

The Drazin inverse for closed linear operators

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Abstract

The paper defines and studies the Drazin inverse for a closed linear operator A in a Banach space X in the case that 0 is an isolated spectral point of A . Results include an integral representation for the Drazin inverse of the infinitesimal generator of a C_0 -semigroup and its application to a singular and singularly perturbed differential equation in a Banach space.

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1 Introduction and preliminaries

In 1958, Drazin [4] introduced a pseudoinverse in associative rings and semigroups that now carries his name. The inverse was extensively studied and applied in matrix setting (see the monograph by Campbell and Meyer [1]), as well as in the setting of bounded linear operators and elements of Banach algebras (see Caradus [2] and Nashed [11]). The conventional Drazin inverse was extended to closed linear operators by Nashed and Zhao in [12]; it exists if and only if 0 is at most a pole of the resolvent $R(\lambda; A)$ of the operator A .

The purpose of this paper is to introduce the Drazin inverse A^D of a closed linear operator A on a Banach space X which is defined if 0 is merely an isolated spectral point of A , and to investigate basic properties of A^D . For bounded linear operators and elements of a Banach algebra such inverse was introduced and studied by Koliha in [6], and further investigated in [3, 7, 8].

A special attention is paid to the case when A is the infinitesimal generator of a C_0 -semigroup. For this situation an integral representation of A^D is derived, which is then used to study the asymptotic behaviour of the solutions to a singular and singularly perturbed differential equation in a Banach space.

For basic concepts of operator theory of closed linear operators we rely on [15]. By $\mathcal{C}(X)$ we denote the space of all closed linear operators A with domain and range in a Banach space X ; $\mathcal{D}(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the domain, nullspace and range of A , respectively. By $\mathcal{B}(X)$ we denote the space of all bounded linear operators defined on all of X . If $A \in \mathcal{C}(X)$, then $\rho(A)$ denotes the resolvent set of A and $\sigma(A)$ the spectrum of A . By $\text{iso } \sigma(A)$ and $\text{acc } \sigma(A)$ we define the set of all isolated and accumulation spectral points of A . The extended spectrum of $A \in \mathcal{C}(X)$ is denoted by $\sigma_e(A)$; for $\lambda \in \rho(A)$, $R(\lambda; A)$ denotes the resolvent $(\lambda I - A)^{-1}$ of A .

Let $A \in \mathcal{C}(X)$ with $\sigma(A) \neq \mathbb{C}$. A subset σ of $\sigma_e(A)$ is called a *spectral set* of A if it is both open and closed in the relative topology of $\sigma_e(A)$ as a subset of $\mathbb{C} \cup \{\infty\}$. Let σ_1 be a bounded spectral set of A with the complement σ_2 in $\sigma_e(A)$. By [15, Theorem V.9.2], X is the direct sum $X = X_1 \oplus X_2$ of closed A -invariant subspaces, so that $A = A_1 \oplus A_2$ with respect to this sum, $\sigma(A_i) = \sigma_i$ for $i = 1, 2$, and A_1 is continuous. The projection $P \in \mathcal{B}(X)$ with $\mathcal{R}(P) = X_1$ and $\mathcal{N}(P) = X_2$ is called the *spectral projection* of A corresponding to σ_1 .

A singleton $\{\mu\}$ is a spectral set of A if and only if μ is an isolated singularity of the resolvent $R(\lambda; A)$ of A . If $\mu \notin \text{acc } \sigma(A)$, then either $\mu \in \text{iso } \sigma(A)$ or μ is a resolvent point of A ; we extend the concept of the spectral projection in the latter case by defining $P = 0$.

The following result from [9] is a generalization of [5, Theorem 1.2] to closed operators. It will play an important role in our development of the Drazin inverse for closed operators.

Theorem 1.1. ([9, Theorem 1.4]) *Let A be a closed linear operator with domain $\mathcal{D}(A)$. The point 0 is an isolated spectral point of A if and only if there exists a nonzero projection P such that*

- (i) $\mathcal{R}(P) \subset \mathcal{D}(A)$,
- (ii) $PAx = APx$ for all $x \in \mathcal{D}(A)$,
- (iii) $\sigma(AP) = \{0\}$,
- (iv) $A + \xi P$ is invertible for some (in fact for all) $\xi \neq 0$.

An operator P satisfying (i)–(iv) is the spectral projection of A at 0.

Many results of this paper involve interaction between closed and bounded oper-

ators. We set out the relevant properties in the following lemma for future reference. No proof is given as the arguments are fairly routine.

Lemma 1.2. *Let $A \in \mathcal{C}(X)$ and $B \in \mathcal{B}(X)$. Then the following are true:*

- (i) $A + B \in \mathcal{C}(X)$ and $\mathcal{D}(A + B) = \mathcal{D}(A)$.
- (ii) If $\mathcal{R}(B) \subset \mathcal{D}(A)$, then $AB \in \mathcal{B}(X)$.
- (iii) If $\mathcal{R}(B) \subset \mathcal{D}(A)$, $ABx = BAx$ for all $x \in \mathcal{D}(A)$ and A is invertible, then $A^{-1}B = BA^{-1}$ in $\mathcal{B}(X)$.
- (iv) Let $0 \in \text{iso } \sigma(A)$ and P be the spectral projection of A corresponding to 0 . Then $\mathcal{R}(P) \subset \mathcal{D}(A^n)$ for all $n \geq 1$. If $\mathcal{R}(B) \subset \mathcal{D}(A)$ and $ABx = BAx$ for all $x \in \mathcal{D}(A)$, then $BP = PB$.

