Hardy's discrete inequality

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Abstract

Copson's extension of Hardy's discrete inequality has been generalised in different directions by Hwang, Hwang–Yang and Pachpatte. In this paper inequalities are obtained which subsume and extend Hwang–Yang's and Pachpatte's inequalities.

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1 Introduction

In [1] Copson established the following Hardy type inequalities involving a series of positive terms (see [3, Inequality 326] and [6, p. 145]).

Theorem 1.1. *If* $p > 1$, $\lambda_n > 0, a_n > 0$, $\Lambda_n = \sum_{i=1}^{n} \lambda_i$, $A_n = \sum_{i=1}^{n} \lambda_i a_i$ *and* $\sum_{n=1}^{\infty} \lambda_n a_n^p$ *converges, then*

$$\sum_{n=1}^{\infty} \lambda_n \left(\frac{A_n}{\Lambda_n} \right)^p \le \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} \lambda_n a_n^p. \tag{1.1}$$

The constant is the best possible.

Theorem 1.2. *Let* p_n, λ_n, a_n, Λ_n *and* A_n *be as in Theorem* 1.1 *and let* $H(u)$ *be a real–valued positive convex function defined for* $u > 0$. *If* $\sum_{n=1}^{\infty} \lambda_n H^p(a_n)$ *converges, then*

$$\sum_{n=1}^{\infty} \lambda_n H^p \left(\frac{A_n}{\Lambda_n} \right) \le \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} \lambda_n H^p(a_n). \tag{1.2}$$

The constant is the best possible.

Generalizations of these theorems were given by Hwang and Yang [5] and Pachpatte [7]. We establish inequalities which subsume and extend those results.

2 Main results

Our first result extends Copson's inequality stated in our Theorem 1.1 as well as Hwang and Yang's result [5, Theorem 1].

Theorem 2.1. *Let* $p > 1$, $q \geq 0$, $\beta_n > 0$, $\lambda_n > 0$, $a_n > 0$ *for all* $n \in \mathbb{N}$, *and define*

$$\Lambda_n = \sum_{i=1}^{n} \beta_i \lambda_i, \quad A_n = \sum_{i=1}^{n} \beta_i \lambda_i a_i, \quad \omega_n = \frac{A_n}{\Lambda_n}, \quad n \in \mathbb{N}. \tag{2.1}$$

Suppose that $\sum_{n=1}^{\infty} \lambda_n a_n^p \omega_n^q$ *converges and that there exists* $\kappa > 0$ *such that*

$$p + q - 1 + \frac{(\beta_{n+1} - \beta_n)\Lambda_n}{(\beta_{n+1}\beta_n)\lambda_n} \geq \frac{p+q}{\kappa}, \quad n \in \mathbb{N}. \tag{2.2}$$

Then

$$\sum_{n=1}^{\infty} \lambda_n \omega_n^{p+q} \leq \kappa^p \sum_{n=1}^{\infty} \lambda_n a_n^p \omega_n^q. \tag{2.3}$$

Proof. Without a loss of generality we may assume that $\omega_0 = \beta_0 = \lambda_0 = 1$. In the initial part of the proof of [5, Theorem 1] replace p by $p + q$ obtaining

$$\sum_{n=1}^{N-1} \lambda_n \omega_n^{p+q} \leq \kappa \sum_{n=1}^{N} \lambda_n a_n \omega_n^{p+q-1}. \tag{2.4}$$

Taking the limit as $N \to \infty$ in (2.4) and applying Hölder's inequality with indices p and $p/(p-1)$, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} \lambda_n \omega_n^{p+q} &\leq \kappa \sum_{n=1}^{\infty} \left(\lambda_n^{(p-1)/p} \omega_n^{p+q-1-q/p}\right) \left(\lambda_n^{1/p} a_n \omega_n^{q/p}\right) \\
&\leq \kappa \left\{ \sum_{n=1}^{\infty} \lambda_n \left(\omega_n^{p+q-1-q/p}\right)^{p/(p-1)} \right\}^{(p-1)/p} \left\{ \sum_{n=1}^{\infty} \lambda_n \left(a_n \omega_n^{q/p}\right)^p \right\}^{1/p} \\
&= \kappa \left\{ \sum_{n=1}^{\infty} \lambda_n \omega_n^{p+q} \right\}^{1-1/p} \left\{ \sum_{n=1}^{\infty} \lambda_n a_n^p \omega_n^q \right\}^{1/p};
\end{aligned}$$

hence

$$\left\{ \sum_{n=1}^{\infty} \lambda_n \omega_n^{p+q} \right\}^{1/p} \leq \kappa \left\{ \sum_{n=1}^{\infty} \lambda_n a_n^p \omega_n^q \right\}^{1/p}.$$

Raising both sides to the pth power, we have (2.3). □

The case $q = 0$ is [5, Theorem 1].

