inequality (III) and properties of lower and upper semi-inner products $(\cdot,\cdot)_p$, we have

$$\begin{aligned}\|x+2ty\|\,\|2x+2ty\|&\ge(x+2ty,2x+2ty)_s\\&=(2x+2ty-x,2x+2ty)_s=\|2x+2ty\|^2-(x,2x+2ty)_i\end{aligned}$$

from where we get

$$\|x+2ty\|-\|2x+2ty\|\ge-\frac{(x,2x+2ty)_i}{\|2x+2ty\|},$$

which is equivalent to

$$2\,\|x+ty\|-\|x+2ty\|\le\frac{(x,x+ty)_i}{\|x+ty\|}$$

for all $t\in\mathbb{R}$, and the first inequality in (2.2) is proved.

The second inequality is obvious.

The third inequality follows from the Cauchy-Schwarz inequality

$$(x, x+ty)_s \leq \|x+ty\| \, \|x\| \, , \quad t \in \mathbb{R}.$$

To prove the first inequality in (2.3), we also use the Cauchy-Schwarz inequality:

$$\begin{aligned} \|2x+2ty\| \, \|x+2ty\| &\geq (2x+2ty, x+2ty)_s \\ &= (x+x+2ty, x+2ty)_s = \|x+2ty\|^2 + (x, x+2ty)_s, \end{aligned}$$

which implies

$$2\|x+ty\| - \|x+2ty\| \geq \frac{(x, x+2ty)_s}{\|x+2ty\|} = \Psi^s_{x,2y}(t)$$

for all $t \in \mathbb{R}$.

Now suppose that $t \geq 0$. Then

$$\|x+2ty\| - 2|t| \, \|y\| = \|x+2ty\| - 2t\,\|y\| \, ;$$

by the Cauchy-Cauchy-Schwarz inequality we have

$$\|x+2ty\| \, \|y\| \geq (x+2ty, y)_s = (x,y)_s + 2t\,\|y\|^2 \, ,$$

from where we get

$$\|x+2ty\| - 2t\,\|y\| \geq \frac{(x,y)_s}{\|y\|}.$$

If $t < 0$, then, by the preceding argument,

$$\|x+2(-t)(-y)\| - 2(-t)\,\|-y\| \geq \frac{(x,-y)_s}{\|-y\|} = -\frac{(x,y)_i}{\|y\|},$$

and the last inequality in (2.3) is also proved.

(iii) The continuity of $\delta_{x,y}$ is obvious.

To calculate the first limit in (2.4), we observe that

$$\lim_{\alpha \to 0+} \frac{\|y+\alpha x\| - \|y\|}{\alpha} = \frac{(x,y)_s}{\|y\|};$$

this follows from the identity

$$\frac{\|y+\alpha x\| - \|y\|}{\alpha} = \frac{\|y+\alpha x\|^2 - \|y\|^2}{2\alpha} \cdot \frac{2}{\|y+\alpha x\| + \|y\|}$$

valid for any $\alpha > 0$. Then

$$\begin{aligned}
\lim_{t\to+\infty}(2\|x+ty\| - \|x+2ty\|) &= \lim_{\alpha\to 0+}(2\|x+y/\alpha\| - \|x+2y/\alpha\|) \\
&= \lim_{\alpha\to 0+}\left(\frac{2\|y+\alpha x\|}{\alpha} - \frac{\left\|y+\frac{1}{2}\alpha x\right\|}{\frac{1}{2}\alpha}\right) \\
&= 2\lim_{\alpha\to 0+}\frac{\|y+\alpha x\| - \|y\|}{\alpha} - \lim_{\alpha\to 0+}\frac{\left\|y+\frac{1}{2}\alpha x\right\| - \|y\|}{\frac{1}{2}\alpha} \\
&= 2\frac{(x,y)_s}{\|y\|} - \frac{(x,y)_s}{\|y\|} = \frac{(x,y)_s}{\|y\|}.
\end{aligned}$$

The second limit is obtained in a similar fashion.

