THE DRAZIN AND MOORE–PENROSE INVERSE IN C^*-ALGEBRAS

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ABSTRACT

In this paper we give an explicit representation of the Moore–Penrose inverse in a C^*-algebra in terms of the Drazin inverse of a quasipolar element, and derive properties of the Moore–Penrose inverse from theory of the Drazin inverse. Results include an alternative proof that regularity implies Moore–Penrose invertibility, and a simplified proof of the continuity theorem for the Moore–Penrose inverse.

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1 The Drazin inverse

Let A be a unital Banach algebra with unit e. If $a \in A$, then $\sigma(a)$ and $\operatorname{acc}\sigma(a)$ denote the spectrum and the set of all accumulation points of $\sigma(a)$, respectively; a is *quasipolar* if $0 \notin \operatorname{acc}\sigma(a)$, and *polar* if it is quasipolar and 0 is at most a pole of the resolvent $R(\lambda; a) = (\lambda e - a)^{-1}$ of a. In particular, a is *simply polar* if 0 is at most a simple pole of $R(\lambda; a)$. Further, a is *regular* if $a \in aAa$, that is, if there is $x \in A$ such that $a = axa$; the set of all regular element of A is denoted by $\overline{A}$. If $B \subset A$, the set of all $x \in A$ such that $bx = xb$ for all $b \in B$ is denoted by $\operatorname{comm}(B)$, and the set of all $x \in A$ such that $bx = xb$ for all $b \in \operatorname{comm}(B)$ by $\operatorname{comm}^2(B)$. By A^{-1}, $\dot{A}$, $QN(A)$ and $N(A)$ we denote the sets of all invertible, idempotent, quasinilpotent and nilpotent elements of A, respectively. This notation and terminology follows essentially Harte [7].

Definition 1.1 [11, Definition 4.1]; see also [7, Section 7.5]. We say that an element $a \in A$ is *Drazin invertible* if there is $x \in A$ such that

$$x \in \operatorname{comm}(a), \quad ax^2 = x, \quad a(e - ax) \in QN(A). \tag{1.1}$$

For any Drazin invertible $a \in A$ such x is unique; we write $x = a^D$, and call it the *Drazin inverse* of a. We write A^D for the set of all Drazin invertible elements of A. The *Drazin index* $i(a)$ of $a \in A^D$ is equal to 0 if $a \in A^{-1}$, to q if $a \notin A^{-1}$ and $a(e - aa^D)$ is nilpotent of index q, and to ∞ otherwise. (Drazin's original definition [4] requires $a(e - aa^D) \in N(A)$.)

Theorem 1.2 [11, Theorem 4.2] *The following conditions are equivalent.*

(i) $a \in A^D$.

(ii) $a \in A$ *is quasipolar.*

(iii) *There exists* $p \in \dot{A} \cap \operatorname{comm}(a)$ *such that* $ap \in QN(A)$ *and* $a + p \in A^{-1}$.

The unique idempotent p satisfying (iii) of the preceding theorem is the *spectral idempotent* of a quasipolar element a at 0 (see also [10]); we will write $p = a^\pi$.

For the main properties of the Drazin inverse we depend on Koliha [11]. If $a \in A^D$ with $p = a^\pi$, then

$$a^D = (a+p)^{-1}(e-p), \qquad p = e - a^D a. \tag{1.2}$$

We mention a representation of a^D based on the holomorphic calculus for a: If a is quasipolar and f a function holomorphic in an open neighborhood of $\sigma(a)$ with $f(\lambda) = 0$ in a neighborhood of 0 and $f(\lambda) = \lambda^{-1}$ in a neighborhood of $\sigma(a) \setminus \{0\}$, then

$$a^D = f(a). \tag{1.3}$$

The representation shows that $a^D \in \operatorname{comm}^2(a)$.