From the preceding theorem we can deduce the following more general result.

Theorem 2.2. *Let* $p > 1$*,* $q > 0$*,* $r \geq 0$*, and let* a_n*,* A_n*,* ω_n*,* β_n λ_n *and* Λ_n *be as in Theorem* 2.1*. Suppose that* $\sum_{n=1}^{\infty} \lambda_n a_n^{p+q} \omega_n^r$ *converges and that there exists* $\kappa > 0$ *such that*

$$p + q + r - 1 + \frac{(\beta_{n+1} - \beta_n)\Lambda_n}{(\beta_{n+1}\beta_n)\lambda_n} \geq \frac{p+q+r}{\kappa}, \quad n \in \mathbb{N}.$$

Then

$$\sum_{n=1}^{\infty} \lambda_n a_n^p \omega_n^{q+r} \leq \kappa^q \sum_{n=1}^{\infty} \lambda_n a_n^{p+q} \omega_n^r. \tag{2.5}$$

Proof. Using Hölder's inequality with indices $(p+q)/p$ and $(p+q)/q$ and then applying Theorem 2.1 with q replaced by $q + r$, we get

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda_n a_n^p \omega_n^{q+r} &= \sum_{n=1}^{\infty} \left(\lambda_n a_n^{p+q} \omega_n^r\right)^{p/(p+q)} \left(\lambda_n \omega_n^{p+q+r}\right)^{q/(p+q)} \\ &\leq \left(\sum_{n=1}^{\infty} \lambda_n a_n^{p+q} \omega_n^r\right)^{p/(p+q)} \left(\sum_{n=1}^{\infty} \lambda_n \omega_n^{p+q+r}\right)^{q/(p+q)} \\ &\leq \left(\sum_{n=1}^{\infty} \lambda_n a_n^{p+q} \omega_n^r\right)^{p/(p+q)} \left(\kappa^p \sum_{n=1}^{\infty} \lambda_n a_n^p \omega_n^{q+r}\right)^{q/(p+q)},\end{aligned}$$

that is,

$$\left(\sum_{n=1}^{\infty} \lambda_n a_n^p \omega_n^{q+r}\right)^{p/(p+q)} \leq \kappa^{pq/(p+q)} \left(\sum_{n=1}^{\infty} \lambda_n a_n^{p+q} \omega_n^r\right)^{p/(p+q)}. \tag{2.6}$$

Raising both sides of (2.6) to the power $(p+q)/p$ yields (2.5). ☐

Corollary 2.3. *Let* $p > 1$*,* $q > 0$*, and let* a_n*,* A_n*,* ω_n*,* β_n λ_n *and* Λ_n *be as in Theorem* 2.1*. Suppose that* $\sum_{n=1}^{\infty} \lambda_n a_n^{p+q}$ *converges and that there exists* $\kappa > 0$ *such that*

$$p + q - 1 + \frac{(\beta_{n+1} - \beta_n)\Lambda_n}{(\beta_{n+1}\beta_n)\lambda_n} \geq \frac{p+q}{\kappa}, \quad n \in \mathbb{N}.$$

Then

$$\sum_{n=1}^{\infty} \lambda_n a_n^p \omega_n^q \leq \kappa^q \sum_{n=1}^{\infty} \lambda_n a_n^{p+q}, \tag{2.7}$$

and

$$\sum_{n=1}^{\infty} \lambda_n \omega_n^{p+q} \le \kappa^{p+q} \sum_{n=1}^{\infty} \lambda_n a_n^{p+q}. \tag{2.8}$$

Proof. Inequality (2.7) follows from (2.5) by setting $r = 0$. Combining (2.3) and (2.7), we obtain (2.8). □

We now present a multilevel version of Theorem 2.1 in the case that $q = 0$.