(iv) It will be convenient to introduce the notation

$$n_{x,y}(t) = \|x+ty\|, \qquad t\in\mathbb{R},$$

and calculate one sided derivatives of $n_{x,y}$ first: Let $t\in\mathbb{R}$. Then

$$\begin{aligned}
\frac{d^+n_{x,y}(t)}{dt} &= \lim_{\substack{\alpha\to t\\ \alpha>t}}\left(\frac{\|x+\alpha y\| - \|x+ty\|}{\alpha - t}\right) \\
&= \lim_{\substack{\beta\to 0\\ \beta>0}}\left(\frac{\|x+ty+\beta y\| - \|x+ty\|}{\beta}\right) \\
&= \lim_{\beta\to 0+}\frac{\|x+ty+\beta y\|^2 - \|x+ty\|^2}{2\beta}\cdot \\
&\quad \cdot\lim_{\beta\to 0+}\frac{2}{\|x+ty+\beta y\| + \|x+ty\|} \\
&= \Phi^s_{x,y}(t),
\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{d^+\delta_{x,y}(t)}{dt} &= 2\,\frac{d^+n_{x,y}(t)}{dt} - \frac{d^+n_{x,y}(2t)}{dt} \\
&= 2\left(\Phi^s_{x,y}(t) - \Phi^s_{x,y}(2t)\right);
\end{aligned}$$

the other derivative is obtained similarly.

(v) We show that the mappings $\Phi^p_{x,y}$, $p\in\{s,i\}$ are increasing on $\mathbb{R}$.

Using properties of the norm derivatives, for any $z\in X$ and any $t>0$ we obtain

$$\|z\|\,\|z+ty\| \ge (z, z+ty)_s = (z+ty-ty, z+ty)_s$$

$$= \|z+ty\|^2 - t(y, z+ty)_i;$$

hence

$$\Phi^i_{z,y}(t) \geq \frac{\|z+ty\| - \|z\|}{t} \quad \text{if } t > 0. \tag{2.7}$$

Similarly we show that

$$\Phi^s_{z,y}(u) \leq \frac{\|z+uy\| - \|z\|}{u} \quad \text{if } u < 0. \tag{2.8}$$

Let $t_1 < t_2$, and write $u := t_1 - t_2 < 0$, $t := t_2 - t_1 > 0$. Then

$$\begin{aligned}\Phi^s_{x,y}(t_1) = \Phi^s_{x+t_2y,y}(u) &\leq \frac{\|x+t_2y+uy\| - \|x+t_2y\|}{u} \\ &= \frac{\|x+t_2y\| - \|x+t_1y\|}{t_2 - t_1} \\ &= \frac{\|x+t_1y+ty\| - \|x+t_1y\|}{t} \\ &\leq \Phi^i_{x+t_1y,y}(t) = \Phi^i_{x,y}(t_2),\end{aligned}$$

where we used (2.8) with $z := x + t_2y$ and (2.7) with $z := x + t_1y$.

We have just proved that, for $p \in \{i, s\}$ and $t_1 < t_2$,

$$\Phi^p_{x,y}(t_1) \leq \Phi^p_{x,y}(t_2). \tag{2.9}$$

If $t < 0$, then $2t < t$ and $\Phi^p_{x,y}(t) \geq \Phi^p_{x,y}(2t)$, which implies that

$$\frac{d^+\delta_{x,y}(t)}{dt} \geq 0 \quad \text{for } t \in (-\infty, 0).$$

If $t \geq 0$, then $2t \geq t$ and $\Phi^p_{x,y}(2t) \geq \Phi^p_{x,y}(t)$, which implies that

$$\frac{d^+\delta_{x,y}(t)}{dt} \leq 0 \quad \text{for } t \in [0, +\infty).$$

By [14, Proposition 5.1.2] we conclude that the mapping $\delta_{x,y}$ is increasing in $(-\infty, 0)$ and decreasing in $[0, +\infty)$.

The theorem is thus proved.

Remark 2.2 We observe that we have also proved that

$$\lim_{t\to+\infty} \Psi^s_{x,y}(t) = \frac{(x,y)_s}{\|y\|}, \qquad \lim_{t\to-\infty} \Psi^i_{x,y}(t) = -\frac{(x,y)_i}{\|y\|}$$

in view of (2.3) and (2.4).