If a is quasipolar with $p = a^\pi$, we have a Laurent expansion for the resolvent $R(\lambda; a)$ in some punctured neighborhood of 0 [11, Theorem 5.1]:

$$R(\lambda; a) = \sum_{n=0}^{\infty} \lambda^{-n-1} a^n p - \sum_{n=0}^{\infty} \lambda^n (a^D)^{n+1}, \quad 0 < |\lambda| < r. \tag{1.4}$$

A quasipolar element a is polar of order not exceeding $k \in \mathbb{N}$ if and only if $a^k p = 0$.

The Drazin inverse and the conditions of Theorem 1.2 can be used to give short proofs of certain spectral results for elements of a C^*-algebra, such as the well known

Lemma 1.5 below. We give three results that will be needed later, assuming for the rest of the section that A is a unital C^*-algebra. Recall that, in addition to the standard properties of involution $x \mapsto x^*$ we have $\|a^*\| = \|a\|$ for all a, and the B^*-condition

$$\|a^*a\| = \|a\|^2.$$

Lemma 1.3 *If $a \in A^D$ then $a^* \in A^D$ and $(a^*)^D = (a^D)^*$.*

Proof Let $x = a^D$. From (1.1) it follows that $x^* \in \operatorname{comm}(a^*)$, $a^*(x^*)^2 = x^*$ and $a^*(e - a^*x^*) \in QN(A)$; the conclusion by the uniqueness of the Drazin inverse.

Lemma 1.4 *The Drazin inverse of a normal (self-adjoint) element is normal (self-adjoint).*

Proof By [11, Theorem 5.5], $ab = ba$ implies $a^Db^D = (ab)^D = b^Da^D$; setting $b = a^*$ and applying the preceding lemma, we get the result. The self-adjoint case follows by direct verification.

Lemma 1.5 *A normal quasipolar element $a \in A$ is simply polar.*

Proof Suppose that a is quasipolar, and let $p = e - aa^D$ be the spectral idempotent of a corresponding to 0. Then p is normal by the preceding lemma; since the spectrum of p is real, p is in fact self-adjoint, and $ap = pa$ is normal. Since $ap \in QN(A)$ by Theorem 1.2, $\|ap\| = r(ap) = 0$, and $ap = 0$. This implies that a is simply polar.

2 Moore–Penrose inverse in a C^* algebra

In this section A is a unital C^*-algebra. We hope to demonstrate that the Drazin inverse is a useful unifying tool offering significant simplification of arguments in spectral theory and the Moore–Penrose inverse theory.

Definition 2.1 An element $a \in A$ is *Moore–Penrose invertible* if there exists $x \in A$ such that

$$xax = x, \quad axa = a, \quad (ax)^* = ax, \quad (xa)^* = xa. \tag{2.1}$$

There is at most one element x satisfying (2.1); this is a well known result of Penrose [14]. If a is Moore–Penrose invertible, the unique solution of (2.1) is called the *Moore–Penrose inverse* of a and is denoted by $a^\dagger$. The set of all Moore–Penrose invertible elements of A is denoted by $A^\dagger$. (By Theorem 2.8 below this notation is redundant as it will be shown that $A^\dagger = \overline{A}$.)

The following proposition gives a necessary and sufficient condition for $a^\dagger$ to commute with a. Our result supplements the conditions obtained by Brock [2] for bounded operators and by Harte and Mbekhta [9, Theorem 10] for C^*-algebras. The result is known for matrices (Campbell and Meyer [3, Theorems 4.3.1 and 7.3.4]).

Proposition 2.2 *Let $a \in A^\dagger$. Then*

$$a^\dagger a = aa^\dagger \iff \left[a \in A^D \text{ and } a^\dagger = a^D\right] \implies a \text{ is simply polar.}$$

Proof Suppose that $a^\dagger a = aa^\dagger$; together with $a^\dagger a a^\dagger = a^\dagger$ and $a(e - a^\dagger a) = 0 \in N(A)$ this shows that $x = a^\dagger$ satisfies (1.1), and establishes the simple polarity of a. If $a^\dagger = a^D$, then $a^\dagger a = aa^\dagger$ follows from $a^D \in \operatorname{comm}(a)$.

