New Hilbert–Pachpatte type integral inequalities

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In this paper we obtain a new class of multivariable integral inequalities of Hilbert type from which we can recover as special cases integral inequalities obtained recently by Pachpatte.

1 Introduction

Hilbert's double series inequality and its integral version [3, Theorem 316] have been generalized in several directions (see [1, 2, 3, 4, 5, 6, 8, 9]). Recently, Pachpatte [10] considered integral inequalities similar to those of Hilbert. A representative sample is the following.

Theorem 1.1. (Pachpatte [10, Theorem 1].) *Let* $n \geq 1$ *and* $0 \leq k \leq n-1$ *be integers. Let* $u \in C^n([0,x])$ *and* $v \in C^n([0,y])$*, where* $x > 0$*,* $y > 0$*, and let* $u^{(j)}(0) = v^{(j)}(0) = 0$ *for* $j \in \{0, \dots, n-1\}$*. Then*

$$\int_0^x \int_0^y \frac{|u^{(k)}(s)||v^{(k)}(t)|}{s^{2n-2k-1} + t^{2n-2k-1}} \, ds \, dt$$
$$\leq M(n,k,x,y) \left(\int_0^x (x-s)|u^{(n)}(s)|^2 \, ds \right)^{1/2} \left(\int_0^y (y-t)|v^{(n)}(t)|^2 \, dt \right)^{1/2} \tag{1.1}$$

where

$$M(n,k,x,y) = \frac{1}{2} \frac{\sqrt{xy}}{[(n-k-1)!]^2 (2n-2k-1)}. \tag{1.2}$$

The purpose of the present paper is to derive a new class of related integral inequalities from which the results of Pachpatte in [10, 11] are obtained by specializing the functions Φ_i in (3.1) below.

2 Notation and preliminaries

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ have their usual meaning, $\mathbb{R}_+$ denotes the interval $[0, \infty)$. The following notation and hypotheses will be used throughout the paper.

$I = \{1, \dots, n\}$	$n \in \mathbb{N}$
$m_i,\ i \in I$	$m_i \in \mathbb{N}$
$k_i,\ i \in I$	$k_i \in \{0, \dots, m_i - 1\}$
$x_i,\ i \in I$	$x_i \in \mathbb{R}$, $x_i > 0$
$p_i,\ q_i,\ i \in I$	$p_i,\ q_i \in \mathbb{R}_+$, $1/p_i + 1/q_i = 1$
$p,\ q$	$1/p = \sum_{i=1}^n (1/p_i)$, $1/q = \sum_{i=1}^n (1/q_i)$
$a_i,\ b_i,\ i \in I$	$a_i,\ b_i \in \mathbb{R}_+$, $a_i + b_i = 1$
$w_i,\ i \in I$	$w_i \in \mathbb{R}$, $w_i > 0$, $\sum_{i=1}^n w_i = 1$
$\alpha_i,\ i \in I$	$\alpha_i = (a_i + b_i q_i)(m_i - k_i - 1)$
$\beta_i,\ i \in I$	$\beta_i = a_i(m_i - k_i - 1)$
$u_i,\ i \in I$	$u_i \in C^{m_i'}([0, x_i])$ for some $m_i' \geq m_i$
$\Phi_i,\ i \in I$	$\Phi_i \in C^1([0, x_i])$, $\Phi_i \geq 0$

Here u_i are given functions of sufficient smoothness, and Φ_i are subject to choice. The coefficients p_i, q_i are conjugate Hölder exponents to be used in applications of Hölder's inequality, and the coefficients a_i, b_i will be used in exponents to factorize integrands. The coefficients w_i will act as weights in applications of the geometric-arithmetic mean inequality; this will enable us to pass from products to sums of terms. The coefficients α_i and β_i arise naturally in the derivation of the inequalities.

The key to the results derived in this paper are the inequalities (3.1). Such inequalities are always available with some continuous nonnegative functions Φ_i provided the u_i are sufficiently smooth and their derivatives at 0 satisfy certain conditions (vanish).

