ON INEQUALITIES IN NORMED LINEAR SPACES AND APPLICATIONS

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Abstract. In this paper we obtain inequalities involving the norms of finite sequences of vectors in normed linear spaces, which are new even for complex numbers.

I. Introduction

Let $(X, \|\cdot\|)$ be a normed linear space. The following inequality is well known in the literature as the polygonal inequality, which is a generalization of the triangle inequality:

$$\Big\|\sum_{i\in I} x_i\Big\| \le \sum_{i\in I}\|x_i\|; \tag{P}$$

I is a finite set of indices and x_i $(i \in I)$ are vectors in X.

In papers [1–3], authors explored relations between the following refinements and generalizations of (P):

Theorem A. *Let $(X, \|\cdot\|)$ be a normed linear space and I be a finite set of indices, $x_i \in X$, $i \in I$ and $p \in \mathbf{R}$, $p \ge 1$. Then we have the following inequalities:*

$$\begin{aligned}\Big\|\sum_{i\in I} x_i\Big\|^p &\le [\mathrm{card}(I)]^{p-k-1}\sum_{i_1,\dots,i_{k+1}\in I}\Big\|\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\Big\|^p\\ &\le [\mathrm{card}(I)]^{p-k}\sum_{i_1,\dots,i_k\in I}\Big\|\frac{x_{i_1}+\cdots+x_{i_k}}{k}\Big\|^p\\ &\le \cdots\\ &\le [\mathrm{card}(I)]^{p-2}\sum_{i_1,i_2\in I}\Big\|\frac{x_{i_1}+x_{i_2}}{2}\Big\|^p\\ &\le [\mathrm{card}(I)]^{p-1}\sum_{i\in I}\|x_i\|^p,\end{aligned} \tag{1}$$

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where $k \in N$, $k \geq 1$.

The following result is a refinement of (P) for weighted mean:

Theorem B. *With the above assumptions, if* g_j $(j = 1, 2, 3, \ldots, k)$ *are nonnegative weights such that* $\sum_{j=1}^{k} g_j > 0$, *then we have the following inequality:*

$$\begin{aligned}
\Big\| \sum_{i \in I} x_i \Big\|^p &\leq [\mathrm{card}(I)]^{p-k} \sum_{i_1, \ldots, i_k \in I} \Big\| \frac{x_{i_1} + \cdots + x_{i_k}}{k} \Big\|^p \\
&\leq [\mathrm{card}(I)]^{p-k} \sum_{i_1, \ldots, i_k \in I} \Big\| \frac{g_1 x_{i_1} + \cdots + g_k x_{i_k}}{g_1 + \cdots + g_k} \Big\|^p \qquad (2) \\
&\leq [\mathrm{card}(I)]^{p-1} \sum_{i \in I} \|x_i\|^p.
\end{aligned}$$

Note that both the above inequalities were obtained from more general results for convex mappings defined on convex subsets in normed linear spaces.

In this paper, using only techniques from normed linear spaces, we obtain new inequalities for the norms of finite sequences.

II. The Results

We start with the following theorem:

Theorem 2.1. *Let* $(X, \|\cdot\|)$ *be a normed linear space,* $x_i \in X$ *with* $i \in I$ $(I$ *is finite) and* $x = \sum_{i \in I} x_i$. *Then we have the following inequalities:*

$$\Big\| \|x\| x - \sum_{i \in I} \|x \pm x_i\| x_i \Big\| \leq \sum_{i \in I} \|x_i\|^2 \leq \tfrac{1}{2} \Big(\|x\| + \sum_{i \in I} \|x_i\| \Big) \sum_{i \in I} \|x_i\|. \qquad (3)$$

