PERTURBATION OF THE DRAZIN INVERSE FOR CLOSED LINEAR OPERATORS

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We investigate the perturbation of the Drazin inverse of a closed linear operator recently introduced by second author and Tran, and derive explicit bounds for the perturbations under certain restrictions on the perturbing operators. We give applications to the solution of perturbed linear equations, to the asymptotic behaviour of C_0-semigroups of linear operators, and to perturbed differential equations. As a special case of our results we recover recent perturbation theorems of Wei and Wang.

1 INTRODUCTION AND PRELIMINARIES

The main purpose of our paper is to study the perturbation of the Drazin inverse A^{D} of a closed linear operator A and to obtain a bound for $\|(A+U)^{\mathrm{D}} - A^{\mathrm{D}}\|$ under certain conditions on the perturbing operator U. Among other results, we give applications to the solution of perturbed linear equations, to operators on Hilbert spaces commuting with their Moore–Penrose inverse, and to the asymptotic behaviour of C_0-semigroups of bounded linear operators.

By $\mathcal{C}(X)$ (respectively $\mathcal{B}(X)$) we denote the set of all closed (respectively bounded) linear operators on X to X, where X is a complex Banach space. We write $\mathcal{D}(A)$, $\mathcal{N}(A)$, $\mathcal{R}(A)$ and $\sigma(A)$ for the domain, nullspace, range and spectrum of an operator $A \in \mathcal{C}(X)$. All relevant concepts from theory of closed linear operators can be found in [15].

The following definition is equivalent to the one given in [8].

Definition 1.1 An operator $A \in \mathcal{C}(X)$ is *Drazin invertible* if it can be expressed in the form

$$A = A_1 \oplus A_2, \text{ where } A_1 \text{ is closed invertible and } A_2 \text{ is bounded quasinilpotent.} \tag{1.1}$$

We call (1.1) the *quasipolar decomposition* of A. The *Drazin index* $i(A)$ of A is 0 if A is invertible; if A is not invertible, $i(A)$ is the least positive integer k for which $A_2^k = 0$, or ∞ if no such integer k exists. The operators

$$A^{\mathrm{D}} = A_1^{-1} \oplus 0 \quad \text{and} \quad A^{\pi} = 0 \oplus I \tag{1.2}$$

are the *Drazin inverse* of A and the *spectral projection* of A corresponding to 0, respectively. We note that $A^{\mathrm{D}} \in \mathcal{B}(X)$.

The original definition of the Drazin inverse [3] was given for the case $i(A) < \infty$ (in the setting of semigroups and algebras). Drazin invertible operators include invertible and quasinilpotent operators; these cases arise when in the direct sum $X = X_1 \oplus X_2$ inducing the decomposition (1.1) we have $X_2 = \{0\}$ and $X_1 = \{0\}$, respectively.

From [15, Theorem V.9.2] it follows that A is Drazin invertible if and only if A is *quasipolar* (that is, if 0 is not an accumulation spectral point of A). The spaces X_1 and X_2 in the direct sum $X = X_1 \oplus X_2$ inducing (1.1) were identified by Mbekhta [9] as $X_1 = \mathcal{K}(A)$ and $X_2 = \mathcal{H}_0(A)$, where

$$\begin{aligned}\mathcal{H}_0(A) &= \{x \in \mathcal{D}_\infty(A) : \limsup_{n\to\infty} \|A^n x\|^{1/n} = 0\},\\ \mathcal{K}(A) &= \{x \in X : \exists x_n \in \mathcal{D}_n(A) \text{ such that}\\ &\qquad Ax_1 = x,\ Ax_{n+1} = x_n \text{ for } n = 1, 2, \dots\\ &\qquad \text{and } \limsup_{n\to\infty} \|x_n\|^{1/n} < \infty\}.\end{aligned}$$

These two spaces are hyperinvariant under A, and

$$\mathcal{N}(A^n) \subset \mathcal{H}_0(A), \quad \mathcal{K}(A) \subset \mathcal{R}(A^n), \quad n = 1, 2, \dots$$

Mbekhta [9] proved that $A \in \mathcal{C}(X)$ is quasipolar if and only if $X = \mathcal{K}(A) \oplus \mathcal{H}_0(A)$, where the spaces $\mathcal{K}(A)$ and $\mathcal{H}_0(A)$ are closed.

2 PERTURBATION OF THE DRAZIN INVERSE

In this section we study the behaviour of the Drazin inverse of a closed linear operator under a perturbation by a bounded linear operator, usually of small norm. The continuity properties of the Drazin inverse have been satisfactorily described in a number of papers (see, for instance [1, 2, 7, 12]).

The error bounds for the perturbed Drazin inverse without restriction on the perturbing small matrix (or operator) are difficult to obtain, as pointed out by Campbell and Meyer [2]. Rong [14] and Wei and Wang [16] gave error bounds assuming certain restrictions on the perturbing matrices. We start by defining a class of operators crucial for our investigation.

Definition 2.1 Let $A \in \mathcal{C}(X)$ be Drazin invertible. An operator $U \in \mathcal{B}(X)$ is called *A-compatible* if it commutes with A^π and if the operator $A^\pi U$ is quasinilpotent and commutes with A. Explicitly,

$$A^\pi U = UA^\pi, \tag{2.1}$$

$$\sigma(A^\pi U) = \{0\}, \tag{2.2}$$

$$AA^\pi U = UAA^\pi. \tag{2.3}$$

We consider two further conditions

$$I + A^{\mathrm{D}}U \text{ is invertible,} \tag{2.4}$$

$$\|A^{\mathrm{D}}U\| < 1. \tag{2.5}$$

If U is A-compatible and satisfies (2.4) (respectively (2.5)), we say that U is *inverse-A-compatible* (respectively *norm-A-compatible*).