3 The main result

The theorem of this section forms an abstract basis for obtaining a class of concrete inequalities by selecting suitable functions Φ_i in (3.1); as noted above, such functions Φ_i always exist under suitable hypotheses on the u_i.

Theorem 3.1. *Let $u_i \in C^{m_i}([0, x_i])$ for $i \in I$. If*

$$\left|u_i^{(k_i)}(s_i)\right| \leq \int_0^{s_i} (s_i - \tau_i)^{m_i - k_i - 1} \Phi_i(\tau_i)\, d\tau_i,\ \ s_i \in [0, x_i],\ \ i \in I, \tag{3.1}$$

then

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_iw_i)}}\,ds_1\cdots ds_n$$
$$\leq U\prod_{i=1}^n x_i^{1/q_i}\prod_{i=1}^n\left(\int_0^{x_i}(x_i-s_i)^{\beta_i+1}\Phi_i(s_i)^{p_i}\,ds_i\right)^{1/p_i} \tag{3.2}$$

where

$$U=\frac{1}{\prod_{i=1}^n[(\alpha_i+1)^{1/q_i}(\beta_i+1)^{1/p_i}]}. \tag{3.3}$$

Proof. Factorize the integrand on the right side of (3.1) as

$$(s_i-\tau_i)^{(a_i/q_i+b_i)(m_i-k_i-1)}\times(s_i-\tau_i)^{(a_i/p_i)(m_i-k_i-1)}\Phi_i(\tau_i)$$

and apply Hölder's inequality [7, p. 106]. Then

$$\left|u_i^{(k_i)}(s_i)\right|\leq\left(\int_0^{s_i}(s_i-\tau_i)^{(a_i+b_iq_i)(m_i-k_i-1)}\,d\tau_i\right)^{1/q_i}\times$$
$$\times\left(\int_0^{s_i}(s_i-\tau_i)^{a_i(m_i-k_i-1)}\Phi_i(\tau_i)^{p_i}\,d\tau_i\right)^{1/p_i}$$
$$=\frac{s_i^{(\alpha_i+1)/q_i}}{(\alpha_i+1)^{1/q_i}}\left(\int_0^{s_i}(s_i-\tau_i)^{\beta_i}\Phi_i(\tau_i)^{p_i}\,d\tau_i\right)^{1/p_i}.$$

Using the inequality of means [7, p. 15]

$$\prod_{i=1}^n s_i^{(\alpha_i+1)/q_i}\leq\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_iw_i)},$$

we get

$$\prod_{i=1}^n|u_i^{(k_i)}(s_i)|\leq W\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_iw_i)}\prod_{i=1}^n\left(\int_0^{s_i}(s_i-\tau_i)^{\beta_i}\Phi_i(\tau_i)^{p_i}\,d\tau_i\right)^{1/p_i}$$

where

$$W=\frac{1}{\prod_{i=1}^n(\alpha_i+1)^{1/q_i}}.$$

In the following estimate we apply Hölder's inequality and, at the end, change the order of integration:

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n|u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_iw_i)}}\,ds_1\cdots ds_n$$

$$\begin{aligned}
&\le W\prod_{i=1}^n\left[\int_0^{x_i}\left(\int_0^{s_i}(s_i-\tau_i)^{\beta_i}\Phi_i(\tau_i)^{p_i}\,d\tau_i\right)^{1/p_i}ds_i\right]\\
&\le W\prod_{i=1}^n x_i^{1/q_i}\left(\int_0^{x_i}\left(\int_0^{s_i}(s_i-\tau_i)^{\beta_i}\Phi_i(\tau_i)^{p_i}\,d\tau_i\right)ds_i\right)^{1/p_i}\\
&=\frac{W}{\prod_{i=1}^n(\beta_i+1)^{1/p_i}}\prod_{i=1}^n x_i^{1/q_i}\prod_{i=1}^n\left(\int_0^{x_i}(x_i-\tau_i)^{\beta_i+1}\Phi_i(\tau_i)^{p_i}\,d\tau_i\right)^{1/p_i}.
\end{aligned}$$