Remark 2.3 We consider the graph of $\delta_{x,y}$ in the case of a normed linear space. In Figures 1 and 2 the graphs are drawn in a dashed line to suggest that we have no information about the convexity of $\delta_{x,y}$. The graph of $\delta_{x,y}$ attains its absolute maximum at $t = 0$, and is asymptotic to $(x, y)_s / \|y\|$ as $t \to \infty$, and to $-(x, y)_i / \|y\|$ as $t \to -\infty$.

Figure 1 depicts the graph in the case that $(x, y)_s \geq 0$. Note that $-(x, y)_i \geq -(x, y)_s$, but $-(x, y)_i$ may be of either sign.

In Figure 2 we give the graph of $\delta_{x,y}$ in the case that $(x, y)_s < 0$. In this case $-(x, y)_i \geq -(x, y)_s > 0$.

We shall now investigate the case of an inner product space.

Proposition 2.4 *If $(X, (\cdot, \cdot))$ is a real linear product space and x, y linearly independent vectors in X, then the mapping $\delta_{x,y}$ is strictly convex on the set $(-\infty, t_1) \cup (t_2, +\infty)$ and strictly concave on (t_1, t_2), where*

$$t_1 = \frac{(\sqrt[3]{4} - 2)(y, x) - \sqrt{\Delta_{x,y}}}{(4 - \sqrt[3]{4}) \|y\|^2}, \quad t_2 = \frac{(\sqrt[3]{4} - 2)(y, x) + \sqrt{\Delta_{x,y}}}{(4 - \sqrt[3]{4}) \|y\|^2}, \tag{2.10}$$

$$\Delta_{x,y} = (2 - \sqrt[3]{4})^2 (y, x)^2 + (4 - \sqrt[3]{4})(\sqrt[3]{4} - 1) \|x\|^2 \|y\|^2 > 0.$$

Proof It is clear by the preceding theorem that

$$\frac{d\delta_{x,y}(t)}{dt} = 2 \left(\Phi_{x,y}(t) - \Phi_{x,y}(2t) \right),$$

where

$$\Phi_{x,y}(t) = \frac{(y, x) + t \|y\|^2}{\|x + ty\|}.$$

The mapping $\Phi_{x,y}$ is differentiable on $\mathbb{R}$ with

$$\begin{aligned} \frac{d\Phi_{x,y}(t)}{dt} &= \frac{\|y\|^2 n_{x,y}(t) - (y, x + ty) n'_{x,y}(t)}{n_{x,y}^2(t)} \\ &= \frac{\|y\|^2 n_{x,y}(t) - (y, x + ty)(y, x + ty)/n_{x,y}(t)}{n_{x,y}^2(t)} \\ &= \frac{\|y\|^2 n_{x,y}^2(t) - (y, x + ty)^2}{n_{x,y}^3(t)} \end{aligned}$$

$$= \frac{\|y\|^2 (\|x\|^2 + 2t(y,x) + t^2 \|y\|^2)}{n^3_{x,y}(t)}$$
$$- \frac{(y,x)^2 + 2t(y,x)\|y\|^2 + t^2 \|y\|^4}{n^3_{x,y}(t)}$$
$$= \frac{\|y\|^2 \|x\|^2 - (y,x)^2}{n^3_{x,y}(t)}.$$

Hence we get

$$\begin{aligned}\frac{d^2\delta_{x,y}(t)}{dt^2} &= 2\left(\frac{d\Phi_{x,y}(t)}{dt} - 2\frac{d\Phi_{x,y}(2t)}{dt}\right)\\ &= 2(\|x\|^2\|y\|^2 - (x,y)^2)\left(\frac{1}{n^3_{x,y}(t)} - \frac{2}{n^3_{x,y}(2t)}\right)\\ &= \frac{2(\|x\|^2\|y\|^2 - (x,y)^2)(n^3_{x,y}(2t) - 2n^3_{x,y}(t))}{n^3_{x,y}(2t)n^3_{x,y}(t)}.\end{aligned} \tag{2.11}$$

The equation $\delta''_{x,y}(t) = 0$ is equivalent to

$$\|x + 2ty\|^2 = \sqrt[3]{4}\,\|x + ty\|^2,$$

that is, to

$$(4 - \sqrt[3]{4})\|y\|^2 t^2 + 2(2 - \sqrt[3]{4})(y,x)t + (1 - \sqrt[3]{4})\|x\|^2 = 0.$$

The solutions to this equation are real numbers t_1, t_2 given by (2.10). Note that $t_1 < 0 < t_2$. The second derivative $\delta''_{x,y}(t)$ changes sign at t_1 and t_2, while $\delta''_{x,y}(0) < 0$; hence

$$\frac{d^2\delta_{x,y}(t)}{dt^2} > 0 \text{ if } t \in (-\infty, t_1) \cup (t_2, +\infty)$$

and

$$\frac{d^2\delta_{x,y}(t)}{dt^2} > 0 \text{ if } t \in (t_1, t_2).$$

This completes the proof of the proposition.