Convention 1.3. Let $A \in \mathcal{C}(X)$ and $B \in \mathcal{B}(X)$ with $\mathcal{R}(B) \subset \mathcal{D}(A)$. For the sake of brevity we will write in the sequel

$$AB = BA \quad \text{to mean} \quad ABx = BAx \text{ for all } x \in \mathcal{D}(A).$$

2 The Drazin inverse for closed linear operators

We start with a definition of the Drazin inverse of a closed operator that subsumes the conventional Drazin inverse defined by Nashed and Zhao [12, Definition 2.1].

Definition 2.1. Let $A \in \mathcal{C}(X)$. An operator $B \in \mathcal{B}(X)$ is called a *Drazin inverse* of A if $\mathcal{R}(B) \subset \mathcal{D}(A)$, $\mathcal{R}(I - AB) \subset \mathcal{D}(A)$, and

$$BAB = B, \quad AB = BA, \quad \sigma(A(I - AB)) = \{0\}. \quad (2.1)$$

The *Drazin index* $i(A)$ is defined to be $i(A) = 0$ if A is invertible, $i(A) = q$ if A is not invertible and $A(I - AB)$ is nilpotent of index q , and $i(A) = \infty$ otherwise. (The index is well defined since there is at most one operator $B \in \mathcal{B}(X)$ satisfying (2.1)—see (2.3).) An operator $A \in \mathcal{C}(X)$ that possesses a Drazin inverse is called *Drazin invertible*, and its Drazin inverse is denoted by A^D .

Lemma 2.2. *Let $A \in \mathcal{C}(X)$ be Drazin invertible with a Drazin inverse $B \in \mathcal{B}(X)$. Then the operator $P = I - AB$ is a continuous projection such that:*

- (i) $AP = PA$.
- (ii) $\mathcal{R}(P) \subset \mathcal{D}(A^n)$ for all $n \geq 1$.

Proof. From $BAB = B$ we obtain $(AB)^2 = ABAB = AB$, which implies $P^2 = P$.

If $y \in \mathcal{D}(A)$, then $AB y = y - P y \in \mathcal{D}(A)$; by the second condition in (2.1),

$$AP y = A(y - AB y) = A(y - B A y) = (I - AB) A y = P A y.$$

This proves (i).

Suppose that $y = P x \in \mathcal{D}(A^{n-1})$ for some $n \geq 2$. Then $P y = y$, and by (i),

$$A y = AP y = P A y \in \mathcal{D}(A),$$

which implies $y \in \mathcal{D}(A^n)$. The result follows by induction. \square

We give some necessary and sufficient conditions for $A \in \mathcal{C}(X)$ to possess a Drazin inverse.

Theorem 2.3. *The following conditions on $A \in \mathcal{C}(X)$ are equivalent.*

- (i) $A \in \mathcal{C}(X)$ is Drazin invertible.
- (ii) $0 \notin \text{acc } \sigma(A)$.
- (iii) $A = A_1 \oplus A_2$, where A_1 is bounded and quasinilpotent and A_2 is closed and invertible.

Proof. Suppose that 0 is not an accumulation point of $\sigma(A)$. If $0 \in \rho(A)$, then A is invertible, and A^{-1} is a Drazin inverse of A . Suppose that $0 \in \sigma(A)$. Then $0 \in \text{iso } \sigma(A)$, and the spectral projection P of A corresponding to 0 satisfies conditions (i)–(iv) of Theorem 1.1 with $\xi = 1$. Set $B = (A + P)^{-1}(I - P)$. Since $\mathcal{D}(A + P) = \mathcal{D}(A)$, $\mathcal{R}(B) = \mathcal{R}((A + P)^{-1}(I - P)) \subset \mathcal{D}(A)$. The commutativity condition of (2.1) is clear.

Further, $AB = A(I - P)(A + P)^{-1} = (A + P)(I - P)(A + P)^{-1} = I - P$, and $ABA = (A + P)^{-1}(I - P)^2 = (A + P)^{-1}(I - P) = B$. Finally, AP is quasinilpotent, that is, $\sigma(A(I - AB)) = \{0\}$. Hence B satisfies (2.1).

Conversely, suppose that B is a Drazin inverse of A , and set $P = I - AB$. By Lemma 2.2, the operator $P = I - AB$ is a continuous projection satisfying the commutativity condition of Theorem 1.1. Further, $AP = A(I - BA)$ is quasinilpotent.

We need to verify that $A + P \in \mathcal{C}(X)$ is invertible. We have

$$(A + P)(B + P) = AB + AP + PB + P = I - P + AP + P = I + AP, \quad (2.2)$$

and $(B + P)(A + P)x = (I + AP)x$ for all $x \in \mathcal{D}(A)$. Since $I + AP$ is an invertible operator, so is $A + P$. By Theorem 1.1, $0 \notin \text{acc } \sigma(A)$. We observe that, for any $\xi \neq 0$, $(A + \xi P)B = I - P$, which implies $B = (A + \xi P)^{-1}(I - P)$.