This proves the theorem. □

Corollary 3.2. *Under the assumptions of Theorem* 3.1*,*

$$\begin{aligned}
&\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n|u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_iw_i)}}\,ds_1\cdots ds_n\\
&\qquad\le p^{1/p}U\prod_{i=1}^n x_i^{1/q_i}\left(\sum_{i=1}^n\frac{1}{p_i}\int_0^{x_i}(x_i-s_i)^{\beta_i+1}\Phi(\tau_i)^{p_i}\,ds_i\right)^{1/p}, \qquad (3.4)
\end{aligned}$$

where U is given by (3.2)*.*

Proof. By the inequality of means, for any $A_i \ge 0$,

$$\prod_{i=1}^n A_i^{1/p_i}\le p^{1/p}\left(\sum_{i=1}^n\frac{1}{p_i}A_i\right)^{1/p}.$$

The corollary then follows from the preceding theorem. □

In the following sections we discuss various choices of the functions Φ_i.

4 The first inequality

Theorem 4.1. *Let $u_i \in C^{m_i}([0,x_i])$ be such that $u_i^{(j)}(0)=0$ for $j\in\{0,\dots,m_i-1\}$, $i\in I$. Then*

$$\begin{aligned}
&\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n|u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_iw_i)}}\,ds_1\cdots ds_n\\
&\qquad\le U_1\prod_{i=1}^n x_i^{1/q_i}\prod_{i=1}^n\left(\int_0^{x_i}(x_i-s_i)^{\beta_i+1}\left|u_i^{(m_i)}(s_i)\right|^{p_i}ds_i\right)^{1/p_i} \qquad (4.1)
\end{aligned}$$

where

$$U_1=\frac{1}{\prod_{i=1}^n[(m_i-k_i-1)!(\alpha_i+1)^{1/q_i}(\beta_i+1)^{1/p_i}]}. \qquad (4.2)$$

Proof. By [10, Equation (7)],

$$u_i^{(k_i)}(s) = \frac{1}{(m_i-k_i-1)!}\int_0^{s_i}(s_i-\tau_i)^{m_i-k_i-1}u_i^{(m)}(\tau_i)\,d\tau_i.$$

Inequality (4.1) is proved when we set

$$\Phi_i(s_i) = \frac{1}{(m_i-k_i-1)!}\int_0^{s_i}(s_i-\tau_i)^{m_i-k_i-1}\big|u_i^{(m_i)}(\tau_i)\big|\,d\tau_i \tag{4.3}$$

in Theorem 3.1. □

Corollary 4.2. *Under the hypotheses of Theorem* 4.1*,*

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n|u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_iw_i)}}\,ds_1\cdots ds_n \\ \leq p^{1/p}U_1\prod_{i=1}^n x_i^{1/q_i}\left(\sum_{i=1}^n\frac{1}{p_i}\int_0^{x_i}(x_i-s_i)^{\beta_i+1}\big|u_i^{(m_i)}(s_i)\big|^{p_i}\,ds_i\right)^{1/p} \tag{4.4}$$

where U_1 is given by (4.2)*.*

We discuss a number of special cases of Theorem 4.1. Similar examples apply also to Corollary 4.2.

Example 4.3. If $a_i = 0$ and $b_i = 1$ for $i \in I$, then (4.1) becomes

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n|u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{(q_im_i-q_ik_i-q_i+1)/(q_iw_i)}}\,ds_1\cdots ds_n \\ \leq \overline{U_1}\prod_{i=1}^n x_i^{1/q_i}\prod_{i=1}^n\left(\int_0^{x_i}(x_i-s_i)\big|u_i^{(m_i)}(s_i)\big|^{p_i}\,ds_i\right)^{1/p_i} \tag{4.5}$$

where

$$\overline{U_1} = \frac{1}{\prod_{i=1}^n[(m_i-k_i-1)!(q_im_i-q_ik_i-q_i+1)^{1/q_i}]}. \tag{4.6}$$