LEMMA 2.2 *An operator $U \in \mathcal{B}(X)$ is A-compatible with a Drazin invertible operator $A \in \mathcal{C}(X)$ if and only if $A = A_1 \oplus A_2$ and $U = U_1 \oplus U_2$ relative to the direct sum $X = \mathcal{H}_0(A) \oplus \mathcal{K}(A)$ with U_2 quasinilpotent and commuting with A_2.*

Proof. Assume that (2.1)–(2.3) hold. Since A is quasipolar, $X = \mathcal{K}(A) \oplus \mathcal{H}_0(A)$ with the component subspaces closed. Recall that $\mathcal{R}(A^\pi) = \mathcal{H}_0(A)$ and $\mathcal{N}(A^\pi) = \mathcal{K}(A)$. By (2.1), the spaces $\mathcal{K}(A)$ and $\mathcal{H}_0(A)$ are invariant under U; hence U can be expressed as $U = U_1 \oplus U_2$ relative to $X = \mathcal{K}(A) \oplus \mathcal{H}_0(A)$. Further, $A^\pi U = (0 \oplus I)(U_1 \oplus U_2) = 0 \oplus U_2$; hence according to (2.2), $\sigma(U_2) = \{0\}$. Finally, commutativity of the operators A_2 and U_2 can be deduced from (2.3); observe that AA^π is a bounded linear operator.

Conversely, suppose that $U = U_1 \oplus U_2$ with U_2 quasinilpotent and commuting with A_2. Equations (2.1)–(2.3) can be then verified directly.

The following theorem together with Theorem 2.10 are the main results of this paper on the perturbation of the Drazin inverse.

THEOREM 2.3 *Let $A \in \mathcal{C}(X)$ be Drazin invertible and let $U \in \mathcal{B}(X)$ be inverse-A-compatible. If $B = A + U$, then B is Drazin invertible, and*

$$\mathcal{K}(B) = \mathcal{K}(A) \text{ and } \mathcal{H}_0(B) = \mathcal{H}_0(A), \tag{2.6}$$

$$B^{\mathrm{D}} = (I + A^{\mathrm{D}}U)^{-1}A^{\mathrm{D}} = A^{\mathrm{D}}(I + UA^{\mathrm{D}})^{-1}, \tag{2.7}$$

$$|i(A) - i(A^\pi U)| + 1 \le i(B) \le i(A) + i(A^\pi U) - 1; \tag{2.8}$$

(2.8) *holds provided $i(A) > 0$ and the difference $i(A) - i(A^\pi U)$ is defined.*

Proof. Employing Lemma 2.2, we have $B = B_1 \oplus B_2 = (A_1 + U_1) \oplus (A_2 + U_2)$ relative to $X = \mathcal{K}(A) \oplus \mathcal{H}_0(A)$. We show that the operator B_1 is invertible. Since $I + A^{\mathrm{D}}U = I \oplus I + (A_1^{-1} \oplus 0)(U_1 \oplus U_2) = (I + A_1^{-1}U_1) \oplus I$ is invertible by hypothesis, so is its $\mathcal{K}(A)$-component $I + A_1^{-1}U_1$. Then $B_1 = A_1 + U_1 = A_1(I + A_1^{-1}U_1)$ is invertible. Since A_2, U_2 are commuting quasinilpotent operators, B_2 is quasinilpotent. Hence B is Drazin invertible by Definition 1.1, and satisfies (2.6).

From the decomposition $B = (A_1 + U_1) \oplus (A_2 + U_2)$ we deduce that $i(B) = i(A_2 + U_2)$. Similarly, $i(A) = i(A_2)$ and $i(A^\pi U) = i(U_2)$. We apply Lemma 8.1 of the Appendix with A_2 in place of A and with U_2 in place of B to obtain (2.8).

Since $\sigma(A^{\mathrm{D}}U) \setminus \{0\} = \sigma(UA^{\mathrm{D}}) \setminus \{0\}$, the operator $I + UA^{\mathrm{D}}$ is also invertible. The equations in (2.7) can be then obtained by simple manipulations of direct sums of operators:

$$\begin{aligned}(I + A^{\mathrm{D}}U)^{-1}A^{\mathrm{D}} &= ((I + A_1^{-1}U_1)^{-1} \oplus I)(A_1^{-1} \oplus 0) = (A_1(I + A_1^{-1}U_1))^{-1} \oplus 0 \\ &= (A_1 + U_1)^{-1} \oplus 0 = B_1^{-1} \oplus 0 = B^{\mathrm{D}}.\end{aligned}$$

The other equation is obtained similarly.

EXAMPLE 2.4 For one special class of inverse-A-compatible operators U, inequality (2.8) always reduces to equality. Consider a Drazin invertible operator A and an operator $U \in \mathcal{B}(X)$ satisfying (2.4) and

$$A^\pi U = 0 = U A^\pi. \tag{2.9}$$

Then U is inverse-A-compatible, and $U = U_1 \oplus 0$ relative to $X = \mathcal{K}(A) \oplus \mathcal{H}_0(A)$. If $A = A_1 \oplus A_2$, then $A + U = (A_1 + U_1) \oplus A_2$, and

$$\begin{gathered} |i(A) - i(A^\pi U)| + 1 = i(A) - i(A^\pi U) + 1 = i(A), \\ i(A+U) = i(A), \\ i(A) + i(A^\pi U) - 1 = i(A). \end{gathered}$$

COROLLARY 2.5 *If $A \in \mathcal{C}(X)$ is Drazin invertible and $B = A + U$ with $U \in \mathcal{B}(X)$ norm-A-compatible, then*

$$\frac{\|B^{\mathrm{D}} - A^{\mathrm{D}}\|}{\|A^{\mathrm{D}}\|} \le \frac{\|A^{\mathrm{D}}U\|}{1 - \|A^{\mathrm{D}}U\|} \tag{2.10}$$

and

$$\frac{\|A^{\mathrm{D}}\|}{1 + \|A^{\mathrm{D}}U\|} \le \|B^{\mathrm{D}}\| \le \frac{\|A^{\mathrm{D}}\|}{1 - \|A^{\mathrm{D}}U\|}. \tag{2.11}$$

Proof. Inequality (2.10) is obtained as follows:

$$\begin{aligned} \|B^{\mathrm{D}} - A^{\mathrm{D}}\| &= \|(I + A^{\mathrm{D}}U)^{-1}A^{\mathrm{D}} - A^{\mathrm{D}}\| \\ &= \|(I + A^{\mathrm{D}}U)^{-1}A^{\mathrm{D}}UA^{\mathrm{D}}\| \\ &= \frac{\|A^{\mathrm{D}}U\|}{1 - \|A^{\mathrm{D}}U\|}\,\|A^{\mathrm{D}}\|. \end{aligned}$$

Inequality (2.11) follows routinely from the equation $A^{\mathrm{D}} = (I + A^{\mathrm{D}}U)B^{\mathrm{D}}$ and from (2.7).