Thus we have proved the equivalence of (i) and (ii). The equivalence of (ii) and (iii) follows from [15, Theorem V.9.2]. \square

From the preceding proof we obtain a useful explicit formula for the Drazin inverse A^{D} in terms of the spectral projection P of A at 0, and a proof of uniqueness of A^{D} :

$$A^{\text{D}} = (A + \xi P)^{-1}(I - P) \quad \text{for any } \xi \neq 0. \quad (2.3)$$

We also observe that $P = I - AA^{\text{D}}$.

If $A = A_1 \oplus A_2$ is the decomposition of a Drazin invertible operator $A \in \mathcal{C}(X)$ described in the preceding theorem, then

$$A^{\text{D}} = 0 \oplus A_2^{-1}. \quad (2.4)$$

Indeed,

$$A^{\text{D}} = (A + P)^{-1}(I - P) = ((A_1 + I)^{-1} \oplus A_2^{-1})(0 \oplus I) = 0 \oplus A_2^{-1}$$

by (2.3) with $\xi = 1$.

As a final result of this section we give a representation of the Drazin inverse in terms of the holomorphic calculus for a closed linear operator (see [15]) and a resulting expression for the spectrum of A^{D} .

Theorem 2.4. *If $A \in \mathcal{C}(X)$ is Drazin invertible, then*

$$A^{\text{D}} = f(A), \quad (2.5)$$

where f is a function holomorphic in an open neighborhood of $\sigma_e(A)$ equal to 0 in an open neighborhood of 0 and at ∞ , and to λ^{-1} for all λ in an open neighborhood of $\sigma(A) \setminus \{0\}$. If $i(A) > 0$, then

$$\sigma(A^{\text{D}}) = \{0\} \cup \{\lambda^{-1} : \lambda \in \sigma(A) \setminus \{0\}\}.$$

Proof. We assume that $0 \in \text{iso } \sigma(A)$. The spectral projection P of A corresponding to 0 can be expressed as $P = h(A)$, where h is a holomorphic function equal to 1 in an open neighborhood of 0 and to 0 in an open neighborhood of $\sigma_e(A) \setminus \{0\}$. According to (2.3), the Drazin inverse is given by $A^D = (A + P)^{-1}(I - P) = f(A)$, where $f(\lambda) = (\lambda + h(\lambda))^{-1}(1 - h(\lambda))$. From this expression for f we glean that $f(\lambda) = 0$ in an open neighborhood of 0 and $f(\lambda) = \lambda^{-1}$ in an open neighborhood of $\sigma_e(A) \setminus \{0\}$.

By the spectral mapping theorem [15, Theorem V.9.5], $\sigma(f(A)) = f(\sigma_e(A))$. If $\infty \in \sigma_e(A)$, then $f(\infty) = 0$, otherwise $\sigma_e(A) = \sigma(A)$. \square

3 Properties of the Drazin inverse

This section studies properties of the Drazin inverse for closed linear operators. For the bounded case we recover many of the results of [6]. However, not all properties of the Drazin inverse for bounded linear operators find their counterpart in the closed operator theory, as witnessed by Theorem 3.4 and Examples 3.5 and 3.6. In [12], the Drazin inverse A^d of a closed linear operator A is defined for the case when A has a finite index $i(A)$, and several properties of A^d are stated without proof. Clearly, if A^d exists, then so does A^D , and $A^d = A^D$. Theorems 2.3 and 2.4 generalize [12, Theorems 2.3 and 2.6], respectively. Further, [12, Theorems 2.5 and 2.9] are recovered from Theorems 3.2 and 3.3, respectively. We give full proofs of these results.

We begin with the Laurent series expansion for the resolvent of A in a punctured neighborhood of an isolated spectral point of A .

Theorem 3.1. *Let $A \in \mathcal{C}(X)$. If A is Drazin invertible, then there exist a punctured neighborhood Δ of 0 such that*

$$R(\lambda; A) = \sum_{n=0}^{\infty} \lambda^{-n-1} A^n P - \sum_{n=0}^{\infty} \lambda^n (A^D)^{n+1}, \quad \lambda \in \Delta, \quad (3.1)$$

where $P = I - AA^D$ is the spectral projection of A corresponding to 0.

Proof. Follows from Theorems 2.3, 1.1 and from the equation

$$(\lambda I - A)x = (\lambda I - AP)Px + (\lambda I - (A + P))(I - P)x$$

valid for all $x \in \mathcal{D}(A)$ and all λ in some punctured neighborhood Δ for which $\lambda I - (A + P)$ is invertible. Then

$$\begin{aligned} R(\lambda; A) &= (\lambda I - AP)^{-1}P + (\lambda I - (A + P))^{-1}(I - P) \\ &= \sum_{n=0}^{\infty} \lambda^{-n-1} A^n P - \sum_{n=0}^{\infty} \lambda^n ((A + P)^{-1}(I - P))^{n+1}. \end{aligned}$$