Example 4.4. If $a_i = 0$, $b_i = 1$, $q_i = n$, $w_i = 1/n$, $p_i = n/(n-1)$, $m_i = m$ and $k_i = k$ for $i \in I$, then

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n|u_i^{(k)}(s_i)|}{\sum_{i=1}^n s_i^{nm-nk-n+1}}\,ds_1\cdots ds_n \\ \leq \frac{1}{n}\frac{\sqrt[n]{x_1\cdots x_n}}{[(m-k-1)!]^n(nm-nk-n+1)}\times$$

$$\times \prod_{i=1}^{n} \left(\int_0^{x_i} (x_i - s_i) \big| u_i^{(m)}(s_i) \big|^{n/(n-1)} \, ds_i \right)^{(n-1)/n} . \tag{4.7}$$

For $q = p = n = 2$ this is [10, Theorem 1]. Setting $q = p = 2$, $k = 0$ and $n = 1$, we recover the result of [11].

Example 4.5. Let $a_i = 1$ and $b_i = 0$ for $i \in I$. Then (4.1) becomes

$$\begin{aligned} \int_0^{x_1} \cdots \int_0^{x_n} & \frac{\prod_{i=1}^{n} |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^{n} w_i s_i^{(m_i - k_i)/(q_i w_i)}} \, ds_1 \cdots ds_n \\ & \leq \widetilde{U_1} \prod_{i=1}^{n} x_i^{1/q_i} \prod_{i=1}^{n} \left(\int_0^{x_i} (x_i - s_i)^{m_i - k_i} \big| u_i^{(m_i)}(s_i) \big|^{p_i} ds_i \right)^{1/p_i} \end{aligned} \tag{4.8}$$

where

$$\widetilde{U_1} = \frac{1}{\prod_{i=1}^{n} [(m_i - k_i - 1)!(m_i - k_i)]} . \tag{4.9}$$

Example 4.6. Set $a_i = 0$, $b_i = 1$, $q_i = n$, $w_i = 1/n$, $p_i = n/(n-1)$, $m_i = m$ and $k_i = k$ for $i \in I$. Then (4.1) becomes

$$\begin{aligned} \int_0^{x_1} \cdots \int_0^{x_n} & \frac{\prod_{i=1}^{n} |u_i^{(k)}(s_i)|}{\sum_{i=1}^{n} s_i^{m-k}} \, ds_1 \cdots ds_n \\ & \leq \frac{1}{n} \frac{\sqrt[n]{x_1 \cdots x_n}}{[(m-k-1)!]^n (m-k)^n} \prod_{i=1}^{n} \left(\int_0^{x_i} (x_i - s_i)^{m-k} \big| u_i^{(m)}(s_i) \big|^{n/(n-1)} ds_i \right)^{(n-1)/n} . \end{aligned} \tag{4.10}$$

5 The second inequality

Theorem 5.1. *Let* $u_i \in C^{m_i+1}([0, x_i])$ *be such that* $u^{(j)}(0) = 0$ *for* $j \in \{0, \ldots, m_i\}$, *and let* $\rho \in C^1([0, \infty))$. *Then*

$$\begin{aligned} \int_0^{x_1} \cdots \int_0^{x_n} & \frac{\prod_{i=1}^{n} |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^{n} w_i s_i^{(\alpha_i + 1)/(q_i w_i)}} \, ds_1 \cdots ds_n \\ & \leq U_2 \prod_{i=1}^{n} x_i^{1/q_i} \prod_{i=1}^{n} \left[\int_0^{x_i} (x_i - s_i)^{\beta_i + 1} \frac{s_i^{p_i - 1}}{\rho(s_i)^{p_i}} \left(\int_0^{s_i} \big| (\rho(\sigma_i) u_i^{(m_i)}(\sigma_i))' \big|^{p_i} \, d\sigma_i \right) ds_i \right]^{1/p_i} \end{aligned} \tag{5.1}$$

where

$$U_2 = \frac{1}{\prod_{i=1}^{n} [(m_i - k_i - 1)!(\alpha_i + 1)^{1/q_i} (\beta_i + 1)^{1/p_i}]} . \tag{5.2}$$