Theorem 2.4. *Let* $p > 1$*,* $m \in \mathbb{N}$*,* $a_j > 0$*,* $\beta_{ij} > 0$*,* $\lambda_j > 0$ *for* $i \in \{1, \dots, m\}$ *and all* $j \in \mathbb{N}$*. For* $i \in \{1, \dots, m\}$ *and* $n \in \mathbb{N}$ *define*

$$\omega_{0n} = a_n, \quad \Lambda_{in} = \sum_{j=1}^{n} \beta_{ij}\lambda_j, \quad A_{in} = \sum_{j=1}^{n} \beta_{ij}\lambda_j \omega_{i-1,j}, \quad \omega_{in} = \frac{A_{in}}{\Lambda_{in}}.$$

Suppose that $\sum_{n=1}^{\infty} \lambda_n a_n^p$ *converge. If there exist* $\kappa_i > 0$ *such that*

$$p - 1 + \frac{(\beta_{i,n+1} - \beta_{in})\Lambda_{in}}{(\beta_{i,n+1}\beta_{in})\lambda_n} \ge \frac{p}{\kappa_i}, \quad i \in \{1, \dots, m\}, \ n \in \mathbb{N},$$

then

$$\sum_{n=1}^{\infty} \lambda_n \omega_{mn}^p \le \left(\prod_{i=1}^{m} \kappa_i \right)^p \sum_{n=1}^{\infty} \lambda_n a_n^p. \tag{2.9}$$

Proof. Applying Theorem 2.1 with $q = 0$, we conclude that $\sum_{n=1}^{\infty} \lambda_n \omega_{1n}^p$ converges, and

$$\sum_{n=1}^{\infty} \lambda_n \omega_{1n}^p \le \kappa_1^p \sum_{n=1}^{\infty} \lambda_n \omega_{0n}^p.$$

A similar argument shows that

$$\sum_{n=1}^{\infty} \lambda_n \omega_{in}^p \le \kappa_i^p \sum_{n=1}^{\infty} \lambda_n \omega_{i-1,n}^p, \quad i \in \{1, \dots, m\}.$$

Continuing this way, we obtain (2.9). □

Remark 2.5. Setting $\beta_{ij} = 1, \lambda_j = 1, \kappa_i = p/(p-1)$ for $j \in \mathbb{N}$ we recover Pachpatte's result [7, Theorem 1].

Theorem 2.6. *Let H be a real-valued positive convex function defined on* $(0,\infty)$ *and let* $p>1$, $q\ge 0$, $\beta_i>0$, $\lambda_i>0$, $a_i>0$ *for all* $i\in\mathbb{N}$. *Let* A_n, ω_n *and* Λ_n *be as in Theorem* 2.1, *and let*

$$F_n=\sum_{i=1}^{n}\beta_i\lambda_i H(a_i),\qquad \Phi_n=\frac{F_n}{\Lambda_n},\quad n\in\mathbb{N}.$$

Suppose that $\sum_{n=1}^{\infty}\lambda_n H^p(a_n)\Phi_n^q$ *converges and that there exists* $\kappa>0$ *such that*

$$p+q-1+\frac{(\beta_{n+1}-\beta_n)\Lambda_n}{(\beta_{n+1}\beta_n)\lambda_n}\ge\frac{p+q}{\kappa}. \tag{2.10}$$

Then

$$\sum_{n=1}^{\infty}\lambda_n H^{p+q}(\omega_n)\le\kappa^p\sum_{n=1}^{\infty}\lambda_n H^p(a_n)\Phi_n^q. \tag{2.11}$$

The constant in (2.11) *is the best possible.*

Proof. We apply Theorem 2.1 with a_i replaced by $H(a_i)$. Then A_n is replaced by F_n, and ω_n by Φ_n. Inequality (2.3) then becomes

$$\sum_{n=1}^{\infty}\lambda_n\Phi_n^{p+q}\le\kappa^p\sum_{n=1}^{\infty}\lambda_n H^p(a_n)\Phi_n^q. \tag{2.12}$$

Since H is convex, we can apply Jensen's inequality to obtain

$$H(\omega_n)=H\left(\frac{A_n}{\Lambda_n}\right)=H\left(\sum_{i=1}^{n}\frac{\beta_i\lambda_i}{\Lambda_n}\,a_i\right)\le\sum_{i=1}^{n}\frac{\beta_i\lambda_i}{\Lambda_n}\,H(a_i)=\Phi_n.$$

Substituting this in (2.12), we get (2.11).