REMARK 2.6 From the preceding corollary it is clear that, for an A-compatible operator U, condition (2.5) is crucial for obtaining numerical error estimates for the perturbed Drazin inverse. It is therefore interesting to observe that the fulfillment of (2.5) may depend only on the norm of U_1, where $U = U_1 \oplus U_2$ is the direct sum decomposition of the perturbing operator U. Indeed, assume that $\|U_1\| < \delta$ for some $\delta > 0$. Then

$$\|A^{\mathrm{D}}U\| = \|A_1^{-1}U_1 \oplus 0\| = \|A_1^{-1}U_1(I - A^\pi)\| \le \|A_1^{-1}\|\|I - A^\pi\|\delta < 1$$

provided δ is small enough. The norm of U_2 (and the norm of U) may be arbitrarily large.

REMARK 2.7 Wei and Wang [16] considered a perturbation of the Drazin inverse of (a matrix) A under the condition on U given by

$$AA^{\mathrm{D}}UAA^{\mathrm{D}} = U \quad \text{and} \quad \|A^{\mathrm{D}}U\| < 1. \tag{$\mathcal{W}$}$$

To analyze condition ($\mathcal{W}$), we first observe that the following are equivalent:

$$
\begin{gathered}
AA^{\mathrm{D}}UAA^{\mathrm{D}} = U, \\
A^{\pi}U = 0 = UA^{\pi}, \\
\mathcal{R}(U) \subset \mathcal{K}(A) \text{ and } \mathcal{H}_0(A) \subset \mathcal{N}(U), \\
U = U_1 \oplus 0 \text{ relative to } X = \mathcal{K}(A) \oplus \mathcal{H}_0(A).
\end{gathered}
$$

This can be deduced from $AA^{\mathrm{D}} = I - A^{\pi}$ and Example 2.4. If A and U satisfy condition ($\mathcal{W}$), then U is norm-A-compatible; condition ($\mathcal{W}$) thus leads to norm-A-compatible perturbations of the form $B = A + U = (A_1 + U_1) \oplus A_2$. The case treated in [16] allows only for the perturbation of the invertible component of $A = A_1 \oplus A_2$, while the quasinilpotent component remains unchanged. Their results [16, Theorems 3.1 and 3.2] follow from Theorem 2.3 and from the preceding corollary when $A^{\pi}U = 0$. Perturbations subject to condition ($\mathcal{W}$) were recently extended to bounded linear operators and to elements of Banach algebras by Rakočević and Wei in [13]. Their results [13, Theorem 2.1 and Corollaries 2.2, 2.3] are also recovered as special cases of our perturbation theorems.

Given a Drazin invertible operator $A \in \mathcal{C}(X)$, we say that $B \in \mathcal{C}(X)$ is an *inverse-A-compatible* (respectively *norm-A-compatible*) *perturbation* of A if $B = A + U$ for some inverse-A-compatible (respectively norm-A-compatible) operator $U \in \mathcal{B}(X)$.

Corollary 2.8 *If $A \in \mathcal{C}(X)$ is Drazin invertible and B an inverse-A-compatible perturbation of A, then*

$$\mathcal{R}(B^{\mathrm{D}}) = \mathcal{R}(A^{\mathrm{D}}) = \mathcal{K}(A) \cap \mathcal{D}(A), \qquad \mathcal{N}(B^{\mathrm{D}}) = \mathcal{N}(A^{\mathrm{D}}) = \mathcal{H}_0(A).$$

Proof. Follows from (2.7) and from $A^{\mathrm{D}} = A_1^{-1} \oplus 0$, where A_1^{-1} is a surjection on $\mathcal{K}(A)$.

The next result exhibits a symmetry of inverse-A-compatibility.

Corollary 2.9 *If $A \in \mathcal{C}(X)$ is Drazin invertible and B an inverse-A-compatible perturbation of A, then B is Drazin invertible and A is an inverse-B-compatible perturbation of B.*

Proof. By Theorem 2.3, B is Drazin invertible. Set $V = -U \in \mathcal{B}(X)$. Then (2.1)–(2.3) are satisfied with V in place of U, and

$$I + B^{\mathrm{D}}V = I - (I + A^{\mathrm{D}}U)^{-1}A^{\mathrm{D}}U = (I + A^{\mathrm{D}}U)^{-1}$$

is invertible.

For the rest of this section, Ω will denote an open interval containing 0 or an open neighbourhood of 0 in the complex plane $\mathbb{C}$. In [7], Koliha and Rakočević gave thirteen equivalent necessary and sufficient conditions for the continuity of the Drazin inverse for bounded linear operators. The following theorem gives sufficient conditions for the continuity of the Drazin inverse for closed linear operators, and, unlike the results of [2, 7, 12], it gives quantitative error bounds for the convergence.

