

HILBERT-PACHPATTE TYPE INTEGRAL INEQUALITIES FOR FRACTIONAL DERIVATIVES*

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Abstract

In this paper we continue our investigation of multivariable integral inequalities of the type considered by Hilbert and recently by Pachpatte by focusing on fractional derivatives. Our results apply to integrable not necessarily continuous functions, and we are able to relax the original conditions to admit negative exponents in the weight functions.

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1 Introduction and preliminaries

The purpose of the present paper is to derive new integral inequalities for fractional derivatives related to those obtained originally by Hilbert, and their recent analogues and generalizations involving classical derivatives due to Pachpatte and other authors. Recall the original Hilbert's double integral inequality:

THEOREM 1.1. [2, Theorem 316] *If $p > 1$, $q = p/(p - 1)$ and*

$$\int_0^\infty f^p(x) dx \leq F, \quad \int_0^\infty g^q(y) dy \leq G,$$

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then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} F^{1/p} G^{1/q}$$

where f, g are nonnegative measurable functions not identically zero.

Pachpatte [4, 5, 6, 7, 8] obtained analogues and generalizations of the theorem in several directions. We are interested in the results represented for classical derivatives by the following theorem.

THEOREM 1.2. (Pachpatte [6, Theorem 1]) *Let $n \geq 1$ and $0 \leq k \leq n - 1$ be integers. Let $u \in C^n([0, x])$ and $v \in C^n([0, y])$, where $x > 0, y > 0$, and let $u^{(j)}(0) = v^{(j)}(0) = 0$ for $j \in \{0, \dots, n - 1\}$. Then*

$$\begin{aligned} & \int_0^x \int_0^y \frac{|u^{(k)}(s)||v^{(k)}(t)|}{s^{2n-2k-1} + t^{2n-2k-1}} ds dt \\ & \leq M(n, k, x, y) \left(\int_0^x (x-s)|u^{(n)}(s)|^2 ds \right)^{1/2} \left(\int_0^y (y-t)|v^{(n)}(t)|^2 dt \right)^{1/2} \end{aligned} \quad (1.1)$$

where

$$M(n, k, x, y) = \frac{1}{2} \frac{\sqrt{xy}}{[(n-k-1)!]^2 (2n-2k-1)}. \quad (1.2)$$

In [1], the present authors obtained extensions and modifications of Pachpatte's results, again with classical derivatives, recovering many of the theorems in [4, 5, 6, 8] as special cases. Our aim is to derive theorems of this type for fractional derivatives, and treat integrable instead of continuously differentiable functions.

For the purpose of our exposition we survey some facts about fractional derivatives needed in the sequel; for more details see the monograph [9, Chapter 1].

Let $x > 0$. By $C^m([0, x])$ we denote the space of all functions on $[0, x]$ which have continuous derivatives up to order m , and $AC([0, x])$ is the space of all absolutely continuous functions on $[0, x]$. By $AC^m([0, x])$ we denote the space of all functions $g \in C^{m-1}([0, x])$ with $g^{(m-1)} \in AC([0, x])$. For any $\alpha \in \mathbb{R}$ we denote by $[\alpha]$ the integral part of α (the integer k satisfying $k \leq \alpha < k + 1$).

Let $\alpha > 0$. For any $f \in L(0, x)$ the *Riemann–Liouville fractional integral* of f of order α is defined by

$$I^\alpha f(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} f(t) dt, \quad s \in [0, x], \quad (1.3)$$

and the *Riemann–Liouville fractional derivative* of f of order α by

$$D^\alpha f(s) = \left(\frac{d}{ds} \right)^m I^{m-\alpha} f(s) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{ds} \right)^m \int_0^s (t-s)^{m-\alpha-1} f(t) dt \quad (1.4)$$

where $m = [\alpha] + 1$. In addition, we stipulate

$$D^0 f := f =: I^0 f, \quad I^{-\beta} f := D^\beta f \text{ if } \beta > 0, \quad D^{-\alpha} f := I^\alpha f \text{ if } 0 < \alpha \leq 1. \quad (1.5)$$

If α is a positive integer, then $D^\alpha f = (d/ds)^\alpha f$.