Remark 2.5 We consider the graph of $\delta_{x,y}$ in the case of an inner product space. The t-intercept is found from the equation $4\|x + ty\|^2 = \|x + 2ty\|^2$; so $\delta_{x,y}(t) = 0$ if and only if $4(x,y)t = -3\|x\|^2$. This time we have a complete description of the convexity of $\delta_{x,y}$ given by the preceding Proposition with the points t_1, t_2 defined by (2.10).

Figure 3 depicts the graph of $\delta_{x,y}$ in the case that $(x, y) > 0$, and Figure 4 gives the graph in the case that $(x, y) < 0$. The t-intercept in both cases is $t_0 = -\frac{3}{4}\|x\|^2/(x, y)$.

The case of two nonzero orthogonal vectors x, y is depicted in Figure 5. The mapping $\delta_{x,y}$ is even (in particular, $t_2 = -t_1$), and its graph is asymptotic to the t-axis.

3 Applications to inequalities in analysis

We start with a refinement of the Cauchy-Schwarz inequality applicable to both normed linear and inner product spaces based on the results of the preceding section.

Proposition 3.1 *Let x, y be linearly independent vectors in a normed linear space $(X, \|\cdot\|)$ and let $u < 0 < t$. Then the following inequalities hold*:

$$\begin{aligned}
-\|x\|\,\|y\| &\le -\frac{\|y\|\,(x, x+uy)_s}{\|x+uy\|} \le -\frac{\|y\|\,(x, x+uy)_i}{\|x+uy\|} \\
&\le -\|y\|\,(2\,\|x+uy\| - \|x+2uy\|) \\
&\le -\frac{\|y\|\,(x, x+2uy)_i}{\|x+2uy\|} \le -\frac{\|y\|\,(x, x+2uy)_s}{\|x+2uy\|} \\
&\le -\|y\|\,(\|x+2uy\| + 2u\,\|y\|) \\
&\le (x, y)_i \le (x, y)_s \\
&\le \|y\|\,(\|x+2ty\| - 2t\,\|y\|) \\
&\le \frac{\|y\|\,(x, x+2ty)_i}{\|x+2ty\|} \le \frac{\|y\|\,(x, x+2ty)_s}{\|x+2ty\|} \\
&\le \|y\|\,(2\,\|x+ty\| - \|x+2ty\|) \\
&\le \frac{\|y\|\,(x, x+ty)_i}{\|x+ty\|} \le \frac{\|y\|\,(x, x+ty)_s}{\|x+ty\|} \\
&\le \|x\|\,\|y\|. \qquad (3.1)
\end{aligned}$$

Proof Follows from Theorem 2.1, in particular from (2.2) and (2.3).

The preceding proposition inserts up to 6 extra terms in the Cauchy-Schwarz inequality for normed linear spaces (4 for smooth normed linear spaces). We apply the result to some concrete normed spaces.

Example 3.2 Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, let $p > 1$, and let $L^p(\Omega)$ be the Banach space of all real valued functions p-integrable with respect to μ. It is known [15] that

$$[x, y]_p = \|y\|_p^{2-p} \int_\Omega |y|^{p-1} \mathrm{sgn}(y)\, x \, d\mu$$

is a semi-inner product on $L^p(\Omega)$ generating the norm $\|x\|_p = (\int_\Omega |x|^p \, d\mu)^{1/p}$. Since the space $L^p(\Omega)$ is smooth,

$$[x, y]_p = (x, y)_s = (x, y)_i, \quad x, y \in L^p(\Omega).$$

Applying the preceding proposition with $t > 0$ to any linearly independent vectors $x, y \in L^p(\Omega)$, we get