THEOREM 2.10 *Let $A \in \mathcal{C}(X)$ be Drazin invertible and let $U : \Omega \to \mathcal{B}(X)$ be an operator valued function such that*

$$U(z) \text{ is } A\text{-compatible for all } z \in \Omega, \tag{2.12}$$

$$\|A^{\mathrm{D}}U(z)\| \to 0 \text{ as } z \to 0, \text{ and } A^{\mathrm{D}}U(0) = 0. \tag{2.13}$$

Define $B(z) = A + U(z)$ for all $z \in \Omega$. Then there exists $\delta > 0$ such that, for all $z \in \mathbb{C}$ satisfying $|z| < \delta$,

$$\mathcal{H}_0(B(z)) = \mathcal{H}_0(A) \text{ and } \mathcal{K}(B(z)) = \mathcal{K}(A),$$
$$B^{\mathrm{D}}(z) = (I + A^{\mathrm{D}}U(z))^{-1}A^{\mathrm{D}} = A^{\mathrm{D}}(I + U(z)A^{\mathrm{D}})^{-1},$$
$$|i(A) - i(A^{\pi}U(z))| + 1 \le i(B(z)) \le i(A) + i(A^{\pi}U(z)) - 1;$$

the last inequality holds provided that $i(A) > 0$ and the difference $i(A) - i(A^{\pi}U(z))$ is defined. In addition,

$$\frac{\|B^{\mathrm{D}}(z) - A^{\mathrm{D}}\|}{\|A^{\mathrm{D}}\|} \le \frac{\|A^{\mathrm{D}}U(z)\|}{1 - \|A^{\mathrm{D}}U(z)\|},$$

and

$$\frac{\|A^{\mathrm{D}}\|}{1 + \|A^{\mathrm{D}}U(z)\|} \le \|B^{\mathrm{D}}(z)\| \le \frac{\|A^{\mathrm{D}}\|}{1 - \|A^{\mathrm{D}}U(z)\|}$$

whenever $|z| < \delta$. In particular, $B^{\mathrm{D}}(z) \to A^{\mathrm{D}}$ as $z \to 0$.

Proof. From (2.13) it follows that there is $\delta > 0$ such that $\|A^{\mathrm{D}}U(z)\| < 1$ if $|z| < \delta$. Then Theorem 2.3 and Corollary 2.5 yield the result.

3 PERTURBED LINEAR EQUATION

In this section we consider the linear equation

$$Ax = b, \quad b \in \mathcal{K}(A) \text{ given}, \tag{3.1}$$

with A Drazin invertible and $x \in \mathcal{D}(A)$ to be found. There is a unique $\widetilde{x} \in \mathcal{K}(A)$ satisfying (3.1); we observe that $\widetilde{x} = A^{\mathrm{D}}b$. Indeed, $A^{\mathrm{D}}b \in \mathcal{K}(A)$, and $A(A^{\mathrm{D}}b) = (I - A^{\pi})b = b$ since $\mathcal{R}(I - A^{\pi}) = \mathcal{K}(A)$; the uniqueness follows from the fact that the restriction of A to $\mathcal{K}(A)$ is an invertible closed linear operator. The general solution to (3.1) is of the form $x = A^{\mathrm{D}}b + h$, where $h \in \mathcal{N}(A)$.

As in [16] and [13] we obtain a result on the solution of the perturbed equation (3.1). Since the operator A is in general unbounded, our estimate does not involve the Drazin condition number of A, which in the case of a bounded operator may be defined as $\kappa_{\mathrm{D}}(A) = \|A\|\|A^{\mathrm{D}}\|$ (see [13, p. 184]).

THEOREM 3.1 *Let $A \in \mathcal{C}(X)$ be Drazin invertible, $B = A + U$ a norm-A-compatible perturbation of A, and $b, u \in \mathcal{K}(A)$. If $x \in X$ satisfies $Ax = b$ and y satisfies $By = b + u$, then*

$$\frac{\|(I - A^{\pi})(y - x)\|}{\|(I - A^{\pi})x\|} \le \frac{\|A^{\mathrm{D}}\|}{1 - \|A^{\mathrm{D}}U\|}\left(\|U\| + \frac{\|u\|}{\|A^{\mathrm{D}}b\|}\right). \tag{3.2}$$

Proof. First we assume that $\widetilde{x} = A^{\mathrm{D}}b$ and $\widetilde{y} = B^{\mathrm{D}}(b+u)$. By (2.7), $B^{\mathrm{D}}(I+UA^{\mathrm{D}}) = A^{\mathrm{D}}$, and $B^{\mathrm{D}} - A^{\mathrm{D}} = -B^{\mathrm{D}}UA^{\mathrm{D}}$. Therefore

$$\widetilde{y} - \widetilde{x} = (B^{\mathrm{D}} - A^{\mathrm{D}})b + B^{\mathrm{D}}u = -B^{\mathrm{D}}U\widetilde{x} + B^{\mathrm{D}}u,$$

and $\|\widetilde{y} - \widetilde{x}\| \le \|B^{\mathrm{D}}\|(\|U\widetilde{x}\| + \|u\|)$. The norm of B^{D} can be estimated with the help of (2.11), while $\|\widetilde{x}\| = \|A^{\mathrm{D}}b\|$. Hence

$$\frac{\|\widetilde{y} - \widetilde{x}\|}{\|\widetilde{x}\|} \le \frac{\|A^{\mathrm{D}}\|}{1 - \|A^{\mathrm{D}}U\|}\left(\|U\| + \frac{\|u\|}{\|A^{\mathrm{D}}b\|}\right), \tag{3.3}$$

which is (3.2) with $x = \widetilde{x}$ and $y = \widetilde{y}$.

If $Ax = b$ and $By = b+u$, then $x = \widetilde{x} + f$ and $y = \widetilde{y} + g$, where $f \in \mathcal{N}(A) \subset \mathcal{H}_0(A)$, and $g \in \mathcal{N}(B) \subset \mathcal{H}_0(B) = \mathcal{H}_0(A)$. Note that $f - g \in \mathcal{H}_0(A)$. Then

$$(I - A^\pi)(y - x) = (I - A^\pi)(\widetilde{y} - \widetilde{x}) + (I - A^\pi)(f - g) = \widetilde{y} - \widetilde{x},$$
$$(I - A^\pi)x = (I - A^\pi)\widetilde{x} + (I - A^\pi)f = \widetilde{x}$$

since $\mathcal{R}(I - A^\pi) = \mathcal{K}(A)$ and $\mathcal{N}(I - A^\pi) = \mathcal{H}_0(A)$. Estimate (3.2) follows from (3.3).