Let $\alpha > 0$ and $m = [\alpha] + 1$. The space $I^\alpha(L_1)$ consists of all functions f on $[0, x]$ of the form $f = I^\alpha \varphi$ for some $\varphi \in L(0, x)$ (see [9, Chapter I, Definition 2.3]). According to [9, Theorem 1.2.3], this is equivalent to the condition

$$I^{m-\alpha} f \in AC^m([0, x]), \quad (1.6)$$

$$\left(\frac{d}{ds}\right)^j I^{m-\alpha} f(0) = 0, \quad j = 0, 1, \dots, m-1. \quad (1.7)$$

A function $f \in L(0, x)$ satisfying (1.6) is said to have an *integrable fractional derivative* $D^\alpha f$ [9, Chapter I, Definition 2.4]. In particular, $D^\alpha f$ is an integrable fractional derivative if $D^\alpha f$ exists at each point s of $[0, x]$. We find it convenient to express these conditions in terms of fractional derivatives.

LEMMA 1.3. *Let $\alpha > 0$ and $m = [\alpha] + 1$. A function $f \in L(0, x)$ has an integrable fractional derivative $D^\alpha f$ if and only if*

$$D^{\alpha-k} f \in C([0, x]), \quad k = 1, \dots, m, \quad \text{and} \quad D^{\alpha-1} f \in AC([0, x]). \quad (1.8)$$

Further, $f \in I^\alpha(L_1)$ if and only if f has an integrable fractional derivative $D^\alpha f$ and satisfies the conditions

$$D^{\alpha-k} f(0) = 0 \text{ for } k = 1, \dots, m. \quad (1.9)$$

P r o o f. Observe that, in view of the definition of fractional derivative and of the equation $[\alpha - m + k] + 1 = k$,

$$\left(\frac{d}{ds}\right)^k I^{m-\alpha} f = \left(\frac{d}{ds}\right)^k I^{k-(\alpha-m+k)} f = D^{\alpha-m+k} f.$$

Then (1.8) is equivalent to (1.6) and (1.9) is equivalent to (1.7). (For $k = m$ we use the stipulation $D^{\alpha-m} f = I^{m-\alpha} f$ in (1.8).) \square

DEFINITION 1.4. We say that $f \in L(0, x)$ has an L^∞ *fractional derivative* $D^\alpha f$ in $[0, x]$ if conditions (1.8) are satisfied and $D^\alpha f \in L^\infty(0, x)$.

The next result is a version of Taylor's theorem for fractional derivatives with an integral remainder, which will be needed later in the paper. For a generalization of this result see [10].

LEMMA 1.5. [9, Chapter I, Theorem 2.2] *Let $\alpha \geq 0$, $m = [\alpha] + 1$ and $f \in AC^m([0, x])$. Then the fractional derivative $D^\alpha f$ exists almost everywhere in $[0, x]$ and*

$$D^\alpha f(s) = \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)} s^{k-\alpha} + I^{m-\alpha} f^{(m)}(s), \quad s \in [0, x]. \quad (1.10)$$

We will also need the following result on the law of indices for fractional integration and differentiation using the unified notation (1.5) and restricting ourselves to real exponents.

LEMMA 1.6. [9, Chapter I, Theorem 2.5] *The law of indices*

$$I^u I^v f = I^{u+v} f \quad (1.11)$$

is valid in the following cases:

- (i) $v > 0$, $u + v > 0$ and $f \in L(0, x)$;
- (ii) $v < 0$, $u > 0$ and $f \in I^{-v}(L_1)$;
- (iii) $u < 0$, $u + v < 0$ and $f \in I^{-u-v}(L_1)$.

2 Preparatory inequalities

We start with an inequality needed in the proof of the main theorem.