(Note that by Lemma 2.2, $A^n P$ is defined and bounded for all $n \geq 1$.) □

Theorem 3.2. *Let $A \in \mathcal{C}(X)$ be Drazin invertible and let $B \in \mathcal{B}(X)$ be such that $\mathcal{R}(B) \subset \mathcal{D}(A)$ and $AB = BA$. Then $A^D B = B A^D$ in $\mathcal{B}(X)$.*

Proof. Let P be the spectral projection of A at 0. By Lemma 1.2 (iv), $BP = PB$ in $\mathcal{B}(X)$. Hence

$$A^D B = (A + P)^{-1}(I - P)B = B(A + P)^{-1}(I - P) = B A^D$$

by (2.3) and Lemma 1.2 (iii). □

Theorem 3.3. *Let $A \in \mathcal{C}(X)$ be Drazin invertible. Then, for each $n \geq 1$, A^n is Drazin invertible, and $(A^n)^D = (A^D)^n$.*

Proof. Let $A = A_1 \oplus A_2$ with A_1 (bounded) quasinilpotent and A_2 (closed) invertible (see Theorem 2.3 (iii)). Then $A^n = A_1^n \oplus A_2^n$, for $n = 1, 2, \dots$, where A_1^n is quasinilpotent and A_2^n invertible. Hence A^n is Drazin invertible by Theorem 2.3, and

$$(A^n)^D = 0 \oplus (A_2^n)^{-1} = 0 \oplus (A_2^{-1})^n = (0 \oplus A_2^{-1})^n = (A^D)^n$$

for any $n \geq 1$. □

Theorem 3.4. *Let $A \in \mathcal{C}(X)$ be Drazin invertible with the Drazin index $i(A) > 0$. Then A^D is Drazin invertible if and only if $\sigma(A)$ is bounded.*

Proof. By Theorem 2.4, $\sigma(A^D) = \{0\} \cup \{\lambda^{-1} : \lambda \in \sigma(A) \setminus \{0\}\}$. According to Theorem 2.3, the operator A^D is Drazin invertible if and only if there exists a punctured neighborhood $\{\lambda : |\lambda| < r\}$ of 0 disjoint with $\sigma(A^D)$. This occurs if and only if $\sigma(A)$ is contained in $\{\lambda : |\lambda| \leq r^{-1}\}$. □

We give an example of an operator for which A^D is not Drazin invertible with $i(A) = \infty$ and with the ‘worst scenario’ spectrum.

Example 3.5. Consider the space ℓ^1 with a generic element $x = (\xi_1, \xi_2, \xi_3, \dots)$. The operator A_1 on ℓ^1 defined by

$$A_1x = (0, \xi_1, \frac{1}{2}\xi_2, \frac{1}{3}\xi_3, \dots)$$

belongs to $B(\ell^1)$, and is quasinilpotent but not nilpotent.

The right shift $T_2x = (0, \xi_1, \xi_2, \xi_3, \dots)$ on ℓ^1 is an injective bounded linear operator with $\sigma(T_2) = \{\lambda : |\lambda| \leq 1\}$. Its algebraic inverse A_2 is a closed linear operator with the domain $\mathcal{D}(A_2) = \{x \in \ell^1 : \xi_1 = 0\}$; A_2 is invertible in $C(\ell^1)$ with $A_2^{-1} = T_2$ and $\sigma(A_2) = \{\lambda : |\lambda| \geq 1\}$.

Define $A = A_1 \oplus A_2$ on $X = \ell^1 \oplus \ell^1$. Then $A \in \mathcal{C}(X)$, $\sigma(A) = \{0\} \cup \{\lambda : |\lambda| \geq 1\}$, A is Drazin invertible with $A^D = 0 \oplus T_2$ and $i(A) = \infty$. But A^D is not Drazin invertible since $\sigma(A^D) = \{\lambda : |\lambda| \leq 1\}$, and 0 is not an isolated spectral point of A .

In contrast with the unbounded case, the relation between a bounded operator and its Drazin inverse is more symmetrical. According to [6, Theorem 5.2], if $A \in \mathcal{B}(X)$ is Drazin invertible with $i(A) > 0$, then A^D is also Drazin invertible, both operators have the same spectral projection corresponding to 0, and $i(A^D) = 1$. In addition, $(A^D)^D = A$ if and only if $i(A) = 1$ [6, Theorem 5.3]. For a Drazin invertible closed operator A with Drazin invertible A^D , the spectral projections of A and A^D at 0 need not be the same. This is demonstrated in the following example.

Example 3.6. Let A_1 be as in Example 3.5, and let T_2 be defined on ℓ^1 by

$$T_2x = (\xi_1, 0, \frac{1}{2}\xi_2, \frac{1}{3}\xi_3, \dots).$$

Then $T_2 \in B(\ell^1)$ and $\sigma(T_2) = \{0, 1\}$. The operator T_2 is injective, and its algebraic inverse A_2 is a closed linear operator with the domain $\mathcal{D}(A_2) = \mathcal{R}(T_2)$; A_2 is invertible in $C(\ell^1)$ with $A_2^{-1} = T_2$ and $\sigma(A_2) = \{1\}$.