If A and U are matrices and $A^\pi U = 0 = UA^\pi$, (3.3) yields as a special case [16, Theorem 4.1] (with a modification for the Drazin condition number $\kappa_{\mathrm{D}}(A)$).

4 TWO APPLICATIONS

Our first result is concerned with the reversal formula for the Drazin inverse of a product.

PROPOSITION 4.1 *Let $A \in \mathcal{C}(X)$ be Drazin invertible, and let $B \in \mathcal{B}(X)$ and $C \in \mathcal{C}(X)$ be inverse-A-compatible perturbations of A such that $BCA^\pi = CBA^\pi$. Then BC is Drazin invertible, $(BC)^\pi = A^\pi$, and*

$$(BC)^{\mathrm{D}} = C^{\mathrm{D}}B^{\mathrm{D}}.$$

Proof. The hypothesis $B \in \mathcal{B}(X)$ is to guarantee the existence of the composition BC. According to Theorem 2.3, B and C are Drazin invertible with quasipolar decompositions relative to $\mathcal{K}(A) \oplus \mathcal{H}_0(A)$. Condition $BCA^\pi = CBA^\pi$ valid for inverse-A-compatible perturbations B and C of A implies $B_2C_2 = C_2B_2$. Then $BC = B_1C_1 \oplus B_2C_2$, where B_1C_1 is invertible being the product of invertible operators, and B_2C_2 is quasinilpotent being the product of commuting quasinilpotent operators. Then BC is Drazin invertible with

$$(BC)^{\mathrm{D}} = C_1^{-1}B_1^{-1} \oplus 0 = (C_1^{-1} \oplus 0)(B_1^{-1} \oplus 0) = C^{\mathrm{D}}B^{\mathrm{D}}.$$

An interesting special case is $C = A$.

Next we give an application of Theorem 2.3 to bounded linear operators on a Hilbert space H which commute with their Moore–Penrose inverse.

PROPOSITION 4.2 *Let H be a Hilbert space and let $A \in \mathcal{B}(H)$ be a Moore–Penrose invertible operator such that $AA^\dagger = A^\dagger A$. If $B \in \mathcal{B}(H)$ satisfies conditions*

$$A^\pi(B - A) = 0 = (B - A)A^\pi, \tag{4.1}$$

$$I + A^\dagger(B - A) \text{ is invertible in } \mathcal{B}(H), \tag{4.2}$$

then B is Moore–Penrose invertible, and $B^\dagger B = BB^\dagger$. In addition, $\mathcal{R}(B) = \mathcal{R}(A)$ and $\mathcal{N}(B) = \mathcal{N}(A)$.

If the stronger condition $\|A^\dagger(B - A)\| < 1$ holds in place of (4.2), then

$$\frac{\|B^\dagger - A^\dagger\|}{\|A^\dagger\|} \le \frac{\|A^\dagger U\|}{1 - \|A^\dagger U\|},$$

where $U = B - A$.

Proof. We observe that $AA^\dagger = A^\dagger A$ if and only if A is simply polar (that is, if 0 is at most a simple pole of the resolvent of A) and the spectral projection A^π is self-adjoint; this characterization is obtained using known properties of the Moore–Penrose inverse. In this case $A^\dagger = A^{\mathrm{D}}$. From Theorem 2.3 we deduce that B is also simply polar with $B^\pi = A^\pi$ self-adjoint, that is, B is Moore–Penrose invertible and $B^\dagger B = BB^\dagger$. Also, $\mathcal{R}(B) = \mathcal{K}(B) = \mathcal{K}(A) = \mathcal{R}(A)$, and $\mathcal{N}(B) = \mathcal{H}_0(B) = \mathcal{H}_0(A) = \mathcal{N}(A)$.

The last statement is established as in the proof of Corollary 2.5.

5 THE CASE OF BANACH ALGEBRAS

Theorems of the preceding sections specialize to give perturbation results for the Drazin inverse of bounded linear operators, but not of elements of Banach algebras. However, the technique of direct sums of operators has a counterpart in Banach algebras, and the proofs of the results given for operators can be adapted to yield corresponding results for Banach algebras. Such approach is described in [6].

Let $\mathcal{A}$ be a complex unital Banach algebra with unit I. Let P be a proper idempotent in $\mathcal{A}$. Then $P\mathcal{A}P$ is a closed subalgebra of $\mathcal{A}$ with unit P. If $A \in \mathcal{A}$ commutes with P, we write $\sigma_{P\mathcal{A}P}(AP)$ for the spectrum of AP relative to the algebra $P\mathcal{A}P$. Let $Q = I - P$. According to [6], A has a decomposition $A = AP + AQ$ and

$$\sigma(A) = \sigma_{P\mathcal{A}P}(AP) \cup \sigma_{Q\mathcal{A}Q}(AQ).$$

Let $\mathcal{A}$ be the full algebra of bounded linear operators on a Banach space X. Suppose that $X = X_1 \oplus X_2$ for closed subspaces X_1, X_2 invariant under an operator $A \in \mathcal{B}(X)$. Let $P \in \mathcal{B}(X)$ be the projection with $\mathcal{R}(A) = X_1$ and $\mathcal{N}(A) = X_2$; let A_i be the restriction of A to X_i $(i = 1, 2)$. Then

$$\sigma(A_1) = \sigma_{P\mathcal{A}P}(AP), \quad \sigma(A_2) = \sigma_{Q\mathcal{A}Q}(AQ),$$

where $Q = I - P$. This result suggests a procedure for converting the results of the preceding section for elements of Banach algebras.