$$\begin{aligned}
\frac{\int_\Omega |y|^{p-1} \mathrm{sgn}(y)\, x \, d\mu}{\left(\int_\Omega |y|^p \, d\mu\right)^{(p-1)/p}} &\leq \left(\int_\Omega |x + 2ty|^p \, d\mu\right)^{1/p} - 2t \left(\int_\Omega |y|^p \, d\mu\right)^{1/p} \\
&\leq \frac{\int_\Omega |x + 2ty|^{p-1} \mathrm{sgn}(x + 2ty)\, x \, d\mu}{\left(\int_\Omega |x + 2ty|^p \, d\mu\right)^{(p-1)/p}} \\
&\leq 2 \left(\int_\Omega |x + ty|^p \, d\mu\right)^{1/p} - \left(\int_\Omega |x + 2ty|^p \, d\mu\right)^{1/p} \\
&\leq \frac{\int_\Omega |x + ty|^{p-1} \mathrm{sgn}(x + ty)\, x \, d\mu}{\left(\int_\Omega |x + ty|^p \, d\mu\right)^{(p-1)/p}} \\
&\leq \left(\int_\Omega |x|^p \, d\mu\right)^{1/p};
\end{aligned}$$

for $t < 0$ the first line in the above string of inequalities gets changed to

$$-\frac{\int_\Omega |y|^{p-1} \mathrm{sgn}(y)\, x \, d\mu}{\left(\int_\Omega |y|^p \, d\mu\right)^{(p-1)/p}} \leq \left(\int_\Omega |x + 2ty|^p \, d\mu\right)^{1/p} + 2t \left(\int_\Omega |y|^p \, d\mu\right)^{1/p}$$

while the other inequalities remain unchanged.

Example 3.3 We consider the space $\ell^1(\mathbb{R})$ consisting of all vectors of the form $x = (x_j)_{j \in \mathbb{N}}$, where $\|x\| = \sum_{j=1}^\infty |x_j| < \infty$. It is known from [10] that

$$(x, y)_i = \sum_{k=1}^\infty |y_k| \left(\sum_{y_j \neq 0} \mathrm{sgn}(y_j) x_j - \sum_{y_j = 0} |x_j| \right)$$

and

$$(x,y)_s = \sum_{k=1}^{\infty} |y_k| \left(\sum_{y_j \neq 0} \operatorname{sgn}(y_j) x_j + \sum_{y_j = 0} |x_j| \right).$$

Applying Proposition 3.1 with $t > 0$ to any $x, y \in \ell^1(\mathbb{R})$, we get

$$\begin{aligned}
\sum_{y_j \neq 0} \operatorname{sgn}(y_j) x_j - \sum_{y_j = 0} |x_j| &\leq \sum_{y_j \neq 0} \operatorname{sgn}(y_j) x_j + \sum_{y_j = 0} |x_j| \\
&\leq \sum_{j=1}^{\infty} |x_j + 2ty_j| - 2t \sum_{j=1}^{\infty} |y_j| \\
&\leq \sum_{x_j + 2ty_j \neq 0} \operatorname{sgn}(x_j + 2ty_j) x_j - \sum_{x_j + 2ty_j = 0} |x_j| \\
&\leq \sum_{x_j + 2ty_j \neq 0} \operatorname{sgn}(x_j + 2ty_j) x_j + \sum_{x_j + 2ty_j = 0} |x_j| \\
&\leq \sum_{j=1}^{\infty} 2|x_j + ty_j| - \sum_{j=1}^{\infty} |x_j + 2ty_j| \\
&\leq \sum_{x_j + ty_j \neq 0} \operatorname{sgn}(x_j + ty_j) x_j - \sum_{x_j + ty_j = 0} |x_j| \\
&\leq \sum_{x_j + ty_j \neq 0} \operatorname{sgn}(x_j + ty_j) x_j + \sum_{x_j + ty_j = 0} |x_j| \\
&\leq \sum_{j=1}^{\infty} |x_j|.
\end{aligned}$$

For $t < 0$ the first two lines in the above string of inequalities are changed to

$$\begin{aligned}
-\sum_{y_j \neq 0} \operatorname{sgn}(y_j) x_j - \sum_{y_j = 0} |x_j| &\leq -\sum_{y_j \neq 0} \operatorname{sgn}(y_j) x_j + \sum_{y_j = 0} |x_j| \\
&\leq \sum_{j=1}^{\infty} |x_j + 2ty_j| + 2t \sum_{j=1}^{\infty} |y_j|
\end{aligned}$$

with the rest remaining the same.