Example 2.3 Simply polar elements are regular and therefore Moore–Penrose invertible (see Theorem 2.8 below), but the simple polarity of a is not sufficient to ensure that $a^\dagger = a^D$. For a counterexample consider an idempotent element $a \in A$; then a is simply polar with $a^D = a$. However, $a^D = a^\dagger$ is equivalent to $a^* = a$: If $a^D = a^\dagger$, then $a = a^2 = a^D a = a^\dagger a$ is selfadjoint. Conversely, if a is selfadjoint, then (2.1) is satisfied with $x = a$, and hence $a^\dagger = a = a^D$.

Our main existence theorem for the Moore–Penrose inverse follows. Equivalence (i) $\iff$ (ii) is [8, Theorem 7]. We recall that quasipolarity is equivalent to the existence of the Drazin inverse.

Theorem 2.4 *The following conditions on $a \in A$ are equivalent:*

(i) *a has a Moore–Penrose inverse.*

(ii) *a^*a (respectively aa^*) has a Moore–Penrose inverse.*

(iii) a^*a (*respectively* aa^*) *is quasipolar.*

(iv) a^*a (*respectively* aa^*) *is simply polar.*

Proof (i) $\Longrightarrow$ (ii) If $x = a^\dagger$, then $ax = x^*a^*$ and $xa = a^*x^*$. We show that xx^* is the Moore–Penrose inverse of a^*a. The equalities

$$a^*axx^*a^*a = a^*axaxa = a^*a, \qquad x^*xaa^*x^*x = x^*xaxax = x^*x$$

prove the first two equations in (2.1). Further,

$$a^*axx^* = a^*x^*a^*x^* = a^*x^* = xa, \qquad xx^*a^*a = xaxa = xa;$$

since xa is self-adjoint, the remaining two equations in (2.1) are true.

(ii) $\Longrightarrow$ (iii) Write $b = a^*a$ and $x = b^\dagger$. A direct verification shows that x^* satisfies the defining relations (2.1) with a replaced by b. By the uniqueness of the Moore–Penrose inverse, $x = x^*$. Then

$$bx = x^*b^* = xb.$$

The preceding lemma implies that $b \in A^D$, and hence b is quasipolar by Theorem 1.2.

(iii) $\Longrightarrow$ (iv) Lemma 1.5.

(iv) $\Longrightarrow$ (i) Let a^*a be simply polar; we prove that the element $x = (a^*a)^D a^*$ is the Moore–Penrose inverse of a. First we observe that

$$xax = (a^*a)^D a^*a(a^*a)^D a^* = (a^*a)^D a^* = x.$$

From the simple polarity of a^*a we get $a^*a = a^*a(a^*a)^D a^*a = a^*axa$. Then

$$\begin{aligned}(a - axa)^*(a - axa) &= (a^* - a^*ax)(a - axa)\\ &= a^*a - a^*axa - a^*axa + a^*axaxa\\ &= a^*a - a^*a - a^*a + a^*a = 0.\end{aligned}$$

By the B^*-condition, $\|a - axa\|^2 = \|(a - axa)^*(a - axa)\| = 0$, so that $a - axa = 0$, and $axa = a$. Identities $(ax)^* = ax$ and $(xa)^* = xa$ follow by direct verification.

The statements about aa^* follow by symmetry; in the last implication we have to prove that $x = a^*(aa^*)^D$ is the Moore–Penrose inverse of a.

The implication (iv) $\Longrightarrow$ (i) in the preceding theorem leads to an explicit formula for the Moore–Penrose inverse in terms of the Drazin inverse which is a generalization of the identity $a^{-1} = (a^*a)^{-1}a^* = a^*(aa^*)^{-1}$ valid for every $a \in A^{-1}$.

