PERTURBATION OF THE DRAZIN INVERSE

BY

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Abstract

We obtain perturbation theorems for the Drazin inverse and explicit error estimates under certain condition on the perturbing matrices. Then we give applications to the solution of a perturbed linear system, an integral representation of the Drazin inverse, and to the asymptotic behaviour of solutions of a perturbed system of differential equations.

1 Introduction

The Drazin inverse of a matrix has many useful applications. Let us mention difference equations, linear systems of differential equations, Markov chains and minimal norm solutions of systems of linear equations. In connection with the solution of singularly perturbed differential equations it is important to know what happens to the Drazin inverse of a matrix under a perturbation.

The results of this section on the Drazin inverse, index, block decompositions, eigenprojections and the continuity properties of the Drazin inverse can be found in the monograph [2] by Campbell and Meyer.

For each singular matrix $A \in \mathbb{C}^{d\times d}$ which is not nilpotent there exists a nonsingular matrix Q such that

$$A = Q^{-1}\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} Q, \quad A_1 \text{ nonsingular, } A_2 \text{ nilpotent.} \tag{1.1}$$

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The nilpotency index of A_2 is the least nonnegative integer m such that $A_2^m = 0$. It is known that the nilpotency index of A_2 is equal to the index of A, written $\operatorname{ind}(A)$, which is the least positive integer k for which $\mathcal{N}(A^k) = \mathcal{N}(A^{k+1})$. We will also use equation (1.1) in the case when A is nonsingular (respectively nilpotent), in which case the block A_2 (respectively A_1) would be empty. The Drazin inverse of A and the eigenprojection of A corresponding to the eigenvalue 0 are given by

$$A^{\mathrm{D}} = Q^{-1} \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q, \quad A^{\pi} = I - AA^{\mathrm{D}} = Q^{-1} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} Q, \tag{1.2}$$

respectively.

The continuity of the Drazin inverse without explicit error bounds was studied by several authors (for example [1, 2, 3] and more recently [4, 6]). The main result on the continuity of the Drazin inverse [4, 6] says that if (A_n) is a sequence of matrices in $\mathbb{C}^{d\times d}$ convergent to a matrix A, then $A_n^{\mathrm{D}} \to A^{\mathrm{D}}$ if and only if $A_n^{\pi} \to A^{\pi}$. However, numerical estimates for $\|A_n^{\mathrm{D}} - A^{\mathrm{D}}\|$ are difficult to obtain unless the matrices A_n are suitably restricted (see [3]). Rong [7] and recently Wei and Wang [9] gave numerical bounds for $\|A_n^{\mathrm{D}} - A^{\mathrm{D}}\|$ with certain restrictions on the perturbing matrices. In this paper we consider matrices A_n satisfying $A_n^{\pi} = A^{\pi}$ and certain other conditions specified in the next paragraph.

2 Main results

We start by specifying the class of matrices that will be used to perturb a given matrix $A \in \mathbb{C}^{d\times d}$.

DEFINITION 2.1 Let $A \in \mathbb{C}^{d\times d}$. A matrix $E \in \mathbb{C}^{d\times d}$ is called a *compatible A-perturbing matrix* if it satisfies the following conditions:

$$AA^{\mathrm{D}}E = EAA^{\mathrm{D}}, \tag{2.1}$$

$$E^m = AA^{\mathrm{D}}E^m \text{ for some } m \in \mathbb{N}, \tag{2.2}$$

$$AE(I - AA^{\mathrm{D}}) = EA(I - AA^{\mathrm{D}}). \tag{2.3}$$

Our aim is to study perturbations of A of the form $A + E$ with compatible A-perturbing matrices E.

LEMMA 2.2 *Let* $A \in \mathbb{C}^{d\times d}$ *be a matrix with a decomposition* (1.1). *Then* E *is a compatible* A*-perturbing matrix if and only if*

$$E = Q^{-1}\begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} Q \tag{2.4}$$

with the block matrix compatible with $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ *and with* E_2 *nilpotent and commuting with* A_2*.*

Proof. Suppose that E is a compatible A-perturbing matrix. Since $A^\pi = I - AA^{\mathrm{D}}$, (2.1) implies $A^\pi E = EA^\pi$. Let

$$QEQ^{-1} = \begin{bmatrix} E_1 & U \\ V & E_2 \end{bmatrix}$$

with the block matrix compatible with $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$. Then

$$\begin{aligned} QEA^\pi Q^{-1} &= (QEQ^{-1})(QA^\pi Q^{-1}) = \begin{bmatrix} E_1 & U \\ V & E_2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & U \\ 0 & E_2 \end{bmatrix}, \\ QA^\pi EQ^{-1} &= (QA^\pi Q^{-1})(QEQ^{-1}) = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} E_1 & U \\ V & E_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ V & E_2 \end{bmatrix}. \end{aligned}$$