The following definition of the Drazin inverse is the Banach algebra analogy of Definition 1.1. We say that an element $A \in \mathcal{A}$ is *Drazin invertible* if there is an idempotent $P \in \mathcal{A}$ commuting with A such that AP is quasinilpotent in $P\mathcal{A}P$ and AQ invertible in $Q\mathcal{A}Q$, where $Q = I - P$. The *Drazin inverse* of A is then defined by

$$A^{\mathrm{D}} = (AQ)^{-1} \quad \text{(the inverse in } Q\mathcal{A}Q).$$

The element P is the *spectral idempotent* of A, written $A^\pi = P$. The Drazin index $i(A)$ is defined analogously as for operators. To show that this definition is equivalent to the one given in [4], we note that according to [6, Lemma 2.3], $(AQ)^{-1} = (A+P)^{-1}Q$.

Let $A \in \mathcal{A}$ be Drazin invertible. As for operators we say that $U \in \mathcal{A}$ is *A-compatible* if the conditions (2.1)–(2.3) are satisfied; these conditions make good sense in Banach algebras. Similarly we define *inverse-A-compatible* and *norm-A-compatible* elements of $\mathcal{A}$. We can then obtain Banach algebra analogues of Theorems 2.3, 2.10 and Corollary 2.5, as well as most of the other results of the preceding sections. As a sample we reformulate and prove Theorem 2.3.

Theorem 5.1 *Let $A \in \mathcal{A}$ be Drazin invertible and let $U \in \mathcal{A}$ be inverse-A-compatible. If $B = A + U$, then B is Drazin invertible, and*

$$B^\pi = A^\pi, \tag{5.1}$$

$$B^{\mathrm{D}} = (I + A^{\mathrm{D}}U)^{-1}A^{\mathrm{D}} = A^{\mathrm{D}}(I + UA^{\mathrm{D}})^{-1}, \tag{5.2}$$

$$|i(A) - i(A^\pi U)| + 1 \le i(B) \le i(A) + i(A^\pi U) - 1; \tag{5.3}$$

(5.3) *holds provided $i(A) > 0$ and the difference $i(A) - i(A^\pi U)$ is defined.*

Proof. Let $P = A^\pi$. Then P is an idempotent commuting with $B = A + U$. Since $BP = (A+U)A^\pi = A^\pi A + UA^\pi$, BP is the sum of two commuting quasinilpotents, and hence itself quasinilpotent. Let $Q = I - P$; then $QB = QBQ \in Q\mathcal{A}Q$. By definition, $(AQ)A^{\mathrm{D}} = Q$ (the inverse in $Q\mathcal{A}Q$); hence

$$QB = QA + QU = QA(I + A^{\mathrm{D}}U) = (QA)[Q(I + A^{\mathrm{D}}U)Q]. \tag{5.4}$$

The element $Q(I+A^{\mathrm{D}}U)Q$ is invertible in $Q\mathcal{A}Q$ since $I+A^{\mathrm{D}}U$ is invertible in $\mathcal{A}$ by hypothesis and commutes with Q. Consequently, QB is invertible in $Q\mathcal{A}Q$, and B is Drazin invertible with $B^\pi = P = A^\pi$. We note that (5.1) is the correct algebra interpretation of (2.6), though the spatial information is lost.

Using (5.4) we get $B^{\mathrm{D}} = (QB)^{-1} = (Q(I+A^{\mathrm{D}}U)Q)^{-1}(QA)^{-1}$ (the inverses in $Q\mathcal{A}Q$). From this we deduce the first formula for B^{D} in (5.2); the second formula is obtained from $BQ = [Q(I + UA^{\mathrm{D}})Q](AQ)$.

The index inequality follows from Lemma 8.1 of the Appendix.

We can now recover most of the perturbation results of Rakočević and Wei [13] for Banach algebras: Theorem 2.1, Corollaries 2.2–2.5, Proposition 2.7 and Theorems 3.4 and 3.5. Our Proposition 4.1 has a Banach algebra counterpart, from which we obtain [13, Proposition 2.7] as a special case with $BA^\pi = CA^\pi = 0$.

6 APPLICATIONS TO SEMIGROUPS OF OPERATORS

We study the effect of a perturbation of the infinitesimal generator of an asymptotically convergent C_0-semigroup. For all relevant concepts of the theory of C_0-semigroups of bounded linear operators we refer the reader to the monograph [11]; for the asymptotic properties of C_0-semigroups see [10].

If $T(t)$ is a C_0-semigroup with the infinitesimal generator A, such that $\|T(t) - P\| \to 0$ as $t \to \infty$ then, according to [8, Theorem 4.2], there are positive constants M, μ such that

$$\|T(t)(I - P)\| \le Me^{-\mu t} \quad \text{for all } t \ge 0. \tag{6.1}$$

We consider the C_0-semigroup $S(t)$ generated by the perturbed operator $A + U$ with $U \in \mathcal{B}(X)$. We show that if U is A-compatible and of sufficiently small norm, then $\|S(t)(I - P)\|$ also decays exponentially, and the Drazin inverse of $A + U$ has an integral representation, similar to the one available for A^{D} (see [8, Theorem 4.2]).

THEOREM 6.1 *Let $T(t)$ be a C_0-semigroup with the infinitesimal generator A such that $T(t) \to P$ in the operator norm as $t \to \infty$, and let* (6.1) *hold. Let $U \in \mathcal{B}(X)$ be an inverse-A-compatible operator satisfying*

$$\|U\| < \mu M^{-1}, \tag{6.2}$$

and let $S(t)$ be the C_0-semigroup with the infinitesimal generator $A + U$. Then the following are true:

(i) *A and $A + U$ are Drazin invertible and $i(A) \le 1$;*

(ii) *for all $t \ge 0$,*

$$\|S(t)(I - P)\| \le Me^{(-\mu + M\|U\|)t}; \tag{6.3}$$

(iii) *for each $x \in X$,*

$$\int_0^\infty S(t)(I - P)x\, dt = -(I + A^{\mathrm{D}}U)^{-1}A^{\mathrm{D}}x = -(A + U)^{\mathrm{D}}x. \tag{6.4}$$

Proof. (i) The fact that A is Drazin invertible and that $i(A) \le 1$ follows from [8, Theorem 4.2]. By Theorem 2.3, $A + U$ is also Drazin invertible.