Theorem 2.5 *An element a of a C^*-algebra A is Moore–Penrose invertible if and only if a^*a (respectively aa^*) is Drazin invertible. If $a \in A^\dagger$, then*

$$a^\dagger = (a^*a)^D a^* = a^*(aa^*)^D. \tag{2.2}$$

The spectral idempotents of a^*a and aa^* are given by $(a^*a)^\pi = e - a^\dagger a$ and $(aa^*)^\pi = e - aa^\dagger$.

Proof We only need to prove the equality involving the spectral idempotents. By (1.2) and (2.2), $(a^*a)^\pi = e - (a^*a)^D a^*a = e - a^\dagger a$. The other equality follows by symmetry.

Note 2.6 The preceding theorem is false when the Moore–Penrose and Drazin inverses are replaced by the ordinary inverse. In that case we have only

$$a \in A^{-1} \iff a^*a \in A^{-1} \text{ and } aa^* \in A^{-1}.$$

An example of an element $a \in A$ for which $a^*a \in A^{-1}$ and $aa^* \notin A^{-1}$ is provided by the right shift a in $A = \ell^2$.

Next we apply Theorem 2.4 to show that the idempotents are Moore–Penrose invertible. This result is essential for establishing the equality of $A^\dagger$ and $\overline{A}$.

Proposition 2.7 *If $p \in A$ is idempotent, then p^*p is simply polar, and $p \in A^\dagger$.*

Proof Set $t = e - (p - p^*)^2 = e + (p - p^*)^*(p - p^*)$; then t is positive, $t \in A^{-1}$ and $t \in \operatorname{comm}(p, p^*, p^*p)$. We show that $w = e - p^*pt^{-1}$ is the spectral projection of p^*p. From $p^*pt = tp^*p = (p^*p)^2$ we deduce $(p^*pt^{-1})^2 = (p^*p)^2t^{-2} = p^*pt^{-1}$. Hence $w^2 = w$ and $p^*pw = 0 = wp^*p$. Further,

$$p^*p + w = e + p^*(e - t^{-1})p = e + (sp)^*sp,$$

where $s = t^{-1/2}(p - p^*)$. Therefore $p^*p + w \in A^{-1}$, and the simple polarity of p^*p follows from Theorem 1.2; hence $p \in A^\dagger$ by Theorem 2.4.

This proposition holds the key to an alternative proof of the following important result of the Moore–Penrose inverse theory.

Theorem 2.8 [8, Theorem 6] *An element of a C^*-algebra A is Moore–Penrose invertible if and only if it is regular, that is, $A^\dagger = \overline{\overline{A}}$.*

Proof If a is regular, then there exists $b \in A$ such that $aba = a$ and $bab = b$; $p = ba$ and $q = ab$ are idempotents. The elements $u = p^\dagger p$ and $v = qq^\dagger$ exist by the preceding proposition. It can be verified directly that u, v are selfadjoint idempotents with $au = a = va$, and that the element $x = ubv$ satisfies conditions (2.1).

Conversely, every $a \in A^\dagger$ is regular.

Note 2.9 If $p \in A$ is idempotent, then $u = p^\dagger p$ is a selfadjoint idempotent satisfying $up = u$ and $pu = p$. This gives a simple solution to the problem of finding a selfadjoint idempotent u such that $Au = Ap$. An explicit construction of $p^\dagger p$ is given in [8, Theorem 6] for C^*-algebras; $pp^\dagger$ is constructed in [1, p. 55] for operators on Hilbert spaces.

Note 2.10 Harte and Mbekhta [8, Theorems 6 and 8] proved that

$$a \in A^\dagger \iff \overline{aA} = aA \iff \overline{Aa} = Aa,$$

where $\overline{B}$ denotes the closure of $B \subset A$. Their elegant proof is based on square roots of positive elements in a C^*-algebra.