Example 3.4 Let $c_0(\mathbb{R})$ be the space of all vectors of the form $x = (x_j)_{j \in \mathbb{N}}$ such that $\lim_{j \to \infty} x_j = 0$, equipped with the norm $\|x\| = \sup_{j \in \mathbb{N}} |x_j| = \max_{j \in \mathbb{N}} |x_j|$. First we show that in $c_0(\mathbb{R})$

$$(x,y)_s = \sup_{|y_j| = \|y\|} x_j y_j, \qquad (x,y)_i = \inf_{|y_j| = \|y\|} x_j y_j. \tag{3.2}$$

We may assume that $x \neq 0$ and $y \neq 0$. Let A be the set of the indices k with $|y_k| = \|y\|$ and let $\alpha = \inf_{j \notin A}(\|y\| - |y_j|)$; then A is finite, and $\alpha > 0$. Let $0 < t < \alpha/(2\,\|x\|)$. If $j \notin A$ and $k \in A$, then

$$\begin{aligned} |y_j + tx_j| \le |y_j| + t|x_j| \le \|y\| - \alpha + t\,\|x\| < \|y\| - \tfrac{1}{2}\alpha \\ \le \|y\| - t\,\|x\| \le |y_k| - t|x_k| \le |y_k + tx_k|, \end{aligned}$$

and

$$\|y + tx\| = \max_{k \in A} |y_k + tx_k| \ \text{ if } \ 0 < t < \alpha/(2\,\|x\|).$$

Then

$$\begin{aligned} (x, y)_s &= \lim_{t \to 0+} \frac{\max_{k \in A}\Big(|y_k + tx_k|^2 - \|y\|^2\Big)}{2t} \\ &= \max_{k \in A} \lim_{t \to 0+} \frac{(y_k + tx_k)^2 - y_k^2}{2t} = \max_{k \in A} x_k y_k. \end{aligned}$$

The second identity in (3.2) follows from $(x, y)_i = -(-x, y)_s$.

We can apply Proposition 3.1 with $t > 0$ to any linearly independent vectors $x, y \in c_0(\mathbb{R})$. For convenience, for any $x \in c_0(\mathbb{R})$ we introduce the notation

$$\Delta(x) = \{k \in \mathbb{N} : |x_k| = \|x\|\}.$$

Then we have

$$\begin{aligned} \frac{\inf_{k \in \Delta(y)} x_k y_k}{\sup_j |y_j|} &\le \frac{\sup_{k \in \Delta(y)} x_k y_k}{\sup_j |y_j|} \le \sup_j |x_j + 2ty_j| - 2t \sup_j |y_j| \\ &\le \frac{\inf_{k \in \Delta(x+2ty)} x_k(x_k + 2ty_k)}{\sup_j |x_j + 2ty_j|} \le \frac{\sup_{k \in \Delta(x+2ty)} x_k(x_k + 2ty_k)}{\sup_j |x_j + 2ty_j|} \\ &\le 2 \sup_j |x_j + ty_j| - \sup_j |x_j + 2ty_j| \\ &\le \frac{\inf_{k \in \Delta(x+ty)} x_k(x_k + ty_k)}{\sup_j |x_j + ty_j|} \le \frac{\sup_{k \in \Delta(x+ty)} x_k(x_k + ty_k)}{\sup_j |x_j + ty_j|} \\ &\le \sup_j |x_j|. \end{aligned}$$

If $t < 0$, the first line in the above string becomes

$$-\frac{\sup_{k \in \Delta(y)} x_k y_k}{\sup_j |y_j|} \le -\frac{\inf_{k \in \Delta(y)} x_k y_k}{\sup_j |y_j|} \le \sup_j |x_j + 2ty_j| + 2t \sup_j |y_j|$$

with the remaining lines unchanged.