Proof. According to [10, Equation (14)],

$$u_i^{(k_i)}(s_i) = \frac{1}{(m_i-k_i-1)!}\int_0^{s_i}(s_i-\tau_i)^{m_i-k_i-1}\left(\frac{1}{\rho(\tau_i)}\int_0^{\tau_i}(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))'\,d\sigma_i\right)d\tau_i.$$

By Hölder's inequality,

$$\int_0^{\tau_i}\left|(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))'\right|d\sigma_i \le \tau_i^{1/q_i}\left(\int_0^{\tau_i}\left|(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))'\right|^{p_i}d\sigma_i\right)^{1/p_i},$$

and inequalities (3.1) hold with

$$\Phi_i(\tau_i) = \frac{1}{(m_i-k_i-1)!}\frac{\tau_i^{1/q_i}}{\rho(\tau_i)}\left(\int_0^{\tau_i}\left|(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))'\right|^{p_i}d\sigma_i\right)^{1/p_i}.$$

The theorem is then proved by an application of Theorem 3.1. □

Corollary 5.2. *Under the assumptions of Theorem* 5.1*,*

$$\begin{aligned}&\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n|u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_iw_i)}}\,ds_1\cdots ds_n\\ &\le p^{1/p}U_2\prod_{i=1}^n x_i^{1/q_i}\times\\ &\quad\times\left(\sum_{i=1}^n\frac{1}{p_i}\int_0^{x_i}(x_i-s_i)^{\beta_i+1}\frac{s_i^{p_i-1}}{\rho(s_i)^{p_i}}\left(\int_0^{s_i}\left|(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))'\right|^{p_i}d\sigma_i\right)ds_i\right)^{1/p}\end{aligned}\tag{5.3}$$

where U_2 *is given by* (5.2)*.*

Example 5.3. Let $a_i = 0$ and $b_i = 1$ for $i \in I$. Then (5.1) becomes

$$\begin{aligned}&\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n|u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{(q_im_i-q_ik_i-q_i+1)/(q_iw_i)}}\,ds_1\cdots ds_n\\ &\le \overline{U_2}\prod_{i=1}^n x_i^{1/q_i}\prod_{i=1}^n\left(\int_0^{x_i}(x_i-s_i)\frac{s_i^{p_i-1}}{\rho(s_i)^{p_i}}\left(\int_0^{s_i}\left|(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))'\right|^{p_i}d\sigma_i\right)ds_i\right)^{1/p_i}\end{aligned}\tag{5.4}$$

where

$$\overline{U_2} = \frac{1}{\prod_{i=1}^n[(m_i-k_i-1)!(q_im_i-q_ik_i-q_i+1)^{1/q_i}]}.\tag{5.5}$$

Example 5.4. Let $a_i = 0$, $b_i = 1$, $q_i = n$, $p_i = n/(n-1)$, $w_i = 1/n$, $m_i = m$ and $k_i = k$ for $i \in I$. Then (5.1) becomes

$$\begin{aligned}
&\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k)}(s_i)|}{\sum_{i=1}^n s_i^{nm-nk-n+1}} ds_1 \cdots ds_n \\
&\leq \frac{1}{n} \frac{\sqrt[n]{x_1 \cdots x_n}}{[(m-k-1)!]^n (nm-nk-n+1)} \times \\
&\times \prod_{i=1}^n \left[\int_0^{x_i} (x_i - s_i) \frac{s_i^{1/(n-1)}}{\rho(s_i)^{n/(n-1)}} \left(\int_0^{s_i} \left|(\rho(\sigma_i) u_i^{(m)}(\sigma_i))'\right|^{n/(n-1)} d\sigma_i \right) ds_i \right]^{(n-1)/n}.
\end{aligned} \tag{5.6}$$

For $q = p = n = 2$ this is [10, Theorem 2].