Theorems 2.4 and 2.5 enable us to use known properties of the Drazin inverse to derive further facts about the Moore–Penrose inverse. We start by recovering two results due to Harte and Mbekhta.

Proposition 2.11 *Let $a \in A$ be normal. Then the following are true.*

(i) $a \in A^\dagger \iff a \in A^D$.

(ii) [8, Theorem 10] *If $a \in A^\dagger$, then $a^\dagger$ is normal and commutes with a.*

Proof Let $a^*a = aa^*$. If $a \in A^\dagger$, then

$$a^\dagger a = (a^*a)^D a^*a = (aa^*)^D aa^* = aa^*(aa^*)^D = aa^\dagger.$$

Hence $a^\dagger = a^D$ by Proposition 2.2, and the normality of $a^\dagger$ follows from Lemma 1.4.

If $a \in A^D$, then $a^* \in A^D$ by Lemma 1.3, and $a^*a \in A^D$ by [11, Theorem 5.5]. Hence $a \in A^\dagger$ by Theorem 2.5.

Proposition 2.12 [8, Theorem 5] *If $a \in A^\dagger$, then $a^\dagger \in \mathrm{comm}^2(a, a^*)$.*

Proof By [11, Theorem 5.5], $(a^*a)^D \in \mathrm{comm}^2(a^*a)$. If $x \in \mathrm{comm}(a, a^*)$, then

$$a^\dagger x = (a^*a)^D a^*x = x(a^*a)^D a^* = xa^\dagger.$$

The following proposition gives sufficient conditions for the product to be Moore–Penrose invertible.

Proposition 2.13 *Let $a, b \in A^\dagger$ with $b \in \mathrm{comm}(a, a^*)$. Then $ab \in A^\dagger$, and*

$$a^\dagger b = ba^\dagger, \quad ab^\dagger = b^\dagger a, \quad (ab)^\dagger = a^\dagger b^\dagger = b^\dagger a^\dagger. \tag{2.3}$$

Proof We observe that $b \in \mathrm{comm}(a, a^*)$ implies $a \in \mathrm{comm}(b, b^*)$, and the first two equalities then follow from the preceding proposition. Since also $b^* \in \mathrm{comm}(a, a^*)$, we have $a^\dagger b^* = b^*a^\dagger$, and $a^\dagger \in \mathrm{comm}(b, b^*)$. By the preceding proposition, $a^\dagger b^\dagger = b^\dagger a^\dagger$. Since the sets $\{a, a^\dagger\}$ and $\{b, b^\dagger\}$ commute,

$$aba^\dagger b^\dagger ab = aa^\dagger abb^\dagger b = ab, \quad a^\dagger b^\dagger aba^\dagger b^\dagger = a^\dagger aa^\dagger b^\dagger bb^\dagger = a^\dagger b^\dagger,$$
$$(a^\dagger b^\dagger ab)^* = (a^\dagger ab^\dagger b)^* = b^\dagger ba^\dagger a = a^\dagger b^\dagger ab,$$
$$(aba^\dagger b^\dagger)^* = (aa^\dagger bb^\dagger)^* = bb^\dagger aa^\dagger = aba^\dagger b^\dagger.$$

This proves $(ab)^\dagger = a^\dagger b^\dagger$.

Proposition 2.14 *Let $a, b \in A^\dagger$ with a normal and $ab = ba$. Then $ab \in A^\dagger$ and* (2.3) *holds.*

Proof Since a is normal and $ab = ba$, then also $a^*b = ba^*$ by Fuglede's theorem [7, Theorem 9.6.7]. Then $b \in \mathrm{comm}(a, a^*)$, and the result follows from the preceding proposition.