Example 3.5 In this example we consider the Hilbert space $\ell^2(\mathbb{R})$ of all vectors $x = (x_j)_{j\in\mathbb{N}}$ with $\|x\| = \big(\sum_{j=1}^{\infty} x_j^2\big)^{1/2} < \infty$. Applying Proposition 3.1 with $t > 0$ to any linearly independent vectors $x, y \in \ell^2(\mathbb{R})$, we get the following inequalities.

$$\begin{aligned}
\frac{\sum_{k=1}^{\infty} x_k y_k}{\big(\sum_{k=1}^{\infty} y_k^2\big)^{1/2}} &\le \Big(\sum_{k=1}^{\infty}(x_k+2ty_k)^2\Big)^{1/2} - 2t\Big(\sum_{k=1}^{\infty} y_k^2\Big)^{1/2} \\
&\le \frac{\sum_{k=1}^{\infty} x_k(x_k+2ty_k)}{(\sum_{k=1}^{\infty}(x_k+2ty_k)^2)^{1/2}} \\
&\le 2\Big(\sum_{k=1}^{\infty}(x_k+ty_k)^2\Big)^{1/2} - \Big(\sum_{k=1}^{\infty}(x_k+2ty_k)^2\Big)^{1/2} \\
&\le \frac{\sum_{k=1}^{\infty} x_k(x_k+ty_k)}{(\sum_{k=1}^{\infty}(x_k+ty_k)^2)^{1/2}} \\
&\le \Big(\sum_{k=1}^{\infty} x_k^2\Big)^{1/2}.
\end{aligned}$$

For $t < 0$, the first line in the above string of inequalities becomes

$$-\frac{\sum_{k=1}^{\infty} x_k y_k}{\big(\sum_{k=1}^{\infty} y_k^2\big)^{1/2}} \le \Big(\sum_{k=1}^{\infty}(x_k+2ty_k)^2\Big)^{1/2} + 2t\Big(\sum_{k=1}^{\infty} y_k^2\Big)^{1/2}$$

while the rest remains unchanged.

Let us consider the case when X is a general inner product space. For any given pair x, y of linearly independent vectors in X we define real numbers $t_1,\ t_2$ by (2.10). Then by Proposition 2.4, $\delta_{x,y}$ is convex on the intervals $(-\infty, t_1]$, and $[t_2, +\infty)$ and concave on $[t_1, t_2]$. In this case we are able to obtain further refinement of the Cauchy-Schwarz inequality.

Proposition 3.6 *Let $(X, (\cdot,\cdot))$ be a real inner product space, x, y linearly independent vectors in X, and let $t_1,\ t_2$ be defined by* (2.10)*.*

(i) *If $t_2 \le a < b$ or $a < b \le t_1$ and $\eta = \operatorname{sgn}(\frac{1}{2}(a+b))$, then the following inequalities hold.*

$$\begin{aligned}
\eta\,(x,y) &\le \frac{\|y\|}{b-a}\int_a^b \delta_{x,y}(t)\,dt \\
&\le \Bigg[\Big\|x + \frac{a+b}{2}y\Big\| + \frac{\|x+ay\| + \|x+by\|}{2}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\left(\|x+(a+b)y\|+\frac{\|x+2ay\|+\|x+2by\|}{2}\right)\Bigg]\|y\| \\
&\leq\left[\|x+ay\|+\|x+by\|-\frac{\|x+2ay\|+\|x+2by\|}{2}\right]\|y\| \\
&\leq\|x\|\,\|y\|\,. \qquad (3.3)
\end{aligned}
$$

(ii) *If* $0\leq a<b\leq t_2$ *or* $t_1\leq a<b\leq 0$ *and* $\eta=\operatorname{sgn}(\frac{1}{2}(a+b))$, *then*

$$
\begin{aligned}
\eta\,(x,y) &\leq \frac{\|y\|}{b-a}\int_a^b \delta_{x,y}(t)\,dt \\
&\leq \left(\|2x+(a+b)y\|-\|x+(a+b)y\|\right)\|y\| \\
&\leq \|x\|\,\|y\|\,. \qquad (3.4)
\end{aligned}
$$

Proof We note that $t_1<0<t_2$ and that $|\delta_{x,y}(t)|\leq\|x\|$ for all $t\in\mathbb{R}$ in view of Theorem 2.1.