Proof. For a fixed $i \in I$, we have

$$\begin{aligned}
\|x_i\| = \Big\| x - \sum_{j \in I, j \neq i} x_j \Big\| &\leq \|x\| + \Big\| \sum_{j \in I, j \neq i} x_j \Big\| \\
&\leq \|x\| + \sum_{j \in I, j \neq i} \|x_j\| = \|x\| + \sum_{j \in I} \|x_j\| - \|x_i\|,
\end{aligned}$$

which implies that, for all $i \in I$,

$$\|x_i\| \leq \tfrac{1}{2} \Big(\|x\| + \sum_{j \in I} \|x_j\| \Big).$$

Multiplying by $\|x_i\| \geq 0$ and summing over $i \in I$, we deduce

$$\sum_{i\in I} \|x_i\|^2 \leq \tfrac{1}{2}\Big(\|x\| + \sum_{j\in I} \|x_j\|\Big) \sum_{i\in I} \|x_i\|,$$

and so the second inequality in (1) is proved. Also, for a fixed $i \in I$, we have

$$\|x_i\| = \|-x_i\| = \|x \pm x_i - x\| \geq \Big|\|x \pm x_i\| - \|x\|\Big|.$$

Multiplying by $\|x_i\| \geq 0$, we have

$$\|x_i\|^2 \geq \Big|\|x \pm x_i\| - \|x\|\Big| \, \|x_i\| = \Big\|(\|x \pm x_i\| - \|x\|)x_i\Big\|$$

for all $i \in I$. Summing over $i \in I$, we have

$$\sum_{i\in I} \|x_i\|^2 \geq \sum_{i\in I} \Big\|\|x \pm x_i\|x_i - \|x\|x_i\Big\| \geq \Big\|\sum_{i\in I} \|x \pm x_i\|x_i - \|x\|x_i\Big\|,$$

and so the theorem is proved.

Corollary 2.2. *With the above assumptions, if* $\sum_{i\in I} x_i = 0$, *then*

$$\sum_{i\in I} \|x_i\|^2 \leq \tfrac{1}{2}\Big(\sum_{i\in I} \|x_i\|\Big)^2. \tag{4}$$

The following result generalizes Theorem 2.1. In its proof we need the formula

$$\sum_{i_1,\dots,i_k\in I} (x_{i_1} + \cdots + x_{i_k}) = k[\operatorname{card}(I)]^{k-1} \sum_{i\in I} x_i, \tag{5}$$

which can be proved by induction on k.

Theorem 2.3. *With the above assumptions, we have the following inequalities:*

$$\begin{aligned}
&k\Big\|\sum_{i_1,\dots,i_k\in I} \|x \pm (x_{i_1} + \cdots + x_{i_k})\|x_{i_1} - n^{k-1}\|x\|x\Big\| \\
&\leq \sum_{i_1,\dots,i_k\in I} \|x_{i_1} + \cdots + x_{i_k}\|^2 \\
&\leq \tfrac{1}{2}\Big(\sum_{i_1,\dots,i_k\in I} \|x_{i_1} + \cdots + x_{i_k}\|^2 + k\sum_{i_1,\dots,i_k\in I} \|x_{i_1}\|\,\|x_{i_1} + \cdots + x_{i_k}\|\Big) \\
&\leq \tfrac{1}{2}\Big(\|x\| + \sum_{i\in I} \|x_i\|\Big) \sum_{i_1,\dots,i_k\in I} \|x_{i_1} + \cdots + x_{i_k}\|,
\end{aligned} \tag{6}$$

where $k \geq 1$ *and* $\operatorname{card}(I) = n$.