3 Representations of the Moore–Penrose inverse

If a is a bounded linear operator on a Hilbert space X, than $a^\dagger$ can be expressed as $a^\dagger = \widetilde{a}^{-1}a^*$ [6], where $\widetilde{a}$ is the restriction of a^*a to the range a^*X; other representations of $a^\dagger$ also rely on the restricted operator $\widetilde{a}$ [6]. In C^*-algebras we have to use a different device. This is provided by Lemma 3.1 below and by ensuring that 0 is always in the domain of any continuous function φ used to calculate $\varphi(a^*a)$ and $\varphi(aa^*)$.

Throughout this section we assume that $a \in A^\dagger$; then a^*a is self-adjoint, positive and simply polar, and $\sigma(a^*a) \cup \{0\} = \sigma(aa^*) \cup \{0\}$. However, 0 may be contained in only one of the sets $\sigma(a^*a)$ and $\sigma(aa^*)$ (see Note 2.6). For that reason we use functions continuous on $\Sigma = \sigma(a^*a) \cup \{0\}$ when applying the Gelfand–Naimark calculus for the self-adjoint elements a^*a and aa^*.

In this section we use the following notation.

Σ	the set $\sigma(a^*a) \cup \{0\} = \sigma(aa^*) \cup \{0\}$
$p = (a^*a)^\pi$	the spectral idempotent of a^*a at 0
$q = (aa^*)^\pi$	the spectral idempotent of aa^* at 0
$\chi(\lambda)$	the characteristic function of $\{0\}$ in Σ
$\varphi(\lambda)$	the function in $C(\Sigma)$ defined by
	$\varphi(0) = 0$ and $\varphi(\lambda) = \lambda^{-1}$ if $\lambda \in \Sigma \setminus \{0\}$

Lemma 3.1 *If $a \in A^\dagger$, then $(e-p)a^* = a^*$ and $a^*(e-q) = a^*$.*

Proof We have $(e-p)a^* = a^\dagger aa^* = a^*(a^\dagger)^*a^* = (aa^\dagger a)^* = a^*$. The second equality is obtained by symmetry.

The Gelfand–Naimark calculus is consistent with the holomorphic calculus for the self-adjoint (and simply polar) element a^*a. Hence $(a^*a)^D = f(a^*a) = \varphi(a^*a)$, where f is holomorphic in an open neighborhood of $\sigma(a^*a)$ with $f(\lambda) = 0$ in a neighborhood of 0 and $f(\lambda) = \lambda^{-1}$ in a neighborhood of $\sigma(a^*a) \setminus \{0\}$ (see formula (1.3)). Similarly, $(aa^*)^D = \varphi(aa^*)$. In view of (2.2),

$$a^\dagger = \varphi(a^*a)a^* = a^*\varphi(aa^*). \tag{3.1}$$

The following result, an application of (3.1), gives a useful representation of the Moore–Penrose inverse in terms of the ordinary inverse.

Theorem 3.2 *If $a \in A^\dagger$, then*

$$a^\dagger = (a^*a + \xi p)^{-1}a^* = a^*(aa^* + \xi q)^{-1} \text{ for any } \xi \neq 0. \tag{3.2}$$

Proof Let $\xi \neq 0$ and let $g(\lambda) = (\lambda + \xi\chi(\lambda))^{-1}(1 - \chi(\lambda))$ for all $\lambda \in \Sigma$. A direct verification shows that $g = \varphi$, where φ is defined above. Using the Gelfand–Naimark calculus, Lemma 3.1 and (3.1), we get

$$(a^*a + \xi p)^{-1}a^* = (a^*a + \xi p)^{-1}(e - p)a^* = g(a^*a)a^* = \varphi(a^*a)a^* = a^\dagger.$$

The other part of the formula is obtained by symmetry.

Our next result generalizes the main representation theorem of Groetsch [6, Theorem 2.3.1].

Theorem 3.3 *Let $a \in A^\dagger$. If (f_α) is a net in $C(\Sigma)$ convergent uniformly on Σ to the function φ, then*

$$a^\dagger = \lim_\alpha f_\alpha(a^*a)a^* = \lim_\alpha a^* f_\alpha(aa^*). \tag{3.3}$$

Proof Follows from (3.1) and the norm continuity of the Gelfand–Naimark calculus.