(i) By (2.3),

$$\frac{(x,y)}{\|y\|}\leq\delta_{x,y}(t)\ \text{ for all }\ t\geq 0.$$

Hence also

$$(x,y)\leq\frac{\|y\|}{b-a}\int_a^b\delta_{x,y}(t)\,dt\ \text{ if }\ 0\leq a<b.$$

If we choose a,b in the interval $[t_2,+\infty)$, then the function $\delta(t)=\delta_{x,y}(t)$ is convex in $[a,b]$, and the Hermite-Hadamard inequality [11, p.10] can be applied to δ:

$$\frac{1}{b-a}\int_a^b\delta(t)\,dt\leq\frac{1}{2}\left[\delta\left(\frac{a+b}{2}\right)+\frac{\delta(a)+\delta(b)}{2}\right]\leq\frac{\delta(a)+\delta(b)}{2}\leq\|x\|\,. \qquad (3.5)$$

Substituting into (3.5) for $\delta(t)=\delta_{x,y}(t)$, we get (3.3) with $\eta=1$.

Similarly, according to (2.3),

$$-\frac{(x,y)}{\|y\|}\leq\delta_{x,y}(t)\ \text{ for all }\ t\leq 0;$$

(3.3) with $\eta=-1$ follows by (3.5).

(ii): The mapping $\delta_{x,y}$ is concave in $[a,b]\subset[t_1,t_2]$, and the other part of the Hermite-Hadamard inequality implies that

$$\delta_{x,y}\left(\frac{a+b}{2}\right)\geq\frac{1}{b-a}\int_a^b\delta_{x,y}(t)\,dt.$$

By (2.3), $\delta_{x,y}(t) \geq (x,y)/\|y\|$ if $0 \leq a < t < b \leq t_2$, and $\delta_{x,y}(t) \geq -(x,y)/\|y\|$ if $t_1 \leq a < t < b \leq 0$; (3.4) then follows.

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References

[1] Cioranescu, I., *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic, Dordrecht, 1990.

[2] Crandall, M. G. and Ligget, T. M., Generation of semi-groups of nonlinear transformations on general Banach spaces, *Amer. Math. J.* **93** (1971), 265–298.

[3] Dragomir, S. S., On continuous sublinear functionals in reflexive Banach spaces and applications, *Riv. Mat. Univ. Parma* **16** (1990), 239–250.

[4] Dragomir, S. S., Approximation of continuous linear functionals in real normed spaces, *Reconditi di Matematica* **12** (1992), 357–364.

[5] Dragomir, S. S., Continuous linear functionals and norm derivatives in real normed spaces, *Univ. Beograd. Publ. Elektrotehn. Fak.* **3** (1992), 5–12.

[6] Dragomir, S. S. and Koliha, J. J., The mapping $v_{x,y}$ in normed linear spaces with applications to inequalities in analysis, *J. Ineq. Appl.* **2** (1998), 37–55.

[7] Dragomir, S. S. and Koliha, J. J., Mappings Φ^p in normed linear spaces and new characterizations of Birkhoff orthogonality, smoothness and best approximants, *Soochow J. Math.* **23** (1997), 227–239.

[8] Dragomir, S. S. and Koliha, J. J., The mapping $\gamma_{x,y}$ in normed linear spaces and applications, *J. Math. Anal. Appl.* **210** (1997), 549–563.

[9] Istrăţescu, V. I., *Inner Product Structures*, D. Reidel, Dordrecht, 1987.

[10] Miličić, P. M., Sur la g-orthogonalité dans des espaces normés, *Mat. Vesnik* **39** (1987), 325–334.

[11] Mitrinović, D. S., Pečarić, J. E. and Fink, A. M., *Classical and New Inequalities in Analysis*, Mathematics and its Applications, vol. 61, Kluwer Academic, Dordrecht, 1993.

[12] Reich, S., Iterative methods for accretive sets, in *Nonlinear Equations in Abstract Spaces* (Lakshmikantham, V., editor) Academic Press, New York, 1978, 317–326.

[13] Reich, S., Review of 'Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems' by I. Cioranescu, *Bull. Amer. Soc.* **26** (1992), 367–370.

[14] Royden, H. L., *Real Analysis*, 3rd edition, Macmillan, New York, 1989.

[15] Vainberg, M. M., *Variational Method and Method of Monotone Operators in the Theory of Nonlinear Operators*, Wiley, New York, 1974.