Example 5.5. Set $a_i = 1$ and $b_i = 0$ for $i \in I$. Inequality (5.1) becomes

$$\begin{aligned}
&\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{(m_i-k_i)/(q_i w_w)}} ds_1 \cdots ds_n \\
&\leq \widetilde{U_2} \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left[\int_0^{x_i} (x_i - s_i)^{m_i - k_i} \frac{s_i^{p_i - 1}}{\rho(s_i)^{p_i}} \left(\int_0^{s_i} \left|(\rho(\sigma_i) u_i^{(m_i)}(\sigma_i))'\right|^{p_i} d\sigma_i \right) ds_i \right]^{1/p_i}
\end{aligned} \tag{5.7}$$

where

$$\widetilde{U_2} = \frac{1}{\prod_{i=1}^n [(m_i - k_i - 1)!(m_i - k_i)]}. \tag{5.8}$$

Example 5.6. Set $a_i = 1$, $b_i = 0$, $q_i = n$, $w_i = 1/n$, $p_i = n/(n-1)$, $m_i = m$ and $k_i = k$ for $i \in I$. Then (5.1) becomes

$$\begin{aligned}
&\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k)}(s_i)|}{\sum_{i=1}^n s_i^{m-k}} ds_1 \cdots ds_n \\
&\leq \frac{1}{n} \frac{\sqrt[n]{x_1 \cdots x_n}}{[(m-k-1)!]^n (m-k)^n} \times \\
&\times \prod_{i=1}^n \left[\int_0^{x_i} (x_i - s_i)^{m-k} \frac{s_i^{1/(n-1)}}{\rho(s_i)^{n/(n-1)}} \left(\int_0^{s_i} \left|(\rho(\sigma_i) u_i^{(m)}(\sigma_i))'\right|^{n/(n-1)} d\sigma_i \right) ds_i \right]^{(n-1)/n}.
\end{aligned} \tag{5.9}$$

6 The third inequality

Theorem 6.1. *Let* $u_i \in C^{2m_i}([0,x_i])$, $\rho \in C^m([0,\infty))$ *with* $m = \max_i m_i$, $u_i^{(j)}(0) = 0$ *and* $(\rho(s_i)u_i^{(m_i)}(s_i))^{(j)} = 0$ *at* $s_i = 0$ *for* $j \in \{0,\dots,m_i-1\}$, $i \in I$. *Then*

$$\int_0^{x_1}\cdots\int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)}}\, ds_1\cdots ds_n \le U_3 \prod_{i=1}^n x_i^{1/q_i} \times$$
$$\times \prod_{i=1}^n \left[\int_0^{x_i} (x_i-s_i)^{\beta_i+1}\, \frac{s_i^{(q_i(m_i-1)+1)(p_i-1)}}{\rho(s_i)^{p_i}} \left(\int_0^{s_i} \left|(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))^{(m_i)}\right|^{p_i} d\sigma_i\right) ds_i\right]^{1/p_i} \tag{6.1}$$

where

$$U_3 = \frac{1}{\prod_{i=1}^n [(m_i-1)!(m_i-k_i-1)!(q_i(m_i-1)+1)^{1/q_i}(\alpha_i+1)^{1/q_i}(\beta_i+1)^{1/p_i}]}. \tag{6.2}$$

Proof. By [10, Equation (21)],

$$u_i^{(k_i)}(s_i) = \frac{1}{(m_i-1)!(m_i-k_i-1)!}\int_0^{s_i}(s_i-\tau_i)^{m_i-k_i-1} \times$$
$$\times \left(\frac{1}{\rho(\tau_i)}\int_0^{\tau_i}(\tau_i-\sigma_i)^{m_i-1}(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))^{(m_i)}\,d\sigma_i\right) d\tau_i.$$

For brevity write

$$F_i(\sigma_i) = \left|(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))^{(m_i)}\right|.$$

