SINGULARLY PERTURBED C_0-SEMIGROUPS AND NONHOMOGENEOUS DIFFERENTIAL EQUATIONS IN BANACH SPACES

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ABSTRACT. In this paper we study a singular perturbation of an asymptotically convergent operator C_0-semigroup, and describe the spectral behaviour and a power series expansion of the perturbed semigroup. As an application of our results we obtain the description of the asymptotic behaviour of the solutions to a nonhomogeneous singularly perturbed differential equation in a Banach space, extending the matrix results of S. Campbell (*Singular Systems of Differential Equations*, Research Notes in Mathematics 40, Pitman, London, 1980) and previous results of the present authors.

KEYWORDS: *Singularly perturbed C_0-semigroup, singularly perturbed differential equation, g-Drazin inverse of a closed operator.*

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1 INTRODUCTION AND PRELIMINARIES

The main aim of this paper is to investigate a singular perturbation of a C_0-semigroup $T(t)$ by replacing its infinitesimal generator A by $\varepsilon^{-1}A + B$, where $\varepsilon > 0$ and B is a bounded linear operator, under the assumption that $T(t) \to P$ in the operator norm.

In Section 2 we describe the spectral behaviour of the perturbed semigroup $S_\varepsilon(t)$ with the generator $\varepsilon^{-1}A + B$, using this in Section 3 to obtain a power series expansion for $S_\varepsilon(t)$ in the powers of ε. In the last section we study the asymptotic convergence of the solution of the singularly perturbed differential equation

$$\begin{aligned} \varepsilon\frac{\mathrm{d}u_\varepsilon(t)}{\mathrm{d}t} &= (A+\varepsilon B)u_\varepsilon(t) + f(t), \quad t \ge 0, \\ u_\varepsilon(0) &= x, \quad \varepsilon > 0. \end{aligned} \tag{1.1}$$

In the case that A and B are finite matrices, this equation has been studied by Campbell in his monograph [1], who also obtained a power series expansion for $\exp((\varepsilon^{-1}A + B)t)$ (in his notation A and B are interchanged). The homogeneous version of the problem was studied by the authors in [3] for the case when A and B are bounded linear operators, and in [5] for the case of C_0-semigroups. The nonhomogeneous version was investigated in [4] for the special case $B = 0$. An explicit description of the asymptotic behaviour of the solutions for the case of a C_0-semigroup was made possible by the introduction of the concept of the g-Drazin inverse of a closed linear operator by the present authors in [4].

By $\mathcal{C}(X)$ we denote the space of all closed linear operators A with domain and range in X; $\mathcal{D}(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the domain, nullspace and range of A, respectively. If $A, B \in \mathcal{C}(X)$, we write $A \subset B$ to mean that $\Gamma(A) \subset \Gamma(B)$, where $\Gamma(T)$ is the graph of $T \in \mathcal{C}(X)$. By $\mathcal{B}(X)$ we denote the set of all $A \in \mathcal{C}(X)$ with $\mathcal{D}(A) = X$; by the closed graph theorem, the operators in $\mathcal{B}(X)$ are bounded on X. An operator $A \in \mathcal{C}(X)$ is *invertible* if there exists an operator $B \in \mathcal{B}(X)$ such that $BA \subset AB = I$; $A^{-1} = B$ is the *inverse* of A.

If $A \in \mathcal{C}(X)$, then $\rho(A)$ denotes the *resolvent set* of A, that is, the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is invertible. The complement of $\rho(A)$ in $\mathbb{C}$ is the *spectrum* $\sigma(A)$ of A. The *extended spectrum* $\sigma_e(A)$ of A is equal to $\sigma(A)$ if $A \in \mathcal{B}(X)$ and to $\sigma(A) \cup \{\infty\}$ otherwise. For $\lambda \in \rho(A)$, $R(\lambda; A)$ denotes the *resolvent operator* $(\lambda I - A)^{-1}$ of A; the closed graph theorem ensures that $R(\lambda; A)$ is always bounded.