From $A^\pi E = EA^\pi$ we obtain $U = V = 0$. Hence (2.4) holds. By (2.2),

$$\begin{bmatrix} E_1^m & 0 \\ 0 & E_2^m \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} E_1^m & 0 \\ 0 & E_2^m \end{bmatrix} = \begin{bmatrix} E_1^m & 0 \\ 0 & 0 \end{bmatrix},$$

and $E_2^m = 0$. Finally, (2.3) is equivalent to $AA^\pi E = EAA^\pi$, which leads to

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 \\ 0 & A_2E_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & E_2A_2 \end{bmatrix},$$

that is, $A_2E_2 = E_2A_2$.

Conversely, suppose that E satisfies (2.4) with E_2 nilpotent and $A_2E_2 = E_2A_2$. Conditions (2.1)–(2.3) can be verified by direct calculations with block matrices. □

We can now present the main perturbation result. By $\|\cdot\|$ we denote a consistent matrix norm on $\mathbb{C}^{d\times d}$ [8, Definition II.2.4].

THEOREM 2.3 *Suppose that* $A \in \mathbb{C}^{d\times d}$*, and that* E *is a compatible* A*-perturbing matrix such that* $\|A^{\mathrm{D}}E\| < 1$*. If* $B = A + E$*, then*

$$B^{\pi} = A^{\pi}, \tag{2.5}$$

$$B^{\mathrm{D}} = (I + A^{\mathrm{D}}E)^{-1}A^{\mathrm{D}} = A^{\mathrm{D}}(I + EA^{\mathrm{D}})^{-1}, \tag{2.6}$$

$$\frac{\|B^{\mathrm{D}} - A^{\mathrm{D}}\|}{\|A^{\mathrm{D}}\|} \le \frac{\|A^{\mathrm{D}}E\|}{1 - \|A^{\mathrm{D}}E\|}, \tag{2.7}$$

$$\frac{\|A^{\mathrm{D}}\|}{1 + \|A^{\mathrm{D}}E\|} \le \|B^{\mathrm{D}}\| \le \frac{\|A^{\mathrm{D}}\|}{1 - \|A^{\mathrm{D}}E\|}. \tag{2.8}$$

Proof. According to Lemma 2.2,

$$I + A^{\mathrm{D}}E = Q^{-1}\begin{bmatrix} I + A_1^{-1}E_1 & 0 \\ 0 & I \end{bmatrix} Q.$$

Matrix $I + A^{\mathrm{D}}E$ is nonsingular in view of the inequality $\|A^{\mathrm{D}}E\| < 1$; hence so are $I + A_1^{-1}E_1 = A_1^{-1}(A_1 + E_1)$ and $A_1 + E_1$. By Lemma 2.2 again, A_2 and E_2 commute and are nilpotent. Then also $A_2 + E_2$ is nilpotent, and

$$B = Q^{-1}\begin{bmatrix} A_1 + E_1 & 0 \\ 0 & A_2 + E_2 \end{bmatrix} Q. \tag{2.9}$$

Consequently, $B^{\pi} = Q^{-1}\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} Q = A^{\pi}$. We calculate

$$\begin{aligned}
(I + A^{\mathrm{D}}E)^{-1}A^{\mathrm{D}} &= Q^{-1}\begin{bmatrix} I + A_1^{-1}E_1 & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q \\
&= Q^{-1}\begin{bmatrix} (I + A_1^{-1}E_1)^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q \\
&= Q^{-1}\begin{bmatrix} (A_1 + E_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q = B^{\mathrm{D}};
\end{aligned}$$

the second equation in (2.6) is obtained similarly. Since $\|A^{\mathrm{D}}E\| < 1$, $\|(I + A^{\mathrm{D}}E)^{-1}\| = \|\sum_{n=0}^{\infty}(-A^{\mathrm{D}}E)^n\| \le \sum_{n=0}^{\infty}\|A^{\mathrm{D}}E\|^n = 1/(1 - \|A^{\mathrm{D}}E\|)$, and

$$\begin{aligned}
\|B^{\mathrm{D}} - A^{\mathrm{D}}\| &= \|(I + A^{\mathrm{D}}E)^{-1}A^{\mathrm{D}} - A^{\mathrm{D}}\| \\
&= \|(I + A^{\mathrm{D}}E)^{-1}A^{\mathrm{D}}EA^{\mathrm{D}}\| \\
&\le \frac{\|A^{\mathrm{D}}E\|}{1 - \|A^{\mathrm{D}}E\|}\,\|A^{\mathrm{D}}\|.
\end{aligned}$$

Inequality (2.8) is obtained similarly from (2.6). □

The following result follows directly from the preceding theorem.