Proof. For fixed $i_1, \ldots, i_k \in I$, we have

$$\begin{aligned}
\|x_{i_1} + \cdots + x_{i_k}\| &= \Big\| x - \sum_{j \notin \{i_1, \ldots, i_k\}} x_j \Big\| \\
&\leq \|x\| + \Big\| \sum_{j \notin \{i_1, \ldots, i_k\}} x_j \Big\| \\
&\leq \|x\| + \sum_{j \notin \{i_1, \ldots, i_k\}} \|x_j\| \\
&= \|x\| + \sum_{i \in I} \|x_i\| - \|x_{i_1}\| - \cdots - \|x_{i_k}\|,
\end{aligned}$$

from which it follows

$$\|x_{i_1} + \cdots + x_{i_k}\| + \|x_{i_1}\| + \cdots + \|x_{i_k}\| \leq \|x\| + \sum_{i \in I} \|x_i\|.$$

Multiplying by $\|x_{i_1} + \cdots + x_{i_k}\| \geq 0$ and summing over $i_1, \ldots, i_k$ in I, we have

$$\begin{aligned}
\sum_{i_1, \ldots, i_k \in I} &\|x_{i_1} + \cdots + x_{i_k}\|^2 + k \sum_{i_1, \ldots, i_k \in I} \|x_{i_1}\| \, \|x_{i_1} + \cdots + x_{i_k}\| \\
&\leq \Big(\|x\| + \sum_{i \in I} \|x_i\| \Big) \sum_{i_1, \ldots, i_k \in I} \|x_{i_1} + \cdots + x_{i_k}\|,
\end{aligned}$$

and so the third inequality in (6) follows. By the triangle inequality

$$\|x_{i_1} + \cdots + x_{i_k}\| \leq \tfrac{1}{2}(\|x_{i_1} + \cdots + x_{i_k}\| + \|x_{i_1}\| + \cdots + \|x_{i_k}\|).$$

Multiplying by $\|x_{i_1} + \cdots + x_{i_k}\|$ and summing over $i_1, \ldots, i_k$ in the index set I, we derive

$$\begin{aligned}
&\sum_{i_1, \ldots, i_k \in I} \|x_{i_1} + \cdots + x_{i_k}\|^2 \\
&\qquad \leq \tfrac{1}{2} \Big(\sum_{i_1, \ldots, i_k \in I} \|x_{i_1} + \cdots + x_{i_k}\|^2 + k \sum_{i_1, \ldots, i_k \in I} \|x_{i_1}\| \|x_{i_1} + \cdots + x_{i_k}\| \Big)
\end{aligned}$$

and so the second inequality in (6) is also proved.

Finally, we prove the first inequality in (6). Since

$$\begin{aligned}\|x_{i_1}+\cdots+x_{i_k}\| &= \|-(x_{i_1}+\cdots+x_{i_k})\| \\ &= \|x\pm(x_{i_1}+\cdots+x_{i_k})-x\| \\ &\geq \Big|\|x\pm(x_{i_1}+\cdots+x_{i_k})\|-\|x\|\Big|,\end{aligned}$$

then

$$\begin{aligned}\|x_{i_1}+\cdots+x_{i_k}\|^2 &\geq \Big\|(\|x\pm(x_{i_1}+\cdots+x_{i_k})\|-\|x\|)(x_{i_1}+\cdots+x_{i_k})\Big\| \\ &= \Big\|\|x\pm(x_{i_1}+\cdots+x_{i_k})\|(x_{i_1}+\cdots+x_{i_k})-\|x\|(x_{i_1}+\cdots+x_{i_k})\Big\|.\end{aligned}$$

Summing over $i_1,\dots,i_k$ in the index set I, we use (5) to obtain

$$\begin{aligned}&\sum_{i_1,\dots,i_k\in I}\|x_{i_1}+\cdots+x_{i_k}\|^2 \\ &\geq \sum_{i_1,\dots,i_k\in I}\Big\|\|x\pm(x_{i_1}+\cdots+x_{i_k})\|(x_{i_1}+\cdots+x_{i_k})-\|x\|(x_{i_1}+\cdots+x_{i_k})\Big\| \\ &\geq \Big\|k\sum_{i_1,\dots,i_k\in I}\|x\pm(x_{i_1}+\cdots+x_{i_k})\|x_{i_1}-kn^{k-1}\|x\|\sum_{i\in I}x_i\Big\| \\ &= k\Big\|\sum_{i_1,\dots,i_k\in I}\|x\pm(x_{i_1}+\cdots+x_{i_k})\|x_{i_1}-n^{k-1})\|x\|x\Big\|\end{aligned}$$

and thus the theorem is proved.

