FREDHOLM PROPERTIES OF THE DIFFERENCE OF ORTHOGONAL PROJECTIONS IN A HILBERT SPACE

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Buckholtz (*Proc. Amer. Math. Soc.* **128** (2000), 1415–1418) gave necessary and sufficient conditions for the invertibility of the difference of two orthogonal projections in a Hilbert space. We generalize this result by investigating when the difference of such projections is a Fredholm operator, and give an explicit formula for its Fredholm inverse.

1 Introduction and notation

The question of the invertibility of the difference $P-Q$, where P, Q are idempotent hermitian matrices or, more generally, orthogonal projections on a Hilbert space H, is of great importance in operator theory as it is connected with the question when the space H is the direct sum $H = R(P) \oplus R(Q)$ of the ranges of P and Q, and with the existence of an idempotent operator F satisfying

$$PF = F, \quad FP = P, \quad Q(I-F) = I-F, \quad (I-F)Q = Q.$$

These problems were considered by many mathematicians, for instance, Ljance [11], Pták [17] and Vidav [19], and more recently by Buckholtz [3, 4] and Wimmer [21, 22]. Rakočević [18] studied the question in the setting of Hilbert spaces, Koliha [12] in the setting of C^*-algebras, and the present authors in the setting of C^*-algebras [13] and rings [15, 14].

In this paper we study the problem of when the difference of two Hilbert space orthogonal projections is a Fredholm operator.

Let H be an infinite dimensional complex Hilbert space. By $\mathcal{B}(H)$ ($\mathcal{F}(H)$, $\mathcal{K}(H)$) we denote the set of all bounded (finite rank, compact) linear operators on H. The fact that $\mathcal{K}(H)$ is a closed two sided ideal in the full C^*-algebra $\mathcal{B}(H)$ enables us to define the Calkin algebra over H as the quotient algebra $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ (see [5]). Then $\mathcal{C}(H)$ is itself a C^*-algebra in the quotient (essential) norm

$$\|T\|_e = \|T + \mathcal{K}(H)\| = \inf_{K \in \mathcal{K}(H)} \|T + K\|.$$

We denote by π the natural homomorphism of $\mathcal{B}(H)$ into $\mathcal{C}(H)$:

$$\pi(T) = T + \mathcal{K}(H), \quad T \in \mathcal{B}(H).$$

Throughout this paper, $N(T)$ and $R(T)$ will denote the nullspace and the range of $T \in \mathcal{B}(H)$, respectively. An operator $U \in \mathcal{B}(H)$ is *idempotent* if $U^2 = U$; an idempotent operator $U \in \mathcal{B}(H)$ is called a *projection* (or *oblique projection*), and a self-adjoint idempotent operator $P \in \mathcal{B}(H)$ is called an *orthogonal projection*. Set $\alpha(T) = \dim N(T)$ and $\beta(T) = \operatorname{codim} R(T) = \dim H/R(T)$. An operator $T \in \mathcal{B}(H)$ is called *Fredholm* if the range $R(T)$ of T is closed and both $\alpha(T)$ and $\beta(T)$ are finite. The set $\Phi(H)$ of all Fredholm operators on H constitutes a multiplicative open semigroup in $\mathcal{B}(H)$ (see [5]). According to the Atkinson theorem [5, Theorem 3.2.8], $T \in \mathcal{B}(H)$ is Fredholm if and only if $\pi(T)$ is invertible in the Calkin algebra $\mathcal{C}(H)$. Hence

$$\Phi(H) = \pi^{-1}(\mathcal{C}(H)^{-1}),$$

using the notation $\mathcal{A}^{-1}$ for the group of invertible elements in an algebra $\mathcal{A}$.

For any operator $T \in \mathcal{B}(H)$ we denote by $\sigma(T)$ and $r(T)$ the spectrum and the spectral radius of T. By $\operatorname{acc}\sigma(T)$ we denote the set of all accumulation points of $\sigma(T)$. We write

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi(H)\}$$

for the *essential* (*Fredholm*) *spectrum* of T; the *essential spectral radius* of T is $r_e(T) = \sup_{\lambda \in \sigma_e(T)} |\lambda|$. The *essential equality of operators* $A, B \in \mathcal{B}(H)$ is defined by

$$A \overset{e}{=} B \iff A - B \in \mathcal{K}(H).$$

2 Preliminary results

In this section we summarize preliminary results needed in the sequel, and include some proofs for completeness.