PROPOSITION 2.1. *Let $\Phi \in L^\infty(0, s)$ be nonnegative on $(0, s)$, where $s > 0$. Let $1/p + 1/q = 1$ with $p, q > 1$, let $r > -1$, and let $a, b \in \mathbb{R}$ satisfy*

$$a \geq 0, \quad b \geq 0, \quad a + b = 1; \quad b > \frac{r+1}{(1-q)r} \text{ if } r < 0. \quad (2.1)$$

Then

$$\int_0^s (s-t)^r \Phi(t) dt \leq \frac{s^{((a+bq)r+1)/q}}{((a+bq)r+1)^{1/q}} \left(\int_0^s (s-t)^{ar} \Phi(t)^p dt \right)^{1/p}. \quad (2.2)$$

P r o o f. First assume that $-1 < r < 0$. We observe that $t \mapsto (s-t)^r \Phi(t) \in L(0, s)$ as $|(s-t)^r \Phi(t)| \leq \text{const}(s-t)^r$ a.e. in $(0, s)$, and $t \mapsto (s-t)^r \in L(0, s)$ if $r > -1$. Factorize the integrand on the left in (2.2) as

$$(s-t)^r \Phi(t) = (s-t)^{(a/q+b)r} \left[(s-t)^{ar/p} \Phi(t) \right]. \quad (2.3)$$

From (2.1) we deduce that $(a+bq)r > -1$. For $a = 0$ or $b = 0$ this is obvious, for $a > 0$ and $b > 0$ we have

$$(a+bq)r = (1-b+bq)r = b(q-1)r + r > -r - 1 + r = -1.$$

Hence the first factor $t \mapsto (s-t)^{(a/q+b)r}$ in (2.3) is in $L^q(0, s)$. Further, $ar > -1$, and the second factor $(s-t)^{ar/p} \Phi(t)$ in (2.3) is in $L^p(0, s)$.

We then apply Hölder's inequality to obtain

$$\int_0^s (s-t)^r \Phi(t) dt = \int_0^s (s-t)^{(a/q+b)r} \left[(s-t)^{ar/p} \Phi(t) \right]$$

$$\leq \left(\int_0^s (s-t)^{(a+bq)r} dt \right)^{1/q} \left(\int_0^s (s-t)^{ar} \Phi(t)^p dt \right)^{1/q},$$

from which (2.2) follows.

Let $r \geq 0$. As $\Phi \in L^\infty(0, s)$ and the exponent r in the weight function $(s-t)^r$ is nonnegative, we can apply Hölder's inequality without restriction, and the result follows. \square

In our main theorem below, u_i are given functions, the coefficients p_i, q_i are conjugate Hölder exponents to be used in applications of Hölder's inequality, and the coefficients a_i, b_i are used in exponents to factorize integrands. The coefficients w_i will act as weights in applications of the geometric–arithmetic mean inequality.

THEOREM 2.2. *For each $i \in \{1, \dots, n\}$ let $x_i > 0$, $u_i \in L(0, x_i)$ and $\Phi_i \in L^\infty(0, x_i)$ be nonnegative, $r_i > -1$, let $p_i, q_i > 1$ satisfy $1/p_i + 1/q_i = 1$, $w_i > 0$ satisfy $\sum_{i=1}^n w_i = 1$, and $a_i, b_i \in [0, 1]$ satisfy $a_i + b_i = 1$; in addition, $b_i > (r_i + 1)/(1 - q_i)r_i$ for those i for which $r_i < 0$. If*

$$|u_i(s_i)| \leq \int_0^{s_i} (s_i - \tau_i)^{r_i} \Phi_i(\tau_i) d\tau_i, \quad s_i \in [0, x_i], \quad i = 1, \dots, n, \quad (2.4)$$

then

$$\begin{aligned} & \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{((a_i+b_iq_i)r_i+1)/(q_iw_i)}} ds_1 \cdots ds_n \\ & \leq \Omega \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)^{a_i r_i + 1} \Phi_i(s_i)^{p_i} ds_i \right)^{1/p_i} \end{aligned} \quad (2.5)$$

where

$$\Omega = \frac{1}{\prod_{i=1}^n [((a_i + b_i q_i) r_i + 1)^{1/q_i} (a_i r_i + 1)^{1/p_i}]} \quad (2.6)$$