The case $q=0$, $\beta_n=1, n\in\mathbb{N}$ and $\kappa=p/(p-1)$ shows the constant in (2.11) to be the best possible. □

The choice $H(u)=u$ in the preceding theorem yields Theorem 2.1. Setting $q=0$, we recover [5, Theorem A]. If $q=0$, $\beta_i=1$ and $\kappa=p/(p-1)$, the preceding theorem reduces to Copson's result in [1] (see our Theorem 1.2).

The next result generalizes Theorem 2.6 in the same way that Theorem 2.2 generalizes Theorem 2.1. The proof is left to the reader.

Theorem 2.7. *Let H be a real-valued positive convex function defined on* $(0,\infty)$ *and let* $p>1$, $q\ge 0$, $\beta_i>0$, $r\ge 0$, $\lambda_i>0$, $a_i>0$ *for all* $i\in\mathbb{N}$. *Let* A_n, ω_n

and Λ_n *be as in Theorem* 2.1*, and* F_n *and* Φ_n *as in Theorem* 2.6*. Suppose that* $\sum_{n=1}^{\infty} \lambda_n H^r(a_n)\Phi_n^{p+q}$ *converges and that there exists* $\kappa > 0$ *such that*

$$p+q-1+\frac{(\beta_{n+1}-\beta_n)\Lambda_n}{(\beta_{n+1}\beta_n)\lambda_n} \geq \frac{p+q+r}{\kappa}. \tag{2.13}$$

Then

$$\sum_{n=1}^{\infty} \lambda_n H^{q+r}(\omega_n)\Phi_n^p \leq \kappa^p \sum_{n=1}^{\infty} \lambda_n H^r(a_n)\Phi_n^{p+q}. \tag{2.14}$$

The constant in (2.14) *is the best possible.*

3 Further inequalities

In this section we obtain extensions of Pachpatte's results from [7].

Theorem 3.1. *Let* $p > 1$*,* $q \geq 0$*,* $b_{mn} > 0$ *for* $m, n \in \mathbb{N}$ *and let*

$$B_{mn} = \frac{1}{mn}\sum_{s=1}^{m}\sum_{t=1}^{n}\frac{1}{st}\sum_{i=1}^{s}\sum_{j=1}^{t} b_{ij} \quad \textit{for} \quad m,n,i,j \in \mathbb{N}. \tag{3.1}$$

If $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn}^p B_{mn}^q$ *converges, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}^{p+q} \leq \left(\frac{p}{p-1}\right)^{3p}\left(\frac{p+q}{p+q-1}\right)^{p}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn}^p B_{mn}^q. \tag{3.2}$$

Proof. The proof is an adaptation of the proof of [7, Theorem 2], and extends an idea used by Elliott in [2].

Let $L, M \in \mathbb{N}$ and define

$$\omega_{mn} = mB_{mn} = \frac{1}{n}\sum_{t=1}^{n}\frac{1}{t}\sum_{s=1}^{m}\frac{1}{s}\sum_{i=1}^{s}\sum_{j=1}^{t} b_{ij}, \tag{3.3}$$

and

$$S_{ML} = \sum_{m=1}^{M}\sum_{n=1}^{L} B_{mn}^{p+q} = \sum_{m=1}^{M} m^{-p-q}\sum_{n=1}^{L}\omega_{mn}^{p+q}. \tag{3.4}$$

Thus the left hand side of (3.2) may be written as

$$\sum_{m=1}^{\infty} m^{-p-q}\sum_{n=1}^{\infty}\omega_{mn}^{p+q}. \tag{3.5}$$

By a procedure similar to the one used in [7], we obtain

$$\begin{aligned}
&\sum_{n=1}^{L} \omega^{p+q}{}_{mn} - \left(\frac{p+q}{p+q-1}\right)\sum_{n=1}^{L}\left\{\frac{1}{n}\sum_{s=1}^{m}\frac{1}{s}\sum_{i=1}^{s}\sum_{j=1}^{t} b_{ij}\right\}\omega_{mn}^{p+q-1}\\
&\qquad= \left(\frac{1}{p+q-1}\right)\sum_{n=1}^{L}\left\{(n-1)\omega_{m,n-1}^{p+q} - n\omega_{mn}^{p+q}\right\}\\
&\qquad= -\left(\frac{1}{p+q-1}\right)L\omega_{m,L}^{p+q} \le 0. \qquad (3.6)
\end{aligned}$$