Lemma 2.1. *Let $A \in \mathcal{B}(H)$ be a self-adjoint operator. Then:*

(i) *$R(A)$ is closed if and only $0 \notin \operatorname{acc}\sigma(A)$.*

(ii) *A is Fredholm if and only if $R(A)$ is closed and $\alpha(A) < \infty$.*

(iii) *A is invertible if and only if $R(A)$ is closed and $N(A) = \{0\}$.*

Lemma 2.2. *Let P, Q be projections in $\mathcal{B}(H)$ and let $\lambda \in \mathbb{C}$. Then:*

$$N(I - PQ) = R(P) \cap R(Q), \tag{2.1}$$

$$N(P - Q) = (R(P) \cap R(Q)) \oplus (N(P) \cap N(Q)), \tag{2.2}$$

$$N(P(I - Q)) = R(Q) \oplus (N(P) \cap N(Q)), \tag{2.3}$$

$$(\lambda - 1 + P)(\lambda - (P - Q))(\lambda + 1 - Q) = \lambda(\lambda^2 - 1 + PQ), \tag{2.4}$$

$$(\lambda - 1 + P)(\lambda - (P + Q))(\lambda - 1 + Q) = \lambda((\lambda - 1)^2 - PQ). \tag{2.5}$$

Proof. For the proof of (2.1) assume first that $x \in N(I-PQ)$. Then $x = PQx = Px \in R(P)$. Further, $\|x - Qx\|^2 = \langle x - Qx, x - Qx\rangle = \langle x - Qx, x\rangle - \langle x - Qx, Qx\rangle = \langle x - Qx, Px\rangle = \langle (I - PQ)x, x\rangle = 0$, and $x = Qx \in R(Q)$. Hence $N(I - PQ) \subset R(P) \cap R(Q)$; the reverse inclusion is clear.

To prove (2.2) we first observe that $(R(P) \cap R(Q)) \cap (N(P) \cap N(Q)) = \{0\}$ and $(R(P) \cap R(Q)) \oplus (N(P) \cap N(Q)) \subset N(P - Q)$. Suppose that $x \in N(P - Q)$. Then $u = Px = Qx = Pu = Qu \in R(P) \cap R(Q)$. Hence $x - u \in N(P) \cap N(Q)$ and $x = u + (x - u) \in (R(P) \cap R(Q)) \oplus (N(P) \cap N(Q))$. Hence (2.2) holds.

The inclusions $R(Q) \subset N(P(I - Q))$ and $N(P) \cap N(Q) \subset N(P(I - Q))$ are easily verified. Conversely, let $x \in N(P(I - Q))$. Then $x = Qx + (I - Q)x$, where $(I - Q)x \in N(P) \cap N(Q)$. This proves (2.3).

Towards (2.4) we note that, for any $\lambda \in \mathbb{C}$,

$$\begin{aligned}(\lambda - 1 + P)(\lambda - (P - Q))(\lambda + 1 - Q) &= ((\lambda - 1)(\lambda + Q) + PQ)(\lambda + 1 - Q)\\ &= \lambda(\lambda^2 - 1 + PQ).\end{aligned}$$

Similarly,

$$\begin{aligned}(\lambda - 1 + P)(\lambda - (P + Q))(\lambda - 1 + Q) &= ((\lambda - 1)(\lambda - Q) - PQ)(\lambda - 1 + Q)\\ &= \lambda((\lambda - 1)^2 - PQ),\end{aligned}$$

and (2.5) follows. ☐

The identities (2.4) and (2.5) were applied in the C^*-algebra setting in [13] and in [20] for orthogonal projections P and Q in a Hilbert space. Groß derived equation (2.3) for complex matrices [9, Proposition 1] using the theory of matrix rank.

Lemma 2.3. (See [13].) *Let P and Q be orthogonal projections in $\mathcal{B}(H)$. Then*:

(i) $\sigma(PQ) = \sigma(PQP) \subset [0, r(PQ)] \subset [0, 1]$.