P r o o f. According to Proposition 2.1,

$$|u_i(s_i)| \leq \frac{s_i^{((a_i+b_iq_i)r_i+1)/q_i}}{((a_i + b_i q_i) r_i + 1)^{1/q_i}} \left(\int_0^{s_i} (s_i - \tau_i)^{a_i r_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{1/p_i}.$$

Using the inequality of means [3, p. 15]

$$\prod_{i=1}^n s_i^{((a_i+b_iq_i)r_i+1)/q_i} \leq \sum_{i=1}^n w_i s_i^{((a_i+b_iq_i)r_i+1)/(q_iw_i)},$$

we get

$$\prod_{i=1}^n |u_i(s_i)| \leq \Theta \sum_{i=1}^n w_i s_i^{((a_i+b_iq_i)r_i+1)/(q_iw_i)} \prod_{i=1}^n \left(\int_0^{s_i} (s_i - \tau_i)^{a_i r_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{1/p_i}$$

where

$$\Theta = \frac{1}{\prod_{i=1}^n ((a_i + b_i q_i) r_i + 1)^{1/q_i}}.$$

In the following estimate we apply Hölder's inequality and, at the end, change the order of integration. The existence and finiteness of integrals is shown as in the proof of Proposition 2.1.

$$\begin{aligned} & \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{((a_i + b_i q_i) r_i + 1)/(q_i w_i)}} ds_1 \cdots ds_n \\ & \leq \Theta \prod_{i=1}^n \left[\int_0^{x_i} \left(\int_0^{s_i} (s_i - \tau_i)^{a_i r_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{1/p_i} ds_i \right] \\ & \leq \Theta \prod_{i=1}^n x_i^{1/q_i} \left(\int_0^{x_i} \left(\int_0^{s_i} (s_i - \tau_i)^{a_i r_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right) ds_i \right)^{1/p_i} \\ & = \frac{\Theta}{\prod_{i=1}^n (a_i r_i + 1)^{1/p_i}} \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - \tau_i)^{a_i r_i + 1} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{1/p_i}. \end{aligned}$$

This proves the theorem. \square

REMARK 2.3. The condition that $\Phi \in L^\infty(0, x)$ in Proposition 2.1 and the preceding theorem can be replaced by the assumption that the function $t \mapsto (s - t)^{ar} \Phi(t)^p$ is integrable on $(0, s)$; this allows unbounded Φ .

COROLLARY 2.4. *Under the assumptions of Theorem 2.2,*

$$\begin{aligned} & \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{((a_i + b_i q_i) r_i + 1)/(q_i w_i)}} ds_1 \cdots ds_n \\ & \leq p^{1/p} \Omega \prod_{i=1}^n x_i^{1/q_i} \left(\sum_{i=1}^n \frac{1}{p_i} \int_0^{x_i} (x_i - s_i)^{a_i r_i + 1} \Phi(\tau_i)^{p_i} ds_i \right)^{1/p}, \quad (2.7) \end{aligned}$$

where Ω is given by (2.5) and $p = p_1 + \cdots + p_n$.

P r o o f. By the inequality of means, for any $A_i \geq 0$,

$$\prod_{i=1}^n A_i^{1/p_i} \leq p^{1/p} \left(\sum_{i=1}^n \frac{1}{p_i} A_i \right)^{1/p}.$$

The corollary then follows from the preceding theorem. \square

3 Inequalities for fractional derivatives

Our first result is an integral representation of the fractional derivative $D^\alpha f$ which will enable us to apply Theorem 2.2.