Let $A = A_1 \oplus A_2$ on $X = \ell^1 \oplus \ell^1$. Then $A \in \mathcal{C}(X)$, $\sigma(A) = \{0, 1\}$ and A is Drazin invertible with $A^D = 0 \oplus T_2$ and $i(A) = \infty$. Since $\sigma(A^D) = \{0, 1\}$, A^D itself is Drazin invertible. Let $P = I \oplus 0$ be the spectral projection of A at 0. We observe that $A^D + P = I \oplus T_2$ is not invertible in $\mathcal{B}(X)$ and, by Theorem 1.1, P is not the spectral projection of A^D .

Theorem 3.7. *Let $T \in \mathcal{C}(X)$ and $S \in \mathcal{B}(X)$ be such that T is Drazin invertible, S quasinilpotent, $\mathcal{R}(S) \subset \mathcal{D}(T)$ and $TS = ST$. Then the operator $T + S \in \mathcal{C}(X)$ is Drazin invertible with*

$$(T + S)^{\text{D}} = (T + S + P)^{-1}(I - P), \quad (3.2)$$

where P is the spectral projection of T corresponding to 0.

Proof. Let $T = T_1 \oplus T_2$ (relative to $X = X_1 \oplus X_2$) be the decomposition of a Drazin invertible operator T described in Theorem 2.3 (iii). By Lemma 1.2 (iv), $SP = PS$. Then $S = S_1 \oplus S_2$ relative to $X = X_1 \oplus X_2$ with S_i quasinilpotent and $T_i S_i = S_i T_i$ for $i = 1, 2$. Hence

$$T + S = (T_1 + S_1) \oplus (T_2 + S_2)$$

is Drazin invertible in view of Theorem 2.3 since $T_1 + S_1$ is quasinilpotent and $T_2 + S_2 = T_2(I + T_2^{-1}S_2)$ invertible. We note that P is the spectral projection of $T + S$ at 0, and hence $T + S + P = (T_1 + S_1 + I) \oplus (T_2 + S_2)$ is invertible. Finally,

$$(T + S + P)^{-1}(I - P) = 0 \oplus (T_2 + S_2)^{-1} = (T + S)^{\text{D}}$$

as $I - P = 0 \oplus I$. □

The proofs of the following two theorems are omitted since they are similar to the proofs of Theorems 5.6 and 5.5 in [5] (plus some consideration of domains).

Theorem 3.8. *Let $T \in \mathcal{C}(X)$, $S \in \mathcal{B}(X)$ be Drazin invertible operators such that $\mathcal{R}(S) \subset \mathcal{D}(T)$ and $TS = ST = 0$. Then $(T + S)^{\text{D}}$ exists and*

$$(T + S)^{\text{D}} = T^{\text{D}} + S^{\text{D}}. \quad (3.3)$$

Theorem 3.9. *Let $T \in \mathcal{C}(X)$, $S \in \mathcal{B}(X)$ be Drazin invertible operators such that $\mathcal{R}(S) \subset \mathcal{D}(T)$ and $TSx = STx$ for all $x \in \mathcal{D}(T)$. Then the operator $TS \in \mathcal{B}(X)$ is Drazin invertible, and*

$$(TS)^{\text{D}} = T^{\text{D}} S^{\text{D}}.$$

We close this section with an important decomposition of a Drazin invertible operator.

Theorem 3.10. *An operator $A \in \mathcal{C}(X)$ is Drazin invertible if and only if there exist $C \in \mathcal{C}(X)$ and $Q \in \mathcal{B}(X)$ such that*

- (i) $\mathcal{D}(C) = \mathcal{D}(A)$ and C is Drazin invertible with $i(C) \leq 1$,
- (ii) $Q \in \mathcal{B}(X)$ is quasinilpotent and $\mathcal{R}(Q) \subset \mathcal{D}(A)$,
- (iii) $A = C + Q$ and $CQ = QC = 0$.

Then $C^{\mathcal{D}} = A^{\mathcal{D}}$ and such a decomposition is unique.

Proof. Suppose first that (i)–(iii) hold. Then Q is Drazin invertible with $Q^{\mathcal{D}} = 0$, and we can apply Theorem 3.8 to conclude that A is Drazin invertible with $A^{\mathcal{D}} = C^{\mathcal{D}} + Q^{\mathcal{D}} = C^{\mathcal{D}}$. If P is the spectral projection of A , then, according to Theorem 1.1, AP is quasinilpotent and $A + P$ invertible. Then $PC = CP = AP - QP$ is quasinilpotent (the sum of commuting quasinilpotent operators), and $C + P = A + P - Q = (A + P)(I - (A + P)^{-1}Q)$ is invertible; hence P is also the spectral projection of C by Theorem 1.1. According to Theorem 2.3 we can write $A = A_1 \oplus A_2$ relative to $X = X_1 \oplus X_2$; then $C = C_1 \oplus C_2$ relative to $X = X_1 \oplus X_2$ with C_2 invertible and with $C_1 = 0$ since $i(C) \leq 1$. From $QP = PQ$ we obtain $Q = Q_1 \oplus Q_2$ relative to $X = X_1 \oplus X_2$. Then $0 = CQ = 0 \oplus C_2Q_2$, and the invertibility of C_2 implies $Q_2 = 0$. Consequently,

$$C = 0 \oplus A_2, \quad Q = A_1 \oplus 0,$$

which shows that the decomposition is unique.