(ii) $r(PQ) = r(PQP) = \|PQP\| = \|PQ\|^2$.

The above relations remain valid when $\sigma(\cdot)$, $r(\cdot)$ and $\|\cdot\|$ are replaced by $\sigma_e(\cdot)$, $r_e(\cdot)$ and $\|\cdot\|_e$, respectively.

Lemma 2.4. *Let P and Q be orthogonal projections in $\mathcal{B}(H)$. Then the following conditions are equivalent*:

(i) $R(P - Q)$ *is closed.*

(ii) $R(P + Q)$ *is closed.*

(iii) $R(P) + R(Q)$ *is closed.*

(iv) $N(P) + N(Q)$ *is closed.*

(v) $R(P(I - Q))$ *is closed.*

(vi) $R((I-P)Q)$ *is closed.*

If any of the conditions (i)–(vi) *is satisfied, then* $R(P+Q) = R(P) + R(Q)$.

Proof. (i) $\Longrightarrow$ (ii): Since $R(P-Q)$ is closed, then $0 \notin \operatorname{acc}\sigma(P-Q)$ by Lemma 2.1. In view of (2.4), $1 \notin \operatorname{acc}\sigma(PQ)$. Then by (2.5), $0 \notin \operatorname{acc}\sigma(P+Q)$. Hence $R(P+Q)$ is closed by Lemma 2.1.

(ii) $\Longrightarrow$ (i): If $R(P+Q)$ is closed, then $0 \notin \operatorname{acc}\sigma(P+Q)$. By (2.5), $1 \notin \operatorname{acc}\sigma(PQ)$. According to (2.4), $0 \notin \operatorname{acc}\sigma(P-Q)$. Then $R(P-Q)$ is closed by Lemma 2.1.

The equivalence of (ii), (iii) and (iv) and the formula $R(P+Q) = R(P)+R(Q)$ can be found in [8]. The equivalence of (ii), (v) and (vi) follows from [10, Proposition 2.4]. □

3 When is $P-Q$ Fredholm?

In this section we give necessary and sufficient conditions which ensure that $P-Q \in \Phi(H)$ when $P, Q \in \mathcal{B}(H)$ are orthogonal projections. Our methods are based on the results of the preceding section, with emphasis on simple proofs.

Theorem 3.1. *Let R and K be closed subspaces of a Hilbert space H and let P and Q be the orthogonal projections with the ranges R and K, respectively. The following are equivalent*:

(i) $P-Q \in \Phi(H)$.

(ii) $I-PQ \in \Phi(H)$ *and* $I-(I-P)(I-Q) = P+Q-PQ \in \Phi(H)$.

(iii) $R+K$ *is closed in* H *and* $\dim\left[(R\cap K)\oplus(R^\perp \cap K^\perp)\right] < \infty$.

(iv) $\|P+Q-I\|_e < 1$.

(v) $P+Q \in \Phi(H)$ *and* $I-PQ \in \Phi(H)$.

Proof. We observe that according to (2.1) and (2.2), $N(I-PQ) = R\cap K$ and $N(P-Q) = (R\cap K)\oplus(R^\perp\cap K^\perp)$.

(i) $\Longrightarrow$ (ii): Suppose that $P-Q \in \Phi(H)$. Then $R(P-Q)$ is closed, so that $0 \notin \operatorname{acc}\sigma(P-Q)$, and $\dim N(P-Q) < \infty$. By (2.4) and Lemma 2.3 (i), $1 \notin \operatorname{acc}\sigma(PQP)$. Thus $R(I-PQP)$ is closed and by [1, Theorem 5], the space $R(I-PQ)$ is closed. Hence $I-PQ \in \Phi(H)$ since $N(I-PQ) = R\cap K$. The same argument applied to $I-P$ and $I-Q$ in place of P and Q implies that $I-(I-P)(I-Q) \in \Phi(H)$.