Let $A \in \mathcal{C}(X)$ with $\sigma(A) \neq \mathbb{C}$. Then a subset σ of $\sigma_e(A)$ is called an *isolated spectral set* of A if it is both open and closed in the relative topology of $\sigma_e(A)$ as a subset of $\mathbb{C} \cup \{\infty\}$. A singleton $\{\mu\}$ is an isolated spectral set of A if and only if μ is an isolated singularity of the resolvent $R(\lambda; A)$ of A. We call μ a *pole of* A if μ is a pole of $R(\lambda; A)$. For further relevant facts of operator theory of closed linear operators see [2] and [8].

The concept of the g-Drazin inverse of a closed linear operator is crucial for our exposition. According to [4, Definition 2.1], $A \in \mathcal{C}(X)$ is *g-Drazin invertible* if there exists an operator $B \in \mathcal{B}(X)$ such that $\mathcal{R}(B) \cap \mathcal{R}(I - AB) \subset \mathcal{D}(A)$, and

$$BA \subset AB, \quad BAB = B, \quad \sigma(A(I - AB)) = \{0\}. \tag{1.2}$$

Such an operator is unique, if it exists. It is called the *g-Drazin inverse* of A, and is denoted by A^{D}.

From [4, Theorem 2.3] we know that A is g-Drazin invertible if and only if 0 is not an accumulation point of $\sigma(A)$. In this case we write A^π for the spectral idempotent of A, a bounded linear operator characterized by the properties

$$(A^\pi)^2 = A^\pi, \quad A^\pi A \subset AA^\pi, \quad \sigma(AA^\pi) = \{0\}, \quad 0 \notin \sigma(A + A^\pi);$$

note that $A^\pi = 0$ if A is invertible. By [4, (2.3)],

$$A^{\mathrm{D}} = (A + A^\pi)^{-1}(I - A^\pi), \qquad A^\pi = I - A^{\mathrm{D}}A. \tag{1.3}$$

The resolvent of a g-Drazin invertible operator A has the following Laurent expansion at the origin [4, Theorem 3.1]:

Lemma 1.1. *Let $A \in \mathcal{C}(X)$. If A is g-Drazin invertible, then there exists $r > 0$ such that*

$$R(\lambda; A) = \sum_{n=0}^{\infty} \lambda^{-n-1} A^n A^\pi - \sum_{n=0}^{\infty} \lambda^n (A^{\mathrm{D}})^{n+1}, \quad 0 < |\lambda| < r. \tag{1.4}$$

We will also need the following auxiliary result in which $T_G(t)$ stands for a C_0-semigroup with the infinitesimal generator G.

Lemma 1.2. *Let $T_A(t)$ be a C_0-semigroup, and let $P \in \mathcal{B}(X)$ be an idempotent operator such that $\mathcal{R}(P) \subset \mathcal{D}(A)$ and $T_A(t)P = PT_A(t)$ for all $t \geq 0$. Then $PA \subset AP \in \mathcal{B}(X)$, and*

$$T_A(t)P = \exp(tAP)P = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n P, \tag{1.5}$$

$$T_{A-P}(t) = T_A(t)(I - P + e^{-t}P), \quad t \geq 0. \tag{1.6}$$

Proof. Since $\mathcal{R}(P) \subset \mathcal{D}(A)$, AP is a bounded linear operator on X. Further,

$$APx = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 T(t)Px = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 PT(t)x = PAx, \quad x \in \mathcal{D}(A).$$

For any given $x \in X$, the differential equation

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = (AP)u(t), \quad t \geq 0, \quad u(0) = Px,$$

has a unique solution

$$u(t) = \exp(tAP)Px, \quad t \geq 0.$$

By the properties of the semigroup $T(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}T(t)Px = AT(t)Px = (AP)T(t)Px, \quad T(0)Px = Px,$$

This proves that $T(t)Px = \exp(tAP)Px$ for each $x \in X$.