COROLLARY 2.4 *Suppose that* $A \in \mathbb{C}^{d\times d}$, *and that* E_n *is a sequence of compatible* A*-perturbing matrices such that* $\delta_n = \|A^{\mathrm{D}}E_n\| \to 0$. *Then*

$$\|(A+E_n)^{\mathrm{D}} - A^{\mathrm{D}}\| \le \frac{\|A^{\mathrm{D}}\|\delta_n}{1-\delta_n} \to 0 \text{ as } n \to \infty. \tag{2.10}$$

Condition $\|A^{\mathrm{D}}E_n\| \to 0$ in the preceding corollary can be replaced by a stronger assumption $\|E_n\| \to 0$; (2.10) then holds with $\delta_n = \|A^{\mathrm{D}}\|\|E_n\|$. However, as we show in the next example, $\|A^{\mathrm{D}}E_n\| \to 0$ need not imply $\|E_n\| \to 0$.

EXAMPLE 2.5 Let

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad E_n = \begin{bmatrix} F_n & 0 \\ 0 & A_2 \end{bmatrix},$$

where A_1 is invertible, A_2 nonzero nilpotent, and $F_n \to 0$. Then each E_n is a compatible A-perturbing matrix, and $E_n \to \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix} \neq 0$. However,

$$A^{\mathrm{D}}E_n = \begin{bmatrix} A_1^{-1}F_n & 0 \\ 0 & 0 \end{bmatrix} \to 0.$$

Theorem 2.3 extends the results of Wei and Wang who studied the perturbation of the Drazin inverse and obtained estimates for $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|$ under the condition

$$AA^{\mathrm{D}}EAA^{\mathrm{D}} = E \quad \text{and} \quad \|A^{\mathrm{D}}E\| < 1. \tag{$\mathcal{W}$}$$

To relate their results to theorems of the present paper, we need the following result.

PROPOSITION 2.6 *The following conditions are equivalent under the assumption that* $A \in \mathbb{C}^{d\times d}$ *is a matrix with* $\operatorname{ind}(A) = k$, *and* $E \in \mathbb{C}^{d\times d}$:

(i) $AA^{\mathrm{D}}EAA^{\mathrm{D}} = E$,

(ii) $A^{\pi}E = 0 = EA^{\pi}$,

(iii) $\mathcal{R}(E) \subset \mathcal{R}(A^k)$ *and* $\mathcal{N}(A^k) \subset \mathcal{N}(E)$,

(iv) *if A has the decomposition* (1.1) *with A_1 nonsingular and A_2 nilpotent, then E has the decomposition* (2.4) *with $E_2 = 0$.*

Proof. (i) $\Longleftrightarrow$ (ii) Recall that $AA^{\mathrm{D}} = I - A^{\pi}$. If (i) holds, then

$$EA^{\mathrm{D}}A = AA^{\mathrm{D}}E(AA^{\mathrm{D}})^2 = AA^{\mathrm{D}}EAA^{\mathrm{D}} = E,$$

and $EA^{\pi} = 0$. Similarly, $A^{\pi}E = 0$. The converse follows easily.

(ii) $\Longleftrightarrow$ (iii) Since $\mathcal{R}(A^{\pi}) = \mathcal{N}(A^k)$ and $\mathcal{N}(A^{\pi}) = \mathcal{R}(A^k)$, $A^{\pi}E = 0$ is equivalent to $\mathcal{R}(E) \subset \mathcal{R}(A^k)$, and $EA^{\pi} = 0$ is equivalent to $\mathcal{N}(A^k) \subset \mathcal{N}(E)$.