(ii) $\Longrightarrow$ (iii): The first condition in (ii) implies that $N(I-PQ) = R\cap K$ is finite dimensional, the second implies $N(I-(I-P)(I-Q) = R^\perp \cap K^\perp$ is finite dimensional. We prove that the space $R+K$ is closed. Since $I-PQ$ is Fredholm, $R(I-PQ)$ is closed. By [1, Theorem 5], $R(I-PQP)$ is closed. By Lemma 2.1, $1 \notin \operatorname{acc}\sigma(PQP)$. By Lemma 2.3 (i), $1 \notin \operatorname{acc}\sigma(PQ)$. Thus, (2.5) implies that $0 \notin \operatorname{acc}\sigma(P+Q)$. From this we conclude that $R(P+Q)$ is closed. By Lemma 2.4, $R+K$ is closed.

(iii) $\Longrightarrow$ (i): By Lemma 2.4, $R(P-Q)$ is closed, and by (2.2), $P-Q$ is Fredholm.

(v) $\Longrightarrow$ (iii): The second condition in (v) implies $\dim(R \cap K) < \infty$. Since $R^\perp \cap K^\perp \subset N(P+Q)$, the first condition in (v) implies $\dim(R^\perp \cap K^\perp) < \infty$. The space $R(P+Q)$ is closed as $P+Q$ is Fredholm, and $R+K$ is closed by Lemma 2.4.

(iii) $\Longrightarrow$ (v): We have proved that (i), (ii) and (iii) are all equivalent, hence $I - PQ \in \Phi(H)$ by (ii). Since $R+K$ is closed, $R(P+Q)$ is closed and $R(P+Q) = R+K$ by Lemma 2.4. Then $N(P+Q) = (R+K)^\perp = R^\perp \cap K^\perp$ is finite dimensional.

(iv) $\Longrightarrow$ (ii): By the Akhiezer–Glazman equality [2, § 34],

$$\|P+Q-I\|_e = \|P-(I-Q)\|_e = \max\{\|PQ\|_e, \|(I-P)(I-Q)\|_e\}.$$

Thus (iv) implies $\|PQ\|_e < 1$ and $\|(I-P)(I-Q)\|_e < 1$, which shows that the elements $\pi(I-PQ)$ and $\pi(I-(I-P)(I-Q))$ are invertible in the Calkin algebra $\mathcal{C}(H)$. Hence we obtain (ii).

(ii) $\Longrightarrow$ (iv): Since $\|PQ\|_e \le \|P\|_e \|Q\|_e \le 1$ and $\sigma_e(PQ) \subset [0, \|PQ\|_e^2]$ by Lemma 2.3, the first part of (ii) implies that $1 \notin \sigma_e(PQ)$. Hence $\|PQ\|_e < 1$. Similarly we show that the second part of (ii) implies $\|(I-P)(I-Q)\|_e < 1$. By the Akhiezer–Glazman equality we obtain (iv). □

As a special case of the preceding theorem we now consider the case when the difference $P-Q$ is invertible. This problem is the subject of Buckholtz's papers [3, 4], and the equivalence of (i), (iii) and (iv) of the following corollary is given in [4, Theorem 1]. In the setting of rings, the equivalence of (i), (ii), (iii) and (v) was proved in [15].

Corollary 3.2. *Let R and K be closed subspaces of a Hilbert space H and let P and Q be the orthogonal projections with the ranges R and K, respectively. The following are equivalent*:

(i) *$P-Q$ is invertible.*

(ii) *$I-PQ$ and $I-(I-P)(I-Q) = P+Q-PQ$ are invertible.*

(iii) *$H = R \oplus K$.*

(iv) *$\|P+Q-I\| < 1$.*

(v) *$P+Q$ and $I-PQ$ are invertible.*

Proof. The proof follows from the preceding theorem and its proof, and from the fact (Lemma 2.1 (iii)) that $P-Q$ is invertible if and only if $R(P-Q)$ is closed and

$$N(P-Q) = (R \cap K) \oplus (R^\perp \cap K^\perp) = \{0\}.$$

(Note that $H = R \oplus K$ if and only if $(R \cap K) \oplus (R^\perp \cap K^\perp) = \{0\}$.) □

The corresponding conditions (iii) of the preceding theorem and its corollary give an interesting geometric insight into the difference between Fredholm and invertible.