Conversely, suppose that A is Drazin invertible. By Theorem 2.3, $A = A_1 \oplus A_2$, where A_1 is (bounded) quasinilpotent and A_2 (closed) invertible. Set $C = 0 \oplus A_2$ and $Q = A_1 \oplus 0$. The spectral projection of A at 0 is $P = I \oplus 0$. Then C is Drazin invertible and P is the spectral projection of C corresponding to 0.

(i) From $C = (I - P)A$ we have $\mathcal{D}(C) = \mathcal{D}(A)$. Further, $C^{\mathcal{D}} = 0 \oplus A_2^{-1} = A^{\mathcal{D}}$, and $i(C) = i(0 \oplus A_2) \leq 1$.

(ii) Since $Q = AP = PAP$, $\mathcal{R}(Q) \subset \mathcal{R}(P) \subset \mathcal{D}(A)$. Also, $\sigma(Q) = \sigma(A_1 \oplus 0) = \sigma(A_1) \cup \sigma(0) = \{0\}$.

(iii) Follows from the definition of C and Q . □

The operator C from the preceding theorem is called the *core part* of the Drazin invertible operator A . Its importance is seen from the following properties:

$$C^{\mathcal{D}} = A^{\mathcal{D}}, \quad \sigma(C) = \sigma(A), \quad R(C) = N(P), \quad N(C) = R(P), \quad (3.4)$$

where P is the spectral projection of A corresponding to 0. Only the spectral equality needs to be proved. We observe that, for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{D}(A)$,

$$\begin{aligned} (\lambda I - A)x &= (\lambda I - C)(I - P)x + \lambda(I - Q)Px, \\ (\lambda I - C)x &= (\lambda I - A)(I - P)x + \lambda Px. \end{aligned}$$

Hence $\lambda I - A$ is invertible whenever $\lambda \in \rho(C) \setminus \{0\}$, and $\lambda I - C$ is invertible whenever $\lambda \in \rho(A) \setminus \{0\}$. Consequently, $\sigma(C) \cup \{0\} = \sigma(A) \cup \{0\}$. Considering separately the cases $0 \in \sigma(A)$ and $0 \notin \sigma(A)$, we conclude that $\sigma(C) = \sigma(A)$.

4 C_0 -semigroups and the Drazin inverse of the infinitesimal generator

First we discuss some facts about C_0 -semigroup that will be needed in the sequel.

Lemma 4.1. *Let $T(t)$ be a C_0 -semigroup with the infinitesimal generator A and let $P \in \mathcal{B}(X)$ be a projection satisfying the commutativity condition $T(t)P = PT(t)$ for all $t \geq 0$. Then*

$$S(t) := T(t) \exp(-tP) = \exp(-tP)T(t), \quad t \geq 0, \quad (4.1)$$

is a C_0 -semigroup with the infinitesimal generator $A - P$.

Proof. The commutativity in (4.1) holds since $\exp(-tP) = I - P + e^{-t}P$ for $t \geq 0$. A direct verification shows that $S(t)$ is a C_0 -semigroup. Further, for each $x \in \mathcal{D}(A)$,

$$\left. \frac{d}{dt} \right|_0 S(t)x = \left[\left. \frac{d}{dt} \right|_0 T(t) \right] \exp(0P)x + T(0) \left. \frac{d}{dt} \right|_0 \exp(-tP) = Ax - Px,$$

which shows that $A - P$ is the infinitesimal generator of $S(t)$. \square

We give a representation of the Drazin inverse of the infinitesimal generator. This result generalizes [6, Theorem 6.3]. It will be applied to the study of the asymptotic behaviour of the solutions of a differential equation. In the following, the convergence of semigroups as $t \rightarrow \infty$ is understood in the operator norm topology.

Theorem 4.2. *Let $T(t)$ be a C_0 -semigroup with the infinitesimal generator A such that $T(t) \rightarrow P$ as $t \rightarrow \infty$. Then the following are true.*

- (i) $0 \notin \text{acc } \sigma(A)$ and P is the spectral projection of A corresponding to 0.
- (ii) There are constants $M > 0$ and $\mu > 0$ such that $\|T(t) - P\| \leq Me^{-\mu t}$ for all $t \geq 0$.
- (iii) For all $x \in X$ we have

$$\int_0^\infty T(t)(I - P)x \, dt = -A^D x. \quad (4.2)$$

Proof. We observe that $P^2 = P$. For any $t > 0$, $T(t)P = \lim_{s \rightarrow \infty} T(t)T(s) = \lim_{s \rightarrow \infty} T(t+s) = P$; hence

$$T(t)P = P = PT(t) \text{ for all } t \geq 0. \quad (4.3)$$

Differentiating this equation at 0, we get $APx = PAx = 0$ for all $x \in \mathcal{D}(A)$. By the preceding lemma, $S(t) = T(t) \exp(-tP)$ is the C_0 -semigroup generated by $A - P$. Expressing the exponential as a series, after a short calculation we obtain

$$T(t) = S(t) + (1 - e^{-t})P \quad \text{and} \quad T(t)(I - P) = S(t)(I - P). \quad (4.4)$$

We observe that $S(t) \rightarrow 0$ as $t \rightarrow \infty$. From [13, Proposition 1.2.2] we can deduce that there exist constants $K > 0$, $\mu > 0$ such that $\|S(t)\| \leq Ke^{-\mu t}$ for all $t \geq 0$. Hence $\|T(t) - P\| = \|T(t)(I - P)\| = \|S(t)(I - P)\| \leq \|I - P\|Ke^{-\mu t} \leq Me^{-\mu t}$ for all $t \geq 0$.