This is [7, Equation (24)] with p replaced by $p+q$. From (3.6), using Hölder's inequality with indices p and $p/(p-1)$, we obtain

$$\begin{aligned}
\sum_{n=1}^{L}\omega_{mn}^{p+q} &\le \left(\frac{p+q}{p+q-1}\right)\sum_{n=1}^{L}\left\{\frac{1}{n}\sum_{s=1}^{m}\frac{1}{s}\sum_{i=1}^{s}\sum_{j=1}^{n} b_{ij}\right\}\omega_{mn}^{p+q-1}\\
&\le \left(\frac{p+q}{p+q-1}\right)\left\{\sum_{n=1}^{L}\left\{\omega_{mn}^{q/p}\left\{\frac{1}{n}\sum_{s=1}^{m}\frac{1}{s}\sum_{i=1}^{s}\sum_{j=1}^{n} b_{ij}\right\}\right\}^{p}\right\}^{1/p}\cdot\\
&\qquad\cdot\left\{\sum_{n=1}^{L}\left\{\omega_{mn}^{p+q-1-q/p}\right\}^{p/(p-1)}\right\}^{(p-1)/p}. \qquad (3.7)
\end{aligned}$$

Dividing both sides of (3.7) by the last term on the right and raising to the pth power, we have

$$\begin{aligned}
\sum_{n=1}^{L}\omega_{mn}^{p+q} &\le \left(\frac{p+q}{p+q-1}\right)^{p}\sum_{n=1}^{L}\omega_{mn}^{q}\left\{\frac{1}{n}\sum_{s=1}^{m}\frac{1}{s}\sum_{i=1}^{s}\sum_{j=1}^{n} b_{ij}\right\}^{p}\\
&= \left(\frac{p+q}{p+q-1}\right)^{p}\sum_{n=1}^{L}\omega_{mn}^{q} m^{p} n^{-p}\beta_{mn}^{p}. \qquad (3.8)
\end{aligned}$$

If we define

$$\beta_{mn} = \frac{1}{m}\sum_{s=1}^{m}\frac{1}{s}\sum_{i=1}^{s}\sum_{j=1}^{n} b_{ij}, \qquad (3.9)$$

then from (3.4) and (3.8),

$$\begin{aligned}
S_{ML} &\le \left(\frac{p+q}{p+q-1}\right)^{p}\sum_{n=1}^{L} n^{-p}\sum_{m=1}^{M} m^{-q}\omega_{mn}^{q}\beta_{mn}^{p}\\
&= \left(\frac{p+q}{p+q-1}\right)^{p}\sum_{m=1}^{M} m^{-q}\sum_{n=1}^{L} n^{-p}\omega_{mn}^{q}\beta_{mn}^{p}, \qquad (3.10)
\end{aligned}$$

and

$$m\beta_{mn} - (m-1)\beta_{m-1,n} = \frac{1}{m}\sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij} \geq 0, \tag{3.11}$$

so that $m\beta_{mn}$ is an increasing function of m. We note that

$$\omega_{mn} - \omega_{m-1,n} = \frac{1}{n}\sum_{t=1}^{n}\frac{1}{mt}\sum_{i=1}^{m}\sum_{j=1}^{t} b_{ij} \geq 0, \tag{3.12}$$

$$\beta_{mn} - \beta_{m,n-1} = \frac{1}{m}\sum_{i=1}^{m} b_{in} \geq 0. \tag{3.13}$$