4 A Fredholm inverse for $P - Q$

By the Atkinson theorem, an operator $T \in \mathcal{B}(H)$ is Fredholm if and only if there exists an operator $S \in \mathcal{B}(H)$ such that $I - TS$ and $I - ST$ are finite rank operators. If $T \in \Phi(H)$, any such operator S will be called a *Fredholm inverse* of T, written T^{Φ}. Since $\pi(S)$ is the unique inverse of $\pi(T)$ in the Calkin algebra $\mathcal{C}(H)$, any two Fredholm inverses A, B of T differ by a finite rank operator, that is, $A \stackrel{e}{=} B$.

We assume basic properties of the Moore–Penrose inverse of a closed range operator $T \in \mathcal{B}(H)$, such as can be found in Groetsch's monograph [7]. In particular, if T has a closed range, then there exists a unique operator $T^\dagger \in \mathcal{B}(H)$ satisfying

$$TT^\dagger T = T, \quad T^\dagger T T^\dagger = T^\dagger, \quad (T^\dagger T)^* = T^\dagger T, \quad (TT^\dagger)^* = TT^\dagger;$$

in addition we have

$$R(T^\dagger) = R(T^*), \qquad N(T^\dagger) = N(T^*). \tag{4.1}$$

The following result was obtained by Penrose [16, Lemma 2.3] for matrices, and is valid also for Hilbert space operators.

Lemma 4.1. *Let $E, F \in \mathcal{B}(H)$ be orthogonal projections such that $R(EF)$ is closed. Then the operator $U = (EF)^\dagger$ is a projection, and $U = FUE$.*

Since the Moore–Penrose inverse of a self-adjoint Fredholm operator is a special case of its Fredholm inverse, we first derive a theorem on the Moore–Penrose inverse of the difference of orthogonal projections. The theorem generalizes to Hilbert spaces a result of Cheng and Tian [6, Theorem 2] obtained for matrices, and adds an explicit description of the projections U and V not given in [6].

We recall that $G \in \mathcal{B}(H)$ is a projection if and only if H is the topological direct sum $H = R \oplus N$, where $R(G) = R$ and $N(G) = N$. We call G the *projection onto R along N*, and write $G = P_{R,N}$.

Theorem 4.2. *Let $P, Q \in \mathcal{B}(H)$ be orthogonal projections with the ranges R, K, respectively, and let $R(P - Q)$ be closed. Then*

$$(P - Q)^\dagger = P_{M,N} - P_{L,S}, \tag{4.2}$$

where

$$M = R \cap (R^\perp + K^\perp), \qquad N = K \oplus (R^\perp \cap K^\perp), \tag{4.3}$$

$$L = R^\perp \cap (R + K), \qquad S = K^\perp \oplus (R \cap K). \tag{4.4}$$

Proof. Since the operator $P-Q$ has closed range, it has the Moore–Penrose inverse $(P-Q)^\dagger$. By Lemma 2.4, the operators $(I - Q)P$ and $Q(I - P)$ have closed ranges, and are Moore–Penrose invertible.

We write $P-Q = (I-Q)P-Q(I-P) = A-B$ and observe that $AB^* = 0 = B^*A$. This implies $A^\dagger B = BA^\dagger = AB^\dagger = B^\dagger A = 0$. It is not difficult to verify that $(A - B)^\dagger = A^\dagger - B^\dagger$, that is,

$$(P - Q)^\dagger = ((I - Q)P)^\dagger - (Q(I - P))^\dagger.$$

We find the ranges and nullspaces of the operators $U = ((I-Q)P)^\dagger$ and $V = (Q(I-P))^\dagger$. In view of (4.1),

$$R(U) = R(A^*) = R(P(I-Q)), \quad N(U) = N(A^*) = N(P(I-Q)),$$
$$R(V) = R(B^*) = R((I-P)Q), \quad N(V) = N(B^*) = N((I-P)Q).$$

To find the range of U we use (2.3) and the relation $R(A) = N(A^*)^\perp$ valid for a closed range operator $A \in \mathcal{B}(H)$:

$$\begin{aligned} R(U) &= R(P(I-Q)) = N((I-Q)P)^\perp \\ &= [N(P) \oplus (R(P) \cap R(Q))]^\perp \\ &= [R^\perp \oplus (R \cap K)]^\perp \\ &= R \cap (R^\perp + K^\perp). \end{aligned}$$

Replacing P, Q in the preceding argument by $I-P$ and $I-Q$, respectively, we obtain

$$R(V) = R((I-P)Q) = N(Q(I-P))^\perp = R^\perp \cap (R+K).$$

Applying (2.3) and then replacing P, Q by $I-P, I-Q$, we obtain

$$N(U) = K \oplus (R^\perp \cap K^\perp), \quad N(V) = K^\perp \oplus (R \cap K).$$

□

We can now give the main theorem of this section.