The preceding theorem is very versatile in producing a variety of representations of the Moore–Penrose inverse by specifying the net (f_α) (see Groetsch [5, 6]). We give several examples.

Example 3.4 If $a \in A^\dagger$, then

$$a^\dagger = \lim_{\alpha\to 0}(a^*a + \alpha e)^{-1}a^* = \lim_{\alpha\to 0} a^*(aa^* + \alpha e)^{-1}. \tag{3.4}$$

To obtain the formula, for any $\alpha \neq 0$ and any $\lambda \in \Sigma$ we set $f_\alpha(\lambda) = (\lambda + \alpha)^{-1}(1 - \chi(\lambda))$. We have $f_\alpha(0) = 0$ for all α, and $\lim_{\alpha\to 0} f_\alpha(\lambda) = \lambda^{-1}$ if $\lambda \neq 0$, with the convergence uniform on Σ. The first part of the formula follows from

$$(a^*a + \alpha e)^{-1}a^* = (a^*a + \alpha e)^{-1}(e - p)a^* = f_\alpha(a^*a),$$

the second part is obtained by symmetry.

Example 3.5 The integral representation of Showalter [17] (see also [6, Corollary 2.3.3]):

$$a^\dagger = \int_0^\infty \exp(-ta^*a)a^*\,dt = \int_0^\infty a^*\exp(-taa^*)\,dt.$$

The first part is obtained from $f_\alpha(\lambda) = \int_0^\alpha e^{-t\lambda}(1-\chi(\lambda))\,dt$ via Theorem 3.3 and Lemma 3.1.

Example 3.6 The series representation [6, 17]:

$$a^\dagger = \beta\sum_{n=0}^\infty (e-\beta a^*a)^n a^* = \beta\sum_{n=0}^\infty a^*(e-\beta aa^*)^n,$$

where $\beta \in (0, 2\|a\|^{-2})$. The first part from $f_\alpha(\lambda) = \beta\sum_{n=0}^\alpha (1-\beta\lambda)^n(1-\chi(\lambda))$ using Theorem 3.3, Lemma 3.1 and [6, pp. 62-63].

The representations of this section were essentially all based on the formula

$$a^\dagger = (a^*a)^D a^* = a^*(aa^*)^D.$$

We address the converse problem of expressing the Drazin inverse in terms of the Moore–Penrose inverse, answering thus a question raised by the referee.

Proposition 3.7 *Let $a \in A^D$ with $i(a)$ finite. If $a^{2m+1} \in A^\dagger$ for some $m \ge i(a)$, then*

$$a^D = a^m(a^{2m+1})^\dagger a^m. \tag{3.5}$$

Proof We can prove a slightly stronger result. Let $a^{2m+1} = a^{2m+1}xa^{2m+1}$ for some $x \in A$. If $b = a^D$, then $ab = ba$ and $a^s b^{s+1} = b$, $a^{m+s}b^s = a^m$ for all $s \in \mathbb{N}$. Hence $a^m x a^m = b^{m+1}a^{2m+1}xa^{2m+1}b^{m+1} = b^{2m+2}a^{2m+1} = b = a^D$.

The preceding result cannot be much improved since, unlike Drazin invertibility, Moore–Penrose invertibility is not preserved under taking the positive powers of the element. However, we can produce a somewhat more satisfying result when we use the concept of the *core part* c of a Drazin invertible element $a \in A$. We recall [11, Theorem 6.4] that each Drazin invertible element a has a unique decomposition $a = c + u$, where $cu = uc = 0$, $i(c) \le 1$, u is quasinilpotent and $a^D = c^D$. Since c has a commuting generalized inverse, $c^m \in \overline{A}$ for all $m \in \mathbb{N}$. Applying the preceding result, we have