LEMMA 3.1. *Let $\alpha \geq 0$, $\beta > \alpha$, let $f \in L(0, x)$ have an L^∞ fractional derivative $D^\beta f$ in $[0, x]$, and let $D^{\beta-k}f(0) = 0$ for $k = 1, \dots, [\beta] + 1$. Then*

$$D^\alpha f(s) = \frac{1}{\Gamma(\beta - \alpha)} \int_0^s (t - s)^{\beta - \alpha - 1} D^\beta f(t) dt, \quad s \in [0, x]. \quad (3.1)$$

P r o o f. Set $u = \beta - \alpha > 0$ and $v = -\beta < 0$. According to Lemma 1.3, $f \in I^{-v}(L_1)$. Then case 2) of Lemma 1.6 guarantees that the law of indices holds for this choice of u, v , namely

$$I^{\beta - \alpha} D^\beta f = I^u I^v f = I^{u+v} f = I^{-\alpha} f = D^\alpha f;$$

this is (3.1). \square

The following theorem is the main result of this section, a new inequality for fractional derivatives derived from (2.5).

THEOREM 3.2. *Let $n \in \mathbb{N}$. For each $i \in \{1, \dots, n\}$ let $x_i > 0$, $\alpha_i \geq 0$, $\beta_i > \alpha_i$. Let $p_i, q_i > 1$ satisfy $1/p_i + 1/q_i = 1$, $w_i > 0$ satisfy $\sum_{i=1}^n w_i = 1$, and $a_i, b_i \in [0, 1]$ satisfy $a_i + b_i = 1$; if $\beta_i < \alpha_i + 1$, let in addition $b_i > (\beta_i - \alpha_i)/(q_i - 1)(1 - \beta_i + \alpha_i)$. Write $r_i = \beta_i - \alpha_i - 1$. If, for each $i \in \{1, \dots, n\}$, $f_i \in L(0, x_i)$ has an L^∞ fractional derivative $D^{\beta_i} f_i$ and $D^{\beta_i - j} f_i(0) = 0$ for $j = 1, \dots, [\beta_i] + 1$, then*

$$\begin{aligned} & \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{\alpha_i} f_i(s_i)|}{\sum_{i=1}^n w_i s_i^{((a_i + b_i q_i) r_i + 1)/(q_i w_i)}} ds_1 \cdots ds_n \\ & \leq \Omega_1 \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)^{a_i r_i + 1} |D^{\beta_i} f_i(s_i)|^{p_i} ds_i \right)^{1/p_i} \end{aligned} \quad (3.2)$$

where

$$\Omega_1 = \frac{1}{\prod_{i=1}^n [\Gamma(r_i + 1) ((a_i + b_i q_i) r_i + 1)^{1/q_i} (a_i r_i + 1)^{1/p_i}]} \quad (3.3)$$

P r o o f. By Lemma 3.1,

$$D^{\alpha_i} f_i(s_i) = \frac{1}{\Gamma(r_i + 1)} \int_0^{s_i} (s_i - t_i)^{r_i} D^{\beta_i} f_i(t_i) dt_i, \quad s_i \in [0, x_i].$$

Set

$$\Phi_i(t_i) = \frac{|D^{\beta_i} f_i(t_i)|}{\Gamma(r_i + 1)}.$$

Then Theorem 2.2 applies with $r_i = \beta_i - \alpha_i - 1 > -1$. \square

REMARK 3.3. Instead of the hypothesis $D^{\beta_i}f_i \in L^\infty(0, x_i)$ in the preceding theorem we may assume that for each $s_i \in [0, x_i]$ the functions defined by $t_i \mapsto (s_i - t_i)^{a_i r_i} |D^{\beta_i}f_i(t_i)|^{p_i}$ are integrable on $(0, x_i)$. This allows unbounded fractional derivatives $D^{\beta_i}f_i$.

COROLLARY 3.4. *Under the assumptions of Theorem 3.2,*

$$\begin{aligned} & \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{\alpha_i}f_i(s_i)|}{\sum_{i=1}^n w_i s_i^{((a_i+b_i q_i)r_i+1)/(q_i w_i)}} ds_1 \cdots ds_n \\ & \leq p^{1/p} \Omega_1 \prod_{i=1}^n x_i^{1/q_i} \left(\sum_{i=1}^n \frac{1}{p_i} \int_0^{x_i} (x_i - s_i)^{a_i r_i + 1} |D^{\beta_i}f_i(s_i)|^{p_i} ds_i \right)^{1/p}, \end{aligned} \quad (3.4)$$

where Ω_1 is given by (3.3) and $p = p_1 + \cdots + p_n$.