Theorem 4.3. *Let $P, Q \in \mathcal{B}(H)$ be orthogonal projections with the ranges R, K, respectively, and let $P - Q \in \Phi(H)$. Then*

$$(P-Q)^\Phi \stackrel{e}{=} U + U^* - I, \tag{4.5}$$

where $U = ((I-Q)P)^\dagger$ is the projection onto $R \cap (R^\perp + K^\perp)$ along $K \oplus (R^\perp \cap K^\perp)$.

Proof. Since $P - Q$ is self-adjoint, $(P-Q)^\dagger$ is a Fredholm inverse of $P - Q$. Let $U = ((I-Q)P)^\dagger$ and $V = (Q(I-P))^\dagger$. In view of Theorem 4.2, we need to prove that $U - V \stackrel{e}{=} U + U^* - I$. For this we have to show that $I - V - U^*$ is a finite rank operator. Let the subspaces S, L, Y, Z of H be defined by

$$S = K^\perp \oplus (R \cap K), \qquad L = R^\perp \cap (R+K),$$
$$Y = K^\perp \cap (R+K), \qquad Z = R^\perp \oplus (R \cap K).$$

Then $(I-V) - U^* = P_{S,L} - P_{Y,Z}$. Decomposing $x = y + z$ with $y \in Y$ and $z \in Z$, we get

$$(I - V - U^*)x = (P_{S,L}y - y) + P_{S,L}z = P_{S,L}z, \quad z \in Z,$$

as $Y \subset S$. Decompose $z = u + v$ with $u \in R^\perp$ and $v \in R \cap K$. Then $P_{S,L}z = P_{S,L}u + v$ as $v \in S$. Write further $u = s + t$, where $s \in R+K$ and $t \in R^\perp \cap K^\perp$. Then $s \in R^\perp \cap (R+K) = L$ and $P_{S,L}s = 0$; also $t \in S$ which implies $P_{S,L}t = t$. Hence

$$(I - V - U^*)x = P_{S,L}u + v = t + v \in (R^\perp \cap K^\perp) \oplus (R \cap K). \tag{4.6}$$

Let P_M be the orthogonal projection onto $M = (R \cap K) \oplus (R^\perp \cap K^\perp)$. In view of (4.6) we have $I - V - U^* = P_M(I - V - U^*) = P_M(I - U^*)$ as $R(V) = L \subset M^\perp = N(P_M)$. Hence $R(I - V - U^*) \subset M$, and by Theorem 3.1 (iii), $I - V - U^*$ is a finite rank operator. □

From the following corollary we recover the result of Buckholtz [3, 4].

Corollary 4.4. *Let $P, Q \in \mathcal{B}(H)$ be orthogonal projections with the ranges R, K, respectively. Then $P - Q$ is invertible if and only if $H = R \oplus K$, in which case*

$$(P - Q)^{-1} = U + U^* - I,$$

where $U = P_{R,K}$ and $U^* = P_{K^\perp, R^\perp}$.

Proof. The statement about the invertibility of $P - Q$ follows from Corollary 3.2. From the proof of the preceding theorem we see that

$$(P - Q)^{-1} = (P - Q)^\dagger = U + U^* - I + P_M(I - U^*) = U + U^* - I.$$

Finally,

$$R(U) = R \cap (R^\perp + K^\perp) = R, \quad N(U) = K \oplus (R^\perp \cap K^\perp) = K.$$

The statement about U^* follows from $R(U^*) = N(U)^\perp$ and $N(U^*) = R(U)^\perp$. □

Acknowledgement. The authors are indebted to Professor J. Ball for suggesting an improvement to the presentation of the paper.

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16B99, 16W10, 47A05, 46C05