(ii) $\Longleftrightarrow$ (iv) If $A^{\pi}E = 0 = EA^{\pi}$, then we can easily verify that (2.1)–(2.3) hold. According to Lemma 2.2, E has the decomposition (2.4). From

$$0 = A^{\pi}E = Q^{-1}\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} Q = Q^{-1}\begin{bmatrix} 0 & 0 \\ 0 & E_2 \end{bmatrix} Q$$

follows $E_2 = 0$. The converse follows by a direct verification. □

If A has index $\operatorname{ind}(A) = k$ and if condition $(\mathcal{W})$ holds, then the hypotheses of Theorem 2.3 are satisfied with the perturbation taking the following form:

$$B = A + E = Q^{-1}\begin{bmatrix} A_1 + E_1 & 0 \\ 0 & A_2 \end{bmatrix} Q.$$

From (2.5) we deduce that $\mathcal{R}(B^k) = \mathcal{R}(A^k)$. Hence we recover [9, Theorems 3.1, 3.2 and Corollaries 3.1, 3.2] as a special case of our main perturbation theorem.

EXAMPLE 2.7 We show that if A and E_n satisfy condition $(\mathcal{W})$, then $\delta_n = \|A^{\mathrm{D}}E_n\| \to 0$ is equivalent to $\|E_n\| \to 0$. Suppose that $\delta_n \to 0$. Then

$$\|E_n\| = \|AA^{\mathrm{D}}E_n\| \le \|A\|\|A^{\mathrm{D}}E_n\| \le \|A\|\delta_n.$$

The converse is clear. This means that, under condition $(\mathcal{W})$,

$$(A + E_n)^{\mathrm{D}} \to A^{\mathrm{D}} \iff E_n \to 0.$$

3 Perturbation of a linear system

In this section we give applications to perturbation of a linear system.

Let $A \in \mathbb{C}^{d\times d}$ be a matrix with $\operatorname{ind}(A) = k$. Then

$$\mathbb{C}^d = \mathcal{N}(A^k) \oplus \mathcal{R}(A^k),$$

where the eigenprojection A^π satisfies $\mathcal{R}(A^\pi) = \mathcal{N}(A^k)$ and $\mathcal{N}(A^\pi) = \mathcal{R}(A^k)$; we note that $\mathcal{R}(A^{\mathrm{D}}) = \mathcal{R}(A^k)$.

In this section we consider the linear equation

$$Ax = b, \quad b \in \mathcal{R}(A^k) \text{ given}, \tag{3.1}$$

with $x \in \mathbb{C}^d$ to be found. We note that there is a unique $x_0 \in \mathcal{R}(A^k)$ satisfying (3.1), namely $x_0 = A^{\mathrm{D}}b$. Indeed, $A^{\mathrm{D}}b \in \mathcal{R}(A^k)$, and $A(A^{\mathrm{D}}b) = (I - A^\pi)b = b$ since $\mathcal{R}(I - A^\pi) = \mathcal{R}(A^k)$; the uniqueness follows from the fact that the restriction of A to $\mathcal{R}(A^k)$ is a bijective linear transformation. The general solution to (3.1) is the set $A^{\mathrm{D}}b + \mathcal{N}(A)$.

As in [9] we obtain a result on the solution of the perturbed equation (3.1). Let $\|\cdot\|$ be a vector norm on $\mathbb{C}^d$ consistent with the given matrix norm [8, Definition II.2.4]; that is, we have $\|Ax\| \le \|A\|\|x\|$ for all $A \in \mathbb{C}^{d\times d}$ and and all $x \in \mathbb{C}^d$. We define the *Drazin inverse condition number* of A by

$$\kappa_{\mathrm{D}}(A) = \|A\|\|A^{\mathrm{D}}\|.$$

THEOREM 3.1 *Let $A \in \mathbb{C}^{d\times d}$ have index $\operatorname{ind}(A) = k$, let $E \in \mathbb{C}^{d\times d}$ be a compatible A-perturbing matrix satisfying $\|A^{\mathrm{D}}E\| < 1$, let $B = A + E$, and let $b, u \in \mathcal{R}(A^k)$. If $x \in \mathbb{C}^{d\times d}$ satisfies $Ax = b$ and y satisfies $By = b + u$, then*

$$\frac{\|(I - A^\pi)(y - x)\|}{\|(I - A^\pi)x\|} \le \frac{\|A^{\mathrm{D}}\|}{1 - \|A^{\mathrm{D}}E\|}\left(\|E\| + \frac{\|u\|}{\|A^{\mathrm{D}}b\|}\right). \tag{3.2}$$