To obtain the power series expansion in (1.5), we show that $\mathcal{R}(P) \subset \mathcal{D}(A^n)$ for all $n \geq 1$. Suppose that $x = Py \in \mathcal{D}(A^{n-1})$ for some $n \geq 2$. Then $Px = x$, and $Ax = APx = PAx \in \mathcal{D}(A)$, that is, $x \in \mathcal{D}(A^n)$. Consequently, $(AP)^n = A^nP$ for $n \geq 1$, and

$$\exp(tAP)P = \sum_{n=0}^{\infty} \frac{t^n}{n!}(AP)^nP = \sum_{n=0}^{\infty} \frac{t^n}{n!}A^nP.$$

To prove the second equation, write $S(t) = T_A(t)(I - P + e^{-t}P)$. It can be verified directly that $S(t)$ is a C_0-semigroup. Differentiating $S(t)x$ at $t = 0$ for any $x \in \mathcal{D}(A)$, we find that the generator of $S(t)$ is $A - P$. □

2 SINGULAR PERTURBATION OF AN ASYMPTOTICALLY CONVERGENT C_0-SEMIGROUP

In this section we obtain generalizations of some results of [1] and [3], essential for the description of the asymptotic convergence of the solutions to (1.1).

A C_0-semigroup $T(t)$ with an infinitesimal generator A is (uniformly) *asymptotically convergent* if $T(t) \to P$ in the operator norm as $t \to \infty$. If this is the case, $A \in \mathcal{C}(X)$ is *semistable*, that is, $\sigma(A) \subset H \cup \{0\}$, where H is the open left half of the complex plane, and 0 is at most a simple pole of A (see [5]). The limit operator P is the spectral projection A^π of A at 0, and $AP = 0$. (The semistability of the generator is not sufficient for the asymptotic convergence of the C_0-semigroup.)

It will be convenient to write for any $r > 0$,

$$\Delta_r = \{\lambda : |\lambda| < r\}, \quad \Omega_r = \{\lambda : |\lambda| > r\}.$$

Theorem 2.1. *Let $T_A(t)$ be an asymptotically convergent C_0-semigroup with the infinitesimal generator A, and let $B \in \mathcal{B}(X)$. Then there exists $r > 0$ such that, for all sufficiently small $\varepsilon > 0$, $\sigma_\varepsilon = \sigma(A + \varepsilon B) \cap \Delta_r$ is an isolated spectral set of $A + \varepsilon B$. Let P_ε be the spectral projection of $A + \varepsilon B$ corresponding to σ_ε, and let $P = A^\pi$. Then $P_\varepsilon \to P$ in the operator norm as $\varepsilon \to 0+$.*

Proof. Since A is semistable, 0 is not an accumulation point of $\sigma(A)$, and there exists $r > 0$ such that $\Delta_r \cap \sigma(A) \subset \{0\}$. By the semicontinuity of the separated parts of the spectrum (see [2, Chapter IV, Theorem 3.16]) there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the spectrum of $A + \varepsilon B$ is split into two isolated spectral sets

$$\sigma(A + \varepsilon B) \cap \Delta_r, \qquad \sigma(A + \varepsilon B) \cap \Omega_r,$$

while $\|P_\varepsilon - P\| \to 0$ as $\varepsilon \to 0+$. □

Convention 2.2. Throughout the paper $T(t) = T_A(t)$ will denote an asymptotically convergent C_0-semigroup with the infinitesimal generator A and the spectral projection $A^\pi = P$, S_ε will denote the C_0-semigroup generated by $\varepsilon^{-1}A + B$, and P_ε will stand for the spectral projection of $A + \varepsilon B$ corresponding to the isolated spectral set $\sigma_\varepsilon = \sigma(A+\varepsilon B) \cap \Delta_r$ for a suitable $r > 0$ and all sufficiently small $\varepsilon > 0$.

We can now prove the following two results.

Theorem 2.3. *Let $T_A(t)$ be an asymptotically convergent C_0-semigroup and let $B \in \mathcal{B}(X)$. Then*

$$\lim_{\varepsilon \to 0+} S_\varepsilon(t) P_\varepsilon = \exp(tPB)P = P \exp(tBP), \tag{2.1}$$

uniformly on compact subsets of $(0, \infty)$.

Proof. Observe that, for all sufficiently small $\varepsilon > 0$, $\mathcal{R}(P_\varepsilon) \subset \mathcal{D}(A + \varepsilon B)$ and that $(A + \varepsilon B)P_\varepsilon \in \mathcal{B}(X)$. The spectral projections P_ε are given by

$$P_\varepsilon = f(A) = \frac{1}{2\pi i} \int_\Gamma R(\lambda; A + \varepsilon B)\, \mathrm{d}\lambda, \tag{2.2}$$

where $f(\lambda)$ is equal to 1 on Δ_r and 0 on Ω_r, and Γ is the positively oriented circle $|\lambda| = r$ for a suitable $r > 0$.