(ii) By [11, Chapter 3, Equation (2.1)],

$$S(t)x = T(t)x + \int_0^t T(t - s)US(s)x\, ds, \quad x \in X.$$

Replacing x by $(I - P)x$ and observing that $US(s)(I - P) = (I - P)US(s)(I - P)$, we obtain

$$S(t)(I - P)x = T(t)(I - P)x + \int_0^t T(t - s)(I - P)US(s)(I - P)x\, ds, \quad x \in X.$$

Then

$$\begin{aligned}\|S(t)(I-P)x\| &\le \|T(t)(I-P)x\| + \int_0^t \|T(t-s)(I-P)\|\|U\|\|S(s)(I-P)x\|\,ds \\ &\le Me^{-\mu t}\|x\| + \int_0^t Me^{-\mu(t-s)}\|U\|\|S(s)(I-P)x\|\,ds.\end{aligned}$$

Writing $\varphi(t) = \|S(t)(I-P)x\|e^{\mu t}$ and $C = M\|U\|$, we get

$$\varphi(t) \le M\|x\| + C\int_0^t \varphi(s)\,ds, \quad t \ge 0.$$

By the Gronwall lemma,

$$\varphi(t) \le M\|x\| + Ce^{Ct}\int_0^t M\|x\|e^{-Cs}\,ds = M\|x\|e^{Ct},$$

that is,

$$\|S(t)(I-P)\|e^{\mu t} \le Me^{M\|U\|t}, \quad t \ge 0.$$

(iii) By (6.2), $\nu = \mu - M\|U\| > 0$, and (6.3) becomes $\|S(t)(I-P)\| \le Me^{-\nu t}$ for all $t \ge 0$. Hence the integral $\int_0^\infty S(t)(I-P)x\,dt$ converges for all $x \in X$. Applying [11, Theorem 1.4.2(b)], we deduce that, for all $s > 0$ and all $x \in X$,

$$(A+U-P)\int_0^s S(t)(I-P)x\,dt = (S(s)-I)(I-P)x.$$

Taking the limit as $s \to \infty$, we obtain

$$\int_0^\infty S(t)(I-P)x\,dt = -(A+U-P)^{-1}(I-P)x, \quad x \in X;$$

$A+U-P$ is invertible since, by Theorem 2.3, P is the spectral projection of $A+U$ at 0. In addition, $(A+U-P)^{-1}(I-P) = (A+U)^{\mathrm{D}}$.

7 APPLICATIONS TO PERTURBED DIFFERENTIAL EQUATIONS

We consider the abstract Cauchy problem for the infinitesimal generator A of a C_0-semigroup with the initial datum x,

$$\frac{du(t)}{dt} = Au(t) + f(t), \quad u(0) = x, \tag{7.1}$$

where f is a given function, and the abstract Cauchy problem for the perturbation $B = A+U$ of the infinitesimal generator A,

$$\frac{dv(t)}{dt} = Bv(t) + f(t), \quad v(0) = x. \tag{7.2}$$

We study the asymptotic behavior of the mild solution of the perturbed problem and compare it with the asymptotic behavior of the mild solution of the exact system.

Let $T(t)$ be a C_0-semigroup with $T(t) \to P$ as $t \to \infty$, and let $S(t)$ be the C_0-semigroup generated by $A + U$, where $U \in \mathcal{B}(X)$. Under the conditions of Theorem 6.1, $\|S(t)(I - P)\|$ decays exponentially. However, the other part of the decomposition $S(t) = S(t)(I - P) + S(t)P$ satisfies $S(t)P = P\exp(tU)$. The next lemma gives a necessary and sufficient condition for $S(t)P$ to converge as $t \to \infty$.

LEMMA 7.1 *Let the conditions of Theorem* 6.1 *be fulfilled. Then $S(t)P$ converges in the operator norm as $t \to \infty$ if and only if $UP = 0 = PU$.*

Proof. Suppose that $S(t)P = P\exp(tU)$ converges as $t \to \infty$. Since

$$P\exp(tU) = P - I + \exp(tUP),$$

the semigroup $\exp(tV)$ with $V = UP$ quasinilpotent converges as $t \to \infty$. The spectral projection of V is $V^\pi = I$, and the convergence of $\exp(tV)$ as $t \to \infty$ implies $V = VV^\pi = 0$.

The converse is clear.

In view of the previous lemma it is realistic to consider only perturbations $A+U$ of A with U satisfying $UP = 0 = PU$. The next theorem describes the asymptotic behaviour of the solution of such perturbed equations.

THEOREM 7.2 *Let $T(t)$ be a C_0-semigroup, with the infinitesimal generator A, satisfying* (6.1). *Suppose that $U \in \mathcal{B}(X)$ satisfies the conditions*

$$UP = 0 = PU, \tag{7.3}$$

$$\|U\| < M^{-1}\mu \text{ and } \|A^{\mathrm{D}}U\| < 1. \tag{7.4}$$

Let f be bounded and Lebesgue measurable on $[0, \infty)$, and Pf integrable on $[0, \infty)$. If

$$\lim_{t\to\infty} f(t) = f_0,$$

then the mild solution $v(t)$ of the perturbed differential problem (7.2) *satisfies*

$$\lim_{t\to\infty} v(t) = Px - (I + A^{\mathrm{D}}U)^{-1}A^{\mathrm{D}}f_0 + \int_0^\infty Pf(s)\,ds.$$

Moreover, if $u(t)$ is the mild solution of the exact system (7.1), *then*

$$\lim_{t\to\infty} \|v(t) - u(t)\| \le \frac{\|A^{\mathrm{D}}U\|}{1 - \|A^{\mathrm{D}}U\|}\|A^{\mathrm{D}}f_0\|.$$