Corollary 2.4. *With the above assumptions, if* $\sum_{i\in I}x_i=0$*, then*

$$\begin{aligned}&k\Big\|\sum_{i_1,\dots,i_k\in I}\|x_{i_1}+\cdots+x_{i_k}\|x_{i_1}\Big\| \\ &\leq \sum_{i_1,\dots,i_k\in I}\|x_{i_1}+\cdots+x_{i_k}\|^2 \\ &\leq \tfrac{1}{2}\Big(\sum_{i_1,\dots,i_k\in I}\|x_{i_1}+\cdots+x_{i_k}\|^2+k\sum_{i_1,\dots,i_k\in I}\|x_{i_1}\|\,\|x_{i_1}+\cdots+x_{i_k}\|\Big) \\ &\leq \tfrac{1}{2}\sum_{i\in I}\|x_{i_1}\|\sum_{i_1,\dots,i_k\in I}\|x_{i_1}+\cdots+x_{i_k}\|.\end{aligned} \tag{7}$$

III. Applications to Complex Numbers

The inequalities obtained in Section II can be restated to obtain new inequalities for complex numbers:

1. Let $z_i \in C$, $i \in I$ (I is finite) and let $z = \sum_{i\in I} z_i$. Then

$$\Big||z|z - \sum_{i\in I} |z \pm z_i| z_i\Big| \le \sum_{i\in I} |z_i|^2 \le \tfrac{1}{2}\Big(|z| + \sum_{i\in I} |z_i|\Big) \sum_{i\in I} |z_i|.$$

In particular, if $z = 0$, then

$$\sum_{i\in I} |z_i|^2 \le \tfrac{1}{2}\Big(\sum_{i\in I} |z_i|\Big)^2.$$

2. With the above assumptions, we have

$$\begin{aligned} k\Big| &\sum_{i_1,\dots,i_k\in I} |z \pm (z_{i_1} + \dots + z_{i_k})| z_{i_1} - n^{k-1}|z|z\Big| \\ &\le \sum_{i_1,\dots,i_k\in I} (z_{i_1} + \dots + z_{i_k})^2 \\ &\le \tfrac{1}{2}\Big(\sum_{i_1,\dots,i_k\in I} |z_{i_1} + \dots + x_{i_k}|^2 + k \sum_{i_1,\dots,i_k\in I} |z_{i_1}|\, |z_{i_1} + \dots + z_{i_k}|\Big) \\ &\le \tfrac{1}{2}\Big(|z| + \sum_{i\in I} |z_i|\Big) \sum_{i_1,\dots,i_k\in I} |z_{i_1} + \dots + z_{i_k}|, \end{aligned}$$

where $k \in N$, $k \ge 1$ and $\operatorname{card}(I) = n$.

In particular, if $z = 0$, we have

$$\begin{aligned} k\Big|\sum_{i_1,\dots,i_k\in I} |z_{i_1} + \dots + z_{i_k}| z_{i_1}\Big| &\le \sum_{i_1,\dots,i_k\in I} |z_{i_1} + \dots + z_{i_k}|^2 \\ &\le \tfrac{1}{2}\Big(\sum_{i_1,\dots,i_k\in I} |z_{i_1} + \dots + x_{i_k}|^2 + k \sum_{i_1,\dots,i_k\in I} |z_{i_1}(z_{i_1} + \dots + z_{i_k})|\Big) \\ &\le \tfrac{1}{2} \sum_{i\in I} |z_i| \sum_{i_1,\dots,i_k\in I} |z_{i_1} + \dots + z_{i_k}|. \end{aligned}$$

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