By the spectral inclusion [14, Theorem 2.2.3], $\sigma(A - P)$ lies in the open left half plane, which shows that $A - P$ is invertible. The conditions of Theorem 1.1 are satisfied, $0 \notin \text{acc } \sigma(A)$, and P is the spectral projection of A corresponding to 0. According to [14, Theorem 4.2.4 b)], for all $x \in X$ we have

$$\int_0^\infty S(t)x \, dt = -(A - P)^{-1}x \quad \text{for all } x \in X, \quad (4.5)$$

and, using (2.3) with $\xi = -1$,

$$\int_0^\infty T(t)(I - P)x \, dt = \int_0^\infty S(t)(I - P)x \, dt = -(A - P)^{-1}(I - P)x = -A^D x.$$

□

5 Applications to differential equations

In this section we consider the abstract Cauchy problem for the infinitesimal generator of a C_0 -semigroup and its singular perturbation. All functions f are defined on $[0, \infty)$ with values in X . We apply the results of previous sections to obtain a generalization of asymptotic theorems of Pazy [14, Chapter 4]. In particular, our first theorem generalizes [14, Theorem 4.4.4].

Theorem 5.1. *Let $T(t)$ be a C_0 -semigroup with the infinitesimal generator A such that $T(t) \rightarrow P$. Let f be bounded and Lebesgue measurable on $[0, \infty)$, and let Pf be integrable on $[0, \infty)$. If $\lim_{t \rightarrow \infty} f(t) = f_0$, then the mild solution $u(t)$ of the differential problem*

$$\frac{du}{dt} = Au(t) + f(t), \quad u(0) = x, \quad (5.1)$$

satisfies

$$\lim_{t \rightarrow \infty} u(t) = Px - A^D f_0 + \int_0^\infty Pf(s) ds. \quad (5.2)$$

Proof. The mild solution to the problem is given by

$$u(t) = T(t)x + \int_0^t T(t-s)f(s) ds.$$

By (4.3), $T(t)P = P = PT(t)$ for all $t \geq 0$. Let

$$u_1(t) = P \int_0^t T(t-s)f(s) ds \quad \text{and} \quad u_2(t) = (I - P) \int_0^t T(t-s)f(s) ds.$$

Then

$$u_1(t) = \int_0^t PT(t-s)f(s) ds = \int_0^t Pf(s) ds.$$

Since Pf is integrable on $[0, \infty)$,

$$\lim_{t \rightarrow \infty} u_1(t) = \lim_{t \rightarrow \infty} \int_0^t Pf(s) ds = \int_0^\infty Pf(s) ds.$$

Further,

$$u_2(t) = \int_0^t T(t-s)(I - P)(f(s) - f_0) ds + \int_0^t T(t-s)(I - P)f_0 ds$$

$$= v_1(t) + v_2(t).$$

By Theorem 4.2 there exist positive constants M, μ such that

$$\|T(t)(I - P)\| = \|T(t) - P\| \leq Me^{-\mu t} \quad \text{for all } t \geq 0.$$

Write $\|f\|_\infty = \sup_{t \geq 0} \|f(t)\|$. Let $\eta > 0$ and let t_0 be such that $\|f(s) - f_0\| < \eta$ if $s \geq t_0$. Then

$$\begin{aligned} \|v_1(t)\| &\leq \int_0^t \|T(t-s) - P\| \|f(s) - f_0\| ds \leq \int_0^t Me^{-\mu(t-s)} \|f(s) - f_0\| ds \\ &\leq \int_0^{t_0} Me^{-\mu(t-s)} 2\|f\|_\infty ds + \int_{t_0}^t Me^{-\mu(t-s)} \eta ds \\ &\leq 2M\|f\|_\infty \mu^{-1} (e^{-\mu(t-t_0)} - e^{-\mu t}) + \eta M \mu^{-1} (1 - e^{-\mu(t-t_0)}), \end{aligned}$$

and $\limsup_{t \rightarrow \infty} \|v_1(t)\| \leq \eta M \mu^{-1}$. Since $\eta > 0$ was arbitrary, $\lim_{t \rightarrow \infty} \|v_1(t)\| = 0$.

By Theorem 4.2,

$$\begin{aligned} \lim_{t \rightarrow \infty} v_2(t) &= \int_0^t T(t-s)(I - P)f_0 ds = \lim_{t \rightarrow \infty} \int_0^t T(\tau)(I - P)f_0 d\tau \\ &= \int_0^\infty T(\tau)(I - P)f_0 d\tau = -A^D f_0. \end{aligned}$$

Finally,

$$\lim_{t \rightarrow \infty} u(t) = \lim_{t \rightarrow \infty} T(t)x + \lim_{t \rightarrow \infty} \int_0^t T(t-s)f(s) ds = Px - A^D f_0 + \int_0^\infty Pf(s) ds.$$

□

Next, we derive conditions under which the mild solution of a singularly perturbed problem has limit as $\varepsilon \rightarrow 0+$. This result generalizes [14, Theorem 4.4.5]; see also [10].