Applying Hölder's inequality, we get

$$\int_0^{\tau_i}(\tau_i-\sigma_i)^{m_i-1}F_i(\sigma_i)\,d\sigma_i \le \left(\int_0^{\tau_i}(\tau_i-\sigma_i)^{q_i(m_i-1)}\,d\sigma_i\right)^{1/q_i}\left(\int_0^{\tau_i}F_i(\sigma_i)^{p_i}\,d\sigma_i\right)^{1/p_i}$$
$$= \frac{\tau_i^{(q_i(m_i-1)+1)/q_i}}{(q_i(m_i-1)+1)^{1/q_i}}\left(\int_0^{\tau_i}F_i(\sigma_i)^{p_i}\,d\sigma_i\right)^{1/p_i}.$$

Then the inequalities (3.1) are satisfied with

$$\Phi_i(\tau_i) = W\,\frac{\tau_i^{(q_i(m_i-1)+1)/q_i}}{\rho(\tau_i)}\left(\int_0^{\tau_i}\left|(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))^{(m_i)}\right|^{p_i}d\sigma_i\right)^{1/p_i}$$

where

$$W = \frac{1}{(m_i-1)!(m_i-k_i-1)!(q_i(m_i-1)+1)^{1/q_i}}.$$

The result then follows from Theorem 3.1. □

Corollary 6.2. *Under the hypotheses of Theorem* 6.1,

$$\begin{aligned}\int_0^{x_1}\cdots\int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_iw_i)}}\, ds_1\cdots ds_n \le p^{1/p} U_3 \prod_{i=1}^n x_i^{1/q_i} \times \\ \times \left(\sum_{i=1}^n \frac{1}{p_i}\int_0^{x_i} \frac{(x_i-s_i)^{\beta_i+1}\, s_i^{(q_i(m_i-1)+1)(p_i-1)}}{p_i\rho(s_i)^{p_i}} \int_0^{s_i} \left|(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))^{(m_i)}\right|^{p_i} d\sigma_i ds_i\right)^{1/p}\end{aligned} \tag{6.3}$$

where U_3 *is given by* (6.2).

Example 6.3. Let $a_i = 0$ and $b_i = 1$ for $i \in I$. Inequality (6.1) becomes

$$\begin{aligned}&\int_0^{x_1}\cdots\int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{(q_im_i-q_ik_i-q_i+1)/(q_iw_i)}}\, ds_1\cdots ds_n \\ &\quad\le \overline{U_3}\prod_{i=1}^n x_i^{1/q_i} \times \\ &\qquad\times \prod_{i=1}^n \left[\int_0^{x_i}(x_i-s_i)\frac{s_i^{(q_im_i-q_i+1)(p_i-1)}}{\rho(s_i)^{p_i}}\left(\int_0^{s_i}\left|(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))^{(m_i)}\right|^{p_i} d\sigma_i\right) ds_i\right]^{1/p_i}\end{aligned} \tag{6.4}$$

where

$$\overline{U_3} = \frac{1}{\prod_{i=1}^n [(m_i-1)!(m_i-k_i-1)!(q_im_i-q_ik_i-q_i+1)^{1/q_i}(q_im_i-q_i+1)^{1/q_i}]}. \tag{6.5}$$

Example 6.4. Set $a_i = 0$, $b_i = 1$, $q_i = n$, $w_i = 1/n$, $p_i = n/(n-1)$, $m_i = m$ and $k_i = k$ for $i \in I$. Then (6.1) becomes

$$\begin{aligned}&\int_0^{x_1}\cdots\int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k)}(s_i)|}{\sum_{i=1}^n s_i^{nm-nk-n+1}} ds_1\cdots ds_n \\ &\le \frac{1}{n}\,\frac{\sqrt[n]{x_1\cdots x_n}}{[(m-1)!]^n[(m-k-1)!]^n(nm-nk-n+1)(nm-n+1)} \times \\ &\times \prod_{i=1}^n\left[\int_0^{x_i}(x_i-s_i)\frac{s_i^{(nm-n+1)/(n-1)}}{\rho(s_i)^{n/(n-1)}}\int_0^{s_i}\left|(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))^{(m_i)}\right|^{n/(n-1)} d\sigma_i ds_i\right]^{(n-1)/n}.\end{aligned} \tag{6.6}$$

For $q = p = n = 2$ this becomes [10, Theorem 3].