Proof. Let $S(t)$ be the C_0-semigroup generated by $A + U$. The condition (7.3) ensures the A-compatibility of U and implies the equation

$$S(t)P = P\exp(tU) = P, \quad t \ge 0.$$

The mild solution to the problem is

$$v(t) = S(t)x + \int_0^t S(t-s)f(s)\,ds.$$

Write $v_1(t) = (I-P)v(t)$ and $v_2(t) = Pv(t)$. Using Theorem 6.1 and the assumptions on f, we obtain

$$\lim_{t\to\infty} v_1(t) = -(A+U)^{\mathrm{D}} f_0.$$

The details of this argument are similar to those given in the proof of [8, Theorem 5.1]. Further,

$$\lim_{t\to\infty} v_2(t) = Px + \int_0^\infty Pf(s)\,ds.$$

For the last inequality we calculate

$$\lim_{t\to\infty} \|v(t)-u(t)\| = \|(I+A^{\mathrm{D}}U)^{-1}A^{\mathrm{D}}f_0 - A^{\mathrm{D}}f_0\|,$$

and note that

$$\|(I+A^{\mathrm{D}}U)^{-1}A^{\mathrm{D}}f_0 - A^{\mathrm{D}}f_0\| \le \frac{\|A^{\mathrm{D}}U\|}{1-\|A^{\mathrm{D}}U\|}\|A^{\mathrm{D}}f_0\|.$$

The next result describes the case when both the infinitesimal generator and the given function are perturbed.

Corollary 7.3 *Under the hypotheses of the preceding theorem, we consider the perturbed problem*

$$\frac{dv(t)}{dt} = Bv(t) + h(t), \quad v(0) = z, \tag{7.5}$$

where $B = A+U$, $h(t) = f(t)+g(t)$, $z = x+y$, g *is bounded and Lebesgue measurable on* $[0,\infty)$, *and* Pg *is integrable on* $[0,\infty)$. *If*

$$\lim_{t\to\infty} g(t) = g_0,$$

then the mild solution $v(t)$ *of the perturbed differential problem* (7.5) *satisfies*

$$\lim_{t\to\infty} v(t) = Pz - (I+A^{\mathrm{D}}U)^{-1}A^{\mathrm{D}}(g_0+f_0) + \int_0^\infty Ph(s)\,ds.$$

Moreover, if $u(t)$ *is the mild solution of the exact system* (7.1), *then*

$$\lim_{t\to\infty} \|(I-P)(v(t)-u(t))\| = \|(I+A^{\mathrm{D}}U)^{-1}A^{\mathrm{D}}(f_0+g_0) - A^{\mathrm{D}}f_0\|$$
$$\le \frac{\|A^{\mathrm{D}}U\|}{1-\|A^{\mathrm{D}}U\|}\left((\|A^{\mathrm{D}}f_0\| + \frac{\|A^{\mathrm{D}}g_0\|}{\|A^{\mathrm{D}}U\|}\right).$$

8 APPENDIX

In this Appendix we present a technical lemma needed for the proof of the index inequality in Theorem 2.3. For a quasinilpotent element A of a Banach algebra $\mathcal{A}$ we define its *nilpotency index* $i(A)$ as the least positive integer p such that $A^p = 0$, or $i(A) = \infty$ if no such integer exists. We observe that $i(A)$ coincides with the Drazin index of A as defined in [4]. For any quasinilpotent $A \in \mathcal{A}$, $i(A) \in \mathbb{N} \cup \{\infty\}$, where the arihmetics in $\mathbb{N} \cup \{\infty\}$ obeys the usual rules. Note that $\infty - \infty$ is not defined.

LEMMA 8.1 *Let A, B be commuting quasinilpotent elements of a Banach algebra $\mathcal{A}$. If the difference $i(A) - i(B)$ is defined, then*

$$|i(A) - i(B)| + 1 \le i(A+B) \le i(A) + i(B) - 1. \tag{8.1}$$

Proof. Write $i(A) = p$, $i(B) = q$ and $i(A+B) = n$. The difference $p - q$ is defined if both p, q are finite or exactly one of p, q is infinite.

Let p, q be both finite and let $m = p + q - 1$. Then

$$(A+B)^m = A^p \sum_{k=p}^{m} \binom{m}{k} A^{k-p} B^{m-k} + B^q \sum_{k=q}^{m} \binom{m}{k} A^{m-k} B^{k-q} = 0.$$

This proves that $n \le m = p + q - 1$. Using this result we observe that

$$p \le n + q - 1 \quad \text{and} \quad q \le n + p - 1$$

since $A = (A+B) + (-B)$ and $B = (A+B) + (-A)$. Hence $|p-q| \le n-1$, and $|p-q|+1 \le n$.

Suppose that exactly one of the integers p, q is infinite, say $p = \infty$. If n were finite, by the preceding argument we would get $p \le n + q - 1 < \infty$, which is false.

EXAMPLE 8.2 We show that in the case that $i(A)$, $i(B)$ are both infinite, $i(A+B)$ can be either finite or infinite.

Let $\mathcal{A}$ be a Banach algebra that contains true quasinilpotents. Choose a true quasinilpotent $A \in \mathcal{A}$ and set $B = -A$. Then $i(A) = i(B) = \infty$ and $i(A+B) = i(0) = 1$.

For the second example we again need a Banach algebra $\mathcal{B}$ that contains true quasinilpotents. Chose a true quasinilpotent $T \in \mathcal{B}$ and set $A = T \oplus 0$, $B = 0 \oplus T$ in the Banach algebra $\mathcal{A} = \mathcal{B} \oplus \mathcal{B}$. Then $(A+B)^n = T^n \oplus T^n \ne 0$ for all $n \in \mathbb{N}$, and $i(A+B) = \infty$.

EXAMPLE 8.3 We give an example when the inequality (2.8) is strict. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then A and B are nilpotent and commuting, while $i(A) = i(B) = 2$. Then $|i(A) - i(B)| + 1 = 1$, $i(A) + i(B) - 1 = 3$, and $i(A+B) = 2$ since $A + B \ne 0$ and

$$(A+B)^2 = A^2 + 2AB + B^2 = 2AB = 0.$$

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