From (3.9) and using the inequality

$$u^{k+1} + kv^{k+1} \geq (k+1)uv^{k}, \qquad u, v \geq 0, k \geq 1, \tag{3.14}$$

we deduce that

$$\begin{aligned}
& m^{-q}\omega_{mn}^{q}\beta_{mn}^{p} - \left(\frac{p}{p-1}\right) m^{-q}\omega_{mn}^{q} \left\{\frac{1}{m}\sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}\right\} \beta_{mn}^{p-1} \\
&\quad = m^{-q}\omega_{mn}^{q}\beta_{mn}^{p} - \left(\frac{p}{p-1}\right) m^{-q}\omega_{mn}^{q} \left(m\beta_{mn} - (m-1)\beta_{m-1,n}\right) \beta_{mn}^{p-1} \\
&\quad = \left\{1 - \left(\frac{p}{p-1}\right) m\right\} m^{-q}\omega_{mn}^{q}\beta_{mn}^{p} \\
&\qquad + \left(\frac{p}{p-1}\right)(m-1) m^{-q}\omega_{mn}^{q}\beta_{m-1,n}\beta_{mn}^{p-1} \\
&\quad \leq \left\{1 - m - \left(\frac{1}{p-1}\right) m\right\} m^{-q}\omega_{mn}^{q}\beta_{mn}^{p} \\
&\qquad + \left(\frac{p}{p-1}\right)(m-1)\frac{1}{p} m^{-q}\omega_{mn}^{q} \left(\beta_{m-1,n}^{p} + (p-1)\beta_{mn}^{p}\right) \\
&\quad = \left(\frac{1}{p-1}\right) m^{-q}\omega_{mn}^{q} \left((m-1)\beta_{m-1,n}^{p} - m\beta_{mn}^{p}\right) \leq 0
\end{aligned} \tag{3.15}$$

by (3.11). Keeping n fixed in (3.15) and letting $m = 1, 2, \ldots, M$ and adding the inequalities we have

$$\begin{aligned}
& \sum_{m=1}^{M} m^{-q}\omega_{mn}^{q}\beta_{mn}^{p} - \left(\frac{p}{p-1}\right) \sum_{m=1}^{M} m^{-q}\omega_{mn}^{q} \left\{\frac{1}{m}\sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}\right\} \beta_{mn}^{p-1} \\
&\quad \leq \left(\frac{p}{p-1}\right) \sum_{m=1}^{M} m^{-q}\omega_{mn}^{q} \left((m-1)\beta_{m-1,n}^{p} - m\beta_{mn}^{p}\right) \leq 0
\end{aligned} \tag{3.16}$$

(the sum of negative terms, by (3.11)). Let

$$\gamma_{mn} = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{m} b_{ij}$$

so that

$$n\gamma_{mn} - (n-1)\gamma_{m,n-1} = \sum_{i=1}^{m} b_{in} \geq 0. \tag{3.17}$$

Following similar procedures as used to derive (3.16) from (3.6) with a further application of Hölder's inequality we arrive at

$$S_{ML} \leq \left(\frac{p}{p-1}\right)^p \left(\frac{p+q}{p+q-1}\right)^p \sum_{m=1}^{M} m^{-p-q} \sum_{n=1}^{L} \omega_{mn}^q \gamma_{mn}^p \tag{3.18}$$

and

$$\sum_{n=1}^{L} \omega_{mn}^q \gamma_{mn}^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{L} \omega_{mn}^q \left\{\sum_{i=1}^{m} b_{in}\right\}^p. \tag{3.19}$$

From (3.18) and (3.19) we observe that

$$\begin{aligned} S_{ML} &\leq \left(\frac{p}{p-1}\right)^{2p} \left(\frac{p+q}{p+q-1}\right) \sum_{n=1}^{L} \sum_{m=1}^{M} m^{-p-q} m^p \omega_{mn}^q \delta_{mn}^p \\ &= \left(\frac{p}{p-1}\right)^{2p} \left(\frac{p+q}{p+q-1}\right) \sum_{m=1}^{M} m^{-q} \sum_{n=1}^{L} \omega_{mn}^q \delta_{mn}^p, \end{aligned} \tag{3.20}$$

where

$$\delta_{mn} = \frac{1}{m} \sum_{i=1}^{m} b_{in},$$

and

$$m\delta_{mn} - (m-1)\delta_{m-1,n} = b_{mn} \geq 0. \tag{3.21}$$

Repeating the procedure above (but without a further application of Hölder's inequality), we get

$$\sum_{m=1}^{M} m^{-q} \alpha_{mn}^q \delta_{mn}^p \leq \left(\frac{p}{p-1}\right)^p \sum_{m=1}^{M} m^{-q} \alpha_{mn}^q b_{mn}^p. \tag{3.22}$$