Example 6.5. Set $a_i = 1$ and $b_i = 0$ for $i \in I$. Inequality (6.1) becomes

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{(m_i-k_i)/(q_i w_w)}} ds_1 \cdots ds_n \leq \widetilde{U_3} \prod_{i=1}^n x_i^{1/q_i} \times$$
$$\times \prod_{i=1}^n \left[\int_0^{x_i} (x_i - s_i)^{m_i-k_i} \frac{s_i^{(q_i(m_i-1)+1)(p_i-1)}}{\rho(s_i)^{p_i}} \left(\int_0^{s_i} \left| (\rho(\sigma_i) u_i^{(m_i)}(\sigma_i))^{(m_i)} \right|^{p_i} d\sigma_i \right) ds_i \right]^{1/p_i} \tag{6.7}$$

where

$$\widetilde{U_3} = \frac{1}{\prod_{i=1}^n [(m_i-1)!(m_i-k_i-1)!(q_i m_i - q_i + 1)^{1/q_i}(m_i - k_i)]}. \tag{6.8}$$

Example 6.6. Set $a_i = 1$, $b_i = 0$, $q_i = n$, $w_i = 1/n$, $p_i = n/(n-1)$, $m_i = m$ and $k_i = k$ for $i \in I$. Then (5.1) becomes

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k)}(s_i)|}{\sum_{i=1}^n s_i^{m-k}} ds_1 \cdots ds_n$$
$$\leq \frac{1}{n} \frac{\sqrt[n]{x_1 \cdots x_n}}{[(m-1)!]^n [(m-k-1)!]^n (nm-n+1)(m-k)^n} \times$$
$$\times \prod_{i=1}^n \left[\int_0^{x_i} (x_i - s_i)^{m-k} \frac{s_i^{(nm-n+1)/(n-1)}}{\rho(s_i)^{n/(n-1)}} \int_0^{s_i} \left| (\rho(\sigma_i) u_i^{(m)}(\sigma_i))^{(m)} \right|^{n/(n-1)} d\sigma_i ds_i \right]^{(n-1)/n}. \tag{6.9}$$

References

[1] Yang Bicheng, On Hilbert's integral inequality, *J. Math. Anal. Appl.* **220** (1998), 778–785.

[2] Y. C. Chow, On inequalities of Hilbert and Widder, *J. London Math. Soc.* 14 (1939), 151–154.

[3] G. H. Hardy, J. E. Littlewood and G. Polya, "Inequalities", Cambridge University Press, Cambridge, 1934.

[4] G. Mingzhe, An improvement of Hardy–Riesz's extension of the Hilbert inequality, *J. Math. Res. Exp.* **14** (1994), 255–259.

[5] G. Mingzhe, On Hilbert's inequality and its applications, *J. Math. Anal. Appl.* **212** (1997)

[6] D. S. Mitrinović and J. E. Pečarić, On inequalities of Hilbert and Widder, *Proc. Edinburgh Math. Soc.* **34** (1991), 411–414.

[7] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, "Classical and New Inequalities in Analysis", Kluwer Acad. Publ., Dordrecht, 1993.

[8] B. G. Pachpatte, A note on Hilbert type inequality, *Tamkang J. Math.* **29** (1998), 293–298.

[9] B. G. Pachpatte, On some new inequalities similar to Hilbert's inequality, *J. Math. Anal. Appl.* **226** (1998), 166–179.

[10] B. G. Pachpatte, Inequalities similar to the integral analogue of Hilbert's inequality, *Tamkang J. Math.* **30** (1999), 139–146.

[11] B. G. Pachpatte, A note on inequality of Hilbert type, *Demonstratio Math.*, to appear.