If, in addition, $\|A^{\mathrm{D}}\|\|E\| < 1$, then

$$\frac{\|(I - A^\pi)(y - x)\|}{\|(I - A^\pi)x\|} \le \frac{\kappa_{\mathrm{D}}(A)}{1 - \kappa_{\mathrm{D}}(A)\|E\|/\|A\|}\left(\frac{\|E\|}{\|A\|} + \frac{\|u\|}{\|b\|}\right). \tag{3.3}$$

Proof. First we consider $x_0 = A^{\mathrm{D}}b$ and $y_0 = B^{\mathrm{D}}(b+u)$. From (2.6) we get the equation $B^{\mathrm{D}}(I+EA^{\mathrm{D}}) = A^{\mathrm{D}}$, and $B^{\mathrm{D}} - A^{\mathrm{D}} = -B^{\mathrm{D}}EA^{\mathrm{D}}$. Hence $y_0 - x_0 = (B^{\mathrm{D}} - A^{\mathrm{D}})b + B^{\mathrm{D}}u = -B^{\mathrm{D}}Ex_0 + B^{\mathrm{D}}u$, and $\|y_0 - x_0\| \le \|B^{\mathrm{D}}\|(\|Ex_0\| + \|u\|)$. Applying (2.8) under the assumption $\|A^{\mathrm{D}}E\| < 1$, we obtain

$$\frac{\|y_0 - x_0\|}{\|x_0\|} \le \frac{\|A^{\mathrm{D}}\|}{1 - \|A^{\mathrm{D}}E\|}\left(\|E\| + \frac{\|u\|}{\|A^{\mathrm{D}}b\|}\right); \tag{3.4}$$

then (3.2) holds with $x = x_0$ and $y = y_0$.

If $Ax = b$ and $By = b + u$, then $x = x_0 + f$ and $y = y_0 + g$, where $f \in \mathcal{N}(A) \subset \mathcal{N}(A^k)$, and $g \in \mathcal{N}(B) \subset \mathcal{N}(B^k) = \mathcal{N}(A^k)$. Note that $f - g \in \mathcal{N}(A^k)$. Then

$$\begin{aligned} (I - A^\pi)(y - x) &= (I - A^\pi)(y_0 - x_0) + (I - A^\pi)(f - g) = y_0 - x_0, \\ (I - A^\pi)x &= (I - A^\pi)x_0 + (I - A^\pi)f = x_0 \end{aligned}$$

since $\mathcal{R}(I - A^\pi) = \mathcal{R}(A^k)$ and $\mathcal{N}(I - A^\pi) = \mathcal{N}(A^k)$. Estimate (3.2) follows from (3.4).

Finally, assume that $\|A^{\mathrm{D}}\|\|E\| < 1$. Since $b = AA^{\mathrm{D}}b$, $\|b\| \le \|A\|\|A^{\mathrm{D}}b\|$. Then (3.3) follows from (3.2) and the definition of $\kappa_{\mathrm{D}}(A)$. □

Remark 3.2 If in the preceding theorem A and E satisfy $A^\pi E = 0 = EA^\pi$ and if $x, y \in \mathcal{R}(A^k)$, then (3.3) yields as a special case [9, Theorem 4.1].

4 Applications to semistable matrices

Following [5], we say that a matrix $A \in \mathbb{C}^{d\times d}$ is *semistable* if $\operatorname{ind}(A) \le 1$ and the nonzero eigenvalues λ of A satisfy $\operatorname{Re}\lambda < 0$; a semistable matrix with $\operatorname{ind}(A) = 0$ is *stable*. It is well known that $\exp(tA)$ converges as $t \to \infty$ if and only if A is semistable, in which case

$$\lim_{t\to\infty} \exp(tA) = A^\pi.$$

If the matrix A is stable, we have an integral representation for the inverse of A (e.g. [5, Section 6]):

$$A^{-1} = -\int_0^\infty \exp(tA)\,dt. \tag{4.1}$$

In this section we study integral representation for the Drazin inverse of a semistable matrix A and of a perturbed matrix $A+E$. The perturbing matrix E is again of special form; however, this time we need only condition (2.1).

Throughout this section $\|\cdot\|$ denotes the spectral matrix norm on $\mathbb{C}^{d\times d}$, that is, $\|A\|$ is the largest singular value of A (see [8, Theorem II.2.10]).