A useful specialization of Theorem 3.2 is to functions possessing classical derivatives. From the definition of $AC^m([0, x])$ it follows that if $f \in AC^m([0, x])$, the derivative $f^{(m)}$ exists almost everywhere in $[0, x]$ and is integrable there. As before, we first obtain an integral representation of $D^\alpha f$, a consequence of Lemma 1.5.

LEMMA 3.5. *Let $\alpha \geq 0$, $f \in AC^m([0, x])$ and let $f^{(k)}(0) = 0$ for $k = 0, \dots, m-1$, where $m = [\alpha] + 1$. Then the derivative $D^\alpha f$ exists in $[0, x]$, and*

$$D^\alpha f(s) = \frac{1}{\Gamma(m-\alpha)} \int_0^s (s-t)^{m-\alpha-1} f^{(m)}(t) dt, \quad s \in [0, x]. \quad (3.5)$$

THEOREM 3.6. *Let $n \in \mathbb{N}$. For each $i \in \{1, \dots, n\}$ let $x_i > 0$, $\alpha_i \geq 0$, $m_i = [\alpha_i] + 1$, let $p_i, q_i > 1$ satisfy $1/p_i + 1/q_i = 1$, $w_i > 0$ satisfy $\sum_{i=1}^n w_i = 1$, and $a_i, b_i \in [0, 1]$ satisfy $a_i + b_i = 1$; let also $b_i > (m_i - \alpha_i)/(q_i - 1)(1 - m_i + \alpha_i)$ if α_i is not an integer. Write $r_i = m_i - \alpha_i - 1$. If, for each $i \in \{1, \dots, n\}$, $f_i \in AC^{m_i}([0, x_i])$, $f_i^{(m_i)} \in L^\infty(0, x_i)$ and $f_i^{(j)}(0) = 0$ for $j = 0, \dots, m_i - 1$, then the inequality (3.2) holds with $\beta_i = m_i$ and with Ω_1 defined by (3.3).*

If $\alpha_i = k_i \in \mathbb{N}$ and $\beta_i = m_i \in \mathbb{N}$ for $i = 1, \dots, n$, Theorem 3.2 specializes to the following.

THEOREM 3.7. *For each $i \in \{1, \dots, n\}$ let $m_i \in \mathbb{N}$, $u_i \in C^{m_i}([0, x_i])$ be such that $u_i^{(j)}(0) = 0$ for $j \in \{0, \dots, m_i - 1\}$, and let $k_i \in \{0, \dots, m_i - 1\}$. Further, let $p_i, q_i \in (1, \infty)$ satisfy $1/p_i + 1/q_i = 1$, and let $a_i, b_i \in [0, 1]$ satisfy $a_i + b_i = 1$. Write $r_i = m_i - k_i - 1$. Then*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{((a_i+b_i q_i)r_i+1)/(q_i w_i)}} ds_1 \cdots ds_n$$

$$\leq K \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)^{a_i r_i + 1} |u_i^{(m_i)}(s_i)|^{p_i} ds_i \right)^{1/p_i}$$

where

$$K = \frac{1}{\prod_{i=1}^n [r_i! ((a_i + b_i q_i) r_i + 1)^{1/q_i} (a_i r_i + 1)^{1/p_i]}.$$

EXAMPLE 3.8. Suppose that $\beta_i \geq \alpha_i + 1$ for $i \in \{1, \dots, n\}$. Then we can choose $a_i = 1$ and $b_i = 0$ ($i = 1, \dots, n$) in Theorem 3.2. The inequality (3.2) becomes

$$\begin{aligned} & \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{\alpha_i} f_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(r_i+1)/(q_i w_i)}} ds_1 \cdots ds_n \\ & \leq \frac{1}{\prod_{i=1}^n \Gamma(r_i + 2)} \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)^{r_i+1} |D^{\beta_i} f_i(s_i)|^{p_i} ds_i \right)^{1/p_i}. \end{aligned} \quad (3.6)$$