From (3.20) and (3.22) we observe that

$$\begin{aligned} S_{ML} &\le \left(\frac{p}{p-1}\right)^{3p}\left(\frac{p+q}{p+q-1}\right)^{p}\sum_{m=1}^{M} m^{-q}\sum_{n=1}^{L} b_{mn}^{p}\omega_{mn}^{q} \\ &= \left(\frac{p}{p-1}\right)^{3p}\left(\frac{p+q}{p+q-1}\right)^{p}\sum_{m=1}^{M}\sum_{n=1}^{L} b_{mn}^{p}B_{mn}^{q}. \end{aligned} \tag{3.23}$$

By letting L, M tend to infinity in (3.23) we get the desired inequality (3.2). The proof is complete. □

If we set $q = 0$ in Theorem 3.1, we recover [7, Theorem 2].

The following result extends the preceding theorem in a similar way that Theorem 2.2 extends Theorem 2.1.

Theorem 3.2. *Let* $p > 1$, $q \ge 0$, $r \ge 0$, $b_{mn} > 0$ *for* $m, n \in \mathbb{N}$ *and let* B_{mn} *be given by* (3.1). *If* $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn}^{p+q}B_{mn}^{r}$ *converges, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn}^{p}B_{mn}^{q+r} \le \left(\frac{p}{p-1}\right)^{3(p+q)}\left(\frac{p+q+r}{p+q+r-1}\right)^{p+q}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn}^{p+q}B_{mn}^{r}. \tag{3.24}$$

Proof. Using Hölder's inequality with indices $(p+q)/p$ and $(p+q)/q$, we get

$$\begin{aligned} \sum_{n=1}^{L}\sum_{m=1}^{M} b_{mn}^{p}B_{mn}^{q+r} &= \sum_{n=1}^{L}\sum_{m=1}^{M} b_{mn}^{p}B_{mn}^{pr/(p+q}B_{mn}^{q+r-pr/(p+q)} \\ &= \sum_{n=1}^{L}\left\{\sum_{m=1}^{M}\left\{b_{mn}^{p+q}B_{mn}^{r}\right\}^{p/(p+q)}\left\{B_{mn}^{p+q+r}\right\}^{q/(p+q)}\right\} \\ &\le \sum_{n=1}^{L}\left\{\left\{\sum_{m=1}^{M} b_{mn}^{p+q}B_{mn}^{r}\right\}^{p/(p+q)}\left\{\sum_{m=1}^{M} B_{mn}^{p+q+r}\right\}^{q/(p+q)}\right\} \\ &\le \sum_{n=1}^{L}\left\{\left\{\sum_{m=1}^{M} b_{mn}^{p+q}B_{mn}^{r}\right\}^{p/(p+q)}\cdot\right. \\ &\qquad \left.\cdot\left\{\left(\frac{p}{p-1}\right)^{3p}\left(\frac{p+q+r}{p+q+r-1}\right)^{p}\sum_{m=1}^{M} b_{mn}^{p}B_{mn}^{q+r}\right\}^{q/(p+q)}\right\} \\ &\le \left(\frac{p}{p-1}\right)^{3p}\left(\frac{p+q+r}{p+q+r-1}\right)^{p}\left\{\sum_{n=1}^{L}\sum_{m=1}^{M} b_{mn}^{p+q}B_{mn}^{r}\right\}^{p/(p+q)}. \end{aligned}$$

$$\cdot \left\{ \sum_{n=1}^{L} \sum_{m=1}^{M} b_{mn}^{p} B_{mn}^{q+r} \right\}^{q/(p+q)} . \tag{3.25}$$

Then

$$\left\{ \sum_{n=1}^{L} \sum_{m=1}^{M} b_{mn}^{p} B_{mn}^{q+r} \right\}^{p/(p+q)} \le \left(\frac{p}{p-1} \right)^{3p} \left(\frac{p+q+r}{p+q+r-1} \right)^{p} \left\{ \sum_{n=1}^{L} \sum_{m=1}^{M} b_{mn}^{p+q} B_{mn}^{r} \right\}^{p/(p+q)} . \tag{3.26}$$

Raising both sides of (3.26) to the power $(p+q)/p$ yields (3.24). ☐

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