THEOREM 4.1 *Let $A \in \mathbb{C}^{d\times d}$ be semistable. Then there exists $\delta(A) > 0$ such that, for every matrix E satisfying* (2.1) *and $\|E\| < \delta(A)$, we have the representation*

$$(A+E)^{\mathrm{D}} = -\int_0^\infty \exp(t(A+E))(I-A^\pi)\,dt. \tag{4.2}$$

Proof. If (2.1) is satisfied, then

$$A = Q^{-1}\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} Q, \qquad E = Q^{-1}\begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} Q,$$

where A_1 is nonsingular and A_2 nilpotent, and the two block matrices are compatible. Since A is semistable, A_1 is stable. By the continuity of eigenvalues [8, Section IV.1.1], there exists $\eta > 0$ such that $\|E_1\| < \eta$ implies that $A_1 + E_1$ is also stable. Write $G = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}$. Then $G = QEQ^{-1}$, and

$$\|E_1\| \le \|G\| \le \|Q\|\|E\|\|Q^{-1}\| = \kappa(Q)\|E\|;$$

the first inequality holds since $\|\cdot\|$ is the spectral norm. Set $\delta(A) = \eta/\kappa(Q)$. We have

$$\begin{aligned} \exp(t(A+E))(I-A^\pi) &= Q^{-1}\begin{bmatrix} \exp(t(A_1+E_1)) & 0 \\ 0 & \exp(t(A_2+E_2)) \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Q \\ &= Q^{-1}\begin{bmatrix} \exp(t(A_1+E_1)) & 0 \\ 0 & 0 \end{bmatrix} Q. \end{aligned}$$

If $\|E\| < \delta(A)$, then $\|E_1\| < \eta$, $A_1 + E_1$ is stable, $\exp(t(A+E))(I-A^\pi)$ is integrable on the interval $[0,\infty)$ and, in view of (4.1),

$$\begin{aligned} \int_0^\infty \exp(t(A+E))(I-A^\pi)\,dt &= Q^{-1}\begin{bmatrix} \int_0^\infty \exp(t(A_1+E_1))\,dt(I-A^\pi) & 0 \\ 0 & 0 \end{bmatrix} Q \\ &= Q^{-1}\begin{bmatrix} -(A_1+E_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q = -(A+E)^{\mathrm{D}}. \qquad \square \end{aligned}$$

Setting $E = 0$ in the preceding theorem, we recover [5, Theorem 6.3]. It is of interest to observe, that the perturbed matrix $A + E$ satisfying the conditions of the theorem need not be semistable.

Remark 4.2 We can estimate the quantity $\delta(A)$ using the fact that, for any semistable matrix A, $\|\exp(tA) - A^\pi\|$ decays exponentially, that is,

$$\|\exp(tA) - A^\pi\| \le M e^{-\mu t}, \quad t \ge 0,$$

for some positive constants M, μ. It is enough to choose $\delta(A)$ to satisfy

$$\delta(A) \le M^{-1}\mu.$$

Finally we consider a differential equation

$$\frac{du}{dt} = Au + f, \qquad u(0) = x, \tag{4.3}$$

and its perturbation

$$\frac{dv}{dt} = (A + E)v + f, \quad v(0) = x. \tag{4.4}$$

Combining the preceding theorem, Theorem 2.3 and [5, Theorem 7.3], we obtain the following result, in which the special form of E ensures that $\exp(t(A + E))A^\pi$ converges as $t \to \infty$.

Theorem 4.3 *Let* $A \in \mathbb{C}^{d\times d}$ *be a semistable matrix with a decomposition* (1.1)*, and let*

$$E = Q^{-1}\begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix} Q$$

(*with a block decomposition compatible with* (1.1)) *and such that* $\|E\| < \rho$*, where* $0 < \rho \le \delta(A)$ *and* $\|A^{\mathrm{D}}\|\rho < 1$*. Let* $f : (0, \infty) \to \mathbb{C}^d$ *be bounded and Lebesgue measurable on* $(0, \infty)$*, and* $A^\pi f$ *integrable on* $(0, \infty)$*. If*

$$\lim_{t\to\infty} f(t) = f_0,$$

then the solution $v(t)$ *of the perturbed differential problem* (4.4) *satisfies*

$$\lim_{t\to\infty} v(t) = A^{\pi}x - (A+E)^{\mathrm{D}}f_0 + \int_0^{\infty} A^{\pi}f(s)\,ds.$$

If $u(t)$ *is the solution of the exact system* (4.3)*, then*

$$\lim_{t\to\infty} \|v(t)-u(t)\| \le \frac{\|A^{\mathrm{D}}\|\|E\|}{1-\|A^{\mathrm{D}}\|\|E\|}\,\|A^{\mathrm{D}}f_0\|.$$

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