Proposition 3.8 *Let $a \in A^D$ and let c be the core part of a. Then*

$$a^D = c(c^3)^\dagger c.$$

4 Continuity of the Moore–Penrose inverse

The purpose of this section is to present an elementary proof of the fundamental theorem on the continuity of the Moore–Penrose inverse in a C^*-algebra based on the corresponding result for the Drazin inverse. The simplicity of the arguments in this section underlines the advantages of treating the Moore–Penrose inverse through the Drazin inverse. Koliha and Rakočević [12] recently proved the continuity result for the (generalized) Drazin inverse for Banach algebras, with refinements for bounded linear operators on Banach spaces. However, only the following simple result is needed here.

Theorem 4.1 *Let A be a Banach algebra, $a_n, a \in A^D$, and let $a_n \to a$. Then*

$$a_n^D \to a^D \iff a_n^D a_n \to a^D a.$$

Proof Suppose that $a_n^D \to a^D$. Then $a_n^D a_n \to a^D a$ by the continuity of multiplication in A. Conversely, suppose that $a_n^D a_n \to a^D a$. By (1.2),

$$a_n^D = (a_n + p_n)^{-1}(e - p_n) \to (a + p)^{-1}(e - p) = a^D,$$

where $p_n = e - a_n^D a_n$, $p = e - a^D a$ are the spectral idempotents of a_n, a at 0.

We can now give a simple proof of the main result on the continuity of the Moore–Penrose inverse that appeared with minor variations in [9, 13, 15, 16].

Theorem 4.2 *Let A be a C^*-algebra, let $a_n, a \in A^\dagger$, and let $a_n \to a$. Then the following conditions are equivalent.*

$$a_n^\dagger \to a^\dagger, \tag{4.1}$$

$$\sup_n \|a_n^\dagger\| < \infty, \tag{4.2}$$

$$a_n^\dagger a_n \to a^\dagger a, \tag{4.3}$$

$$a_n a_n^\dagger \to a a^\dagger. \tag{4.4}$$

Proof (4.1) $\Longleftrightarrow$ (4.2) The forward implication is clear. For the converse we use the well known identity for $b^\dagger - a^\dagger$ (see Wedin [18]):

$$b^\dagger - a^\dagger = -b^\dagger(b-a)a^\dagger + (e - b^\dagger b)(b^* - a^*)(a^\dagger)^* a^\dagger + b^\dagger (b^\dagger)^*(b^* - a^*)(e - aa^\dagger).$$

Setting $b = a_n$, we obtain the inequality $\|a_n^\dagger - a^\dagger\| \le 3\max\{\|a^\dagger\|^2, \|a_n^\dagger\|^2\}\|a_n - a\|$, and the result follows.

(4.1) $\Longleftrightarrow$ (4.3). The forward implication follows from the contituity of multiplication in A. Conversely, by Theorem 2.5, $q_n = e - a_n^\dagger a_n$, $q = e - a^\dagger a$ are the spectral idempotents of the simply polar elements $a_n^* a_n$, $a^* a$. Since also $a_n^* \to a^*$, we have $a_n^* a_n \to a^* a$. By the preceding theorem, $q_n \to q$ implies $(a_n^* a_n)^D \to (a^* a)^D$. Finally

$$a_n^\dagger = (a_n^* a_n)^D a_n^* \to (a^* a)^D a^* = a^\dagger.$$

(4.1) $\Longleftrightarrow$ (4.4) follows by symmetry.

Harte and Mbekhta [9, Theorem 2] proved that the conorm $\gamma(a)$ of a in a C^*-algebra A is given by $\gamma(a) = \|a^\dagger\|^{-1}$ if a is nonzero and regular. If we consider only nonzero elements in the preceding theorem, we recover condition (6.2) of [9, Theorem 6]:

$$a_n^\dagger \to a^\dagger \iff \gamma(a_n) \to \gamma(a).$$

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