Theorem 5.2. *Let $T(t)$ be a C_0 -semigroup with the infinitesimal generator A satisfying $T(t) \rightarrow P$ as $t \rightarrow \infty$. Let f be continuous and bounded on $[0, \infty)$. The limit as $\varepsilon \rightarrow 0+$ of the mild solution $u_\varepsilon(t)$ of the singularly perturbed problem*

$$\varepsilon \frac{du_\varepsilon(t)}{dt} = Au_\varepsilon(t) + f(t), \quad u_\varepsilon(0) = x, \quad \varepsilon > 0, \quad (5.3)$$

exists if and only if $Pf(t) = 0$ for all $t \geq 0$. If this is the case, then

$$u(t) := \lim_{\varepsilon \rightarrow 0^+} u_\varepsilon(t) = Px - A^D f(t), \quad (5.4)$$

where A^D is the Drazin inverse of A . The limit u is a solution of the reduced equation

$$0 = Au(t) + f(t), \quad u(0) = Px - A^D f(0). \quad (5.5)$$

Proof. The mild solution to (5.3) is given by

$$u_\varepsilon(t) = T_\varepsilon(t)x + \varepsilon^{-1} \int_0^t T_\varepsilon(t-s)f(s) ds,$$

where $T_\varepsilon(t) = T(t/\varepsilon)$. From (4.3) we deduce that $T_\varepsilon(t)P = P = PT_\varepsilon(t)$ for all $t \geq 0$. Let

$$u_{1\varepsilon}(t) = \varepsilon^{-1} P \int_0^t T_\varepsilon(t-s)f(s) ds, \quad u_{2\varepsilon}(t) = \varepsilon^{-1} (I - P) \int_0^t T_\varepsilon(t-s)f(s) ds.$$

Then $u_{1\varepsilon}(t) = \varepsilon^{-1} \int_0^t Pf(s) ds$, and $\lim_{\varepsilon \rightarrow 0^+} u_{1\varepsilon}(t)$ exists pointwise for $t \geq 0$ if and only if $\int_0^t Pf(s) ds = 0$ for all $t \geq 0$; this occurs if and only if $Pf(t) = 0$ for all $t \geq 0$. Write

$$\begin{aligned} u_{2\varepsilon}(t) &= \varepsilon^{-1} \int_0^t T_\varepsilon(t-s)(I - P)(f(s) - f(t)) ds + \varepsilon^{-1} \int_0^t T_\varepsilon(t-s)(I - P)f(t) ds \\ &= v_{1\varepsilon}(t) + v_{2\varepsilon}(t). \end{aligned}$$

By Theorem 4.2, there are constants $M > 0$, $\mu > 0$ such that

$$\|T_\varepsilon(t)(I - P)\| = \|T_\varepsilon(t) - P\| \leq Me^{-\mu t/\varepsilon} \text{ for all } t \geq 0.$$

Keep $t > 0$ fixed. If $\eta > 0$, choose $t_0 \in (0, t)$ such that $\|f(s) - f(t)\| < \eta$ if $t_0 \leq s \leq t$. Similarly as in the preceding proof,

$$\begin{aligned} \|v_{1\varepsilon}(t)\| &\leq \varepsilon^{-1} \int_0^t Me^{-\mu(t-s)/\varepsilon} \|f(s) - f(t)\| ds \\ &= \varepsilon^{-1} \int_0^{t_0} Me^{-\mu(t-s)/\varepsilon} 2\|f\|_\infty ds + \varepsilon^{-1} \int_{t_0}^t Me^{-\mu(t-s)/\varepsilon} \eta ds \\ &\leq 2M\mu^{-1}\|f\|_\infty (e^{-\mu(t-t_0)/\varepsilon} - e^{-\mu t/\varepsilon}) + \eta M\mu^{-1}(1 - e^{-\mu(t-t_0)/\varepsilon}). \end{aligned}$$

Hence $\limsup_{\varepsilon \rightarrow 0^+} \|v_{1\varepsilon}(t)\| \leq \eta M \mu^{-1}$. Since $\eta > 0$ was arbitrary, we conclude that $\lim_{\varepsilon \rightarrow 0^+} \|v_{1\varepsilon}(t)\| = 0$.

Therefore,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} v_{2\varepsilon}(t) &= \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \int_0^t T_\varepsilon(t-s)(I-P)f(t) ds \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_0^{t/\varepsilon} T(\tau)(I-P)f(t) d\tau \\ &= \int_0^\infty T(\tau)(I-P)f(t) d\tau = -A^D f(t) \end{aligned}$$

by Theorem 4.2. Therefore we conclude that

$$\lim_{\varepsilon \rightarrow 0^+} u_\varepsilon(t) = Px - A^D f(t)$$

if and only if $Pf(t) = 0$ for all $t \geq 0$. Under this assumption we have

$$Au(t) + f(t) = A(Px - A^D f(t)) + f(t) = Pf(t) = 0,$$

and $u(0) = Px - A^D f(0)$, which completes the proof. \square

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