EXAMPLE 3.9. In Theorem 3.2 set $\alpha_i = \alpha$, $\beta_i = \beta$, where $\beta \geq \alpha + 1$. Then we can choose $a_i = 1$ and $b_i = 0$, $q_i = n$, $w_i = 1/n$, $p_i = n/(n-1)$ for $i = 1, \dots, n$. The inequality (3.2) becomes

$$\begin{aligned} & \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{\alpha} f_i(s_i)|}{\sum_{i=1}^n s_i^{\beta-\alpha}} ds_1 \cdots ds_n \\ & \leq \frac{(x_1 \cdots x_n)^{1/n}}{n \Gamma^n(\beta - \alpha + 1)} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)^{\beta-\alpha} |D^{\beta} f_i(s_i)|^{n/(n-1)} ds_i \right)^{(n-1)/n}. \end{aligned} \quad (3.7)$$

EXAMPLE 3.10. In Theorem 3.2 set $a_i = 0$ and $b_i = 1$ for $i = 1, \dots, n$. Then

$$\begin{aligned} & \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{\alpha_i} f_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(q_i r_i + 1)/(q_i w_i)}} ds_1 \cdots ds_n \\ & \leq \frac{1}{\prod_{i=1}^n [\Gamma(r_i + 1) (q_i r_i + 1)^{1/q_i}]} \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i) |D^{\beta_i} f_i(s_i)|^{p_i} ds_i \right)^{1/p_i}. \end{aligned} \quad (3.8)$$

EXAMPLE 3.11. In Theorem 3.2 set $\alpha_i = \alpha$, $\beta_i = \beta$, $a_i = 0$ and $b_i = 1$ for $i = 1, \dots, n$. Set further $q_i = n$, $w_i = 1/n$, and $p_i = n/(n-1)$, for $i = 1, \dots, n$. Then

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{\alpha} f_i(s_i)|}{\sum_{i=1}^n s_i^{n(\beta-\alpha-1)+1}} ds_1 \cdots ds_n$$

$$\leq \frac{(x_1 \dots x_n)^{1/n}}{n\Gamma^n(\beta - \alpha)(n(\beta - \alpha - 1) + 1)} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i) |D^{\beta} f_i(s_i)|^{n/(n-1)} ds_i \right)^{(n-1)/n}. \quad (3.9)$$

Setting $q = p = n = 2$, we obtain [6, Theorem 1] with $k = 0$. If $q = p = 2$ and $n = 1$, we recover the result of [8].

We observe that in the special case when $\alpha_i = 0$ for $i = 1, \dots, n$, the inequalities obtained in Theorems 3.2 and 3.6, Corollary 3.4 and the preceding examples reduce to new inequalities that estimate functions in terms of their fractional derivatives.

References

- [1] G. D. Handley, J. J. Koliha and J. Pečarić, A Hilbert type inequality, *Tamkang J. Math.*, to appear.
- [2] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, Cambridge University Press, 1934.
- [3] D. S. Mitrinović, J. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Acad. Publ., Dordrecht, 1993.
- [4] B. G. Pachpate, A note on Hilbert type inequality, *Tamkang J. Math.* **29** (1998), 293–298.
- [5] B. G. Pachpate, On some new inequalities similar to Hilbert's inequality, *J. Math. Anal. Appl.* **226** (1998), 166–179.
- [6] B. G. Pachpate, Inequalities similar to the integral analogue of Hilbert's inequality, *Tamkang J. Math.* **30** (1999), 139–146.
- [7] B. G. Pachpate, Inequalities similar to certain extensions of Hilbert's inequality, *J. Math. Anal. Appl.* **243** (2000), 217–227.
- [8] B. G. Pachpate, A note on inequality of Hilbert type, *Demonstratio Math.*, to appear.
- [9] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Reading, 1993.

- [10] J. J. T r u j i l l o, M. R i v e r o and B. B o n i l l a, On a Riemann–Liouville generalized Taylor’s formula, *J. Math. Anal. Appl.* **231** (1999), 255–265.

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