Let $\alpha = \sup\{\|R(\lambda; A)\| : \lambda \in \Gamma\}$ and let $\varepsilon > 0$ satisfy $0 \le \varepsilon\alpha\|B\| \le \frac{1}{2}$. For any $\lambda \in \Gamma$, $\varepsilon\|R(\lambda; A)B\| \le \frac{1}{2}$, and

$$R(\lambda; A + \varepsilon B) = (I - R(\lambda; A)\varepsilon B)^{-1} R(\lambda; A) = \sum_{k=0}^{\infty} (R(\lambda; A)B)^k R(\lambda; A)\varepsilon^k. \tag{2.3}$$

Since the series converges uniformly for $\lambda \in \Gamma$,

$$P_\varepsilon = \sum_{k=0}^{\infty}\Big(\frac{1}{2\pi i}\int_\Gamma (R(\lambda;A)B)^k R(\lambda;A)\,\mathrm{d}\lambda\Big)\varepsilon^k = \sum_{k=0}^{\infty} V_k\varepsilon^k \tag{2.4}$$

with $V_k \in \mathcal{B}(X)$, for all sufficiently small $\varepsilon > 0$, and

$$(\varepsilon^{-1}A + B)P_\varepsilon = \sum_{k=0}^{\infty}(\varepsilon^{-1}A+B)V_k\varepsilon^k = \sum_{k=0}^{\infty}(AV_{k+1}+BV_k)\varepsilon^k \tag{2.5}$$

as $AV_0 = AP = 0$ in view of the semistability of A. Further, from (2.5),

$$\lim_{\varepsilon\to+0}(\varepsilon^{-1}A+B)P_\varepsilon = AV_1 + BV_0 = AV_1 + BP.$$

We find AV_1 recalling that, in some punctured neighbourhood of 0, $R(\lambda;A) = \lambda^{-1}P + H(\lambda)$, where $H(\lambda) = -\sum_{n=0}^{\infty}\lambda^n(A^{\mathrm{D}})^{n+1}$ (see (1.4)):

$$\begin{aligned} AV_1 &= \frac{1}{2\pi i}\int_\Gamma AR(\lambda;A)BR(\lambda;A)\,\mathrm{d}\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma A(\lambda^{-1}P + H(\lambda))B(\lambda^{-1}P+H(\lambda))\,\mathrm{d}\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma AH(\lambda)B(\lambda^{-1}P+H(\lambda))\,\mathrm{d}\lambda = AH(0)BP \\ &= -AA^{\mathrm{D}}BP = -(I-P)BP = -BP + PBP \end{aligned}$$

using (1.3). Thus

$$\lim_{\varepsilon\to+0}(\varepsilon^{-1}A+B)P_\varepsilon = AV_1 + BP = PBP. \tag{2.6}$$

Using Lemma 1.2 and Theorem 2.1 , we get

$$\lim_{\varepsilon\to+0} S_\varepsilon(t)P_\varepsilon = \lim_{\varepsilon\to+0}\exp(t(\varepsilon^{-1}A+B)P_\varepsilon)P_\varepsilon = \exp(tPBP)P, \tag{2.7}$$

where the convergence is uniform for t in compact subsets of $(0,\infty)$. From the power series expansion of the exponential we get

$$\exp(tPBP)P = \exp(tPB)P = P\exp(tBP). \qquad \square$$

Theorem 2.4. *Let $T_A(t)$ be an aymptotically convergent C_0-semigroup and let $B \in \mathcal{B}(X)$. Then there exist a positive constant μ and positive functions $\tau \mapsto M_\tau$, $\tau \mapsto \varepsilon_\tau$ on $(0,\infty)$ such that*

$$\|S_\varepsilon(t)(I-P_\varepsilon)\| \le M_\tau e^{-\mu t/\varepsilon} \text{ for all } t \in (0,\tau) \text{ and all } \varepsilon \in (0,\varepsilon_\tau). \tag{2.8}$$

Proof. Since P_ε and $A+\varepsilon B$ commute, we can use Lemma 1.2 to obtain

$$\begin{aligned} S_\varepsilon(t)(I-P_\varepsilon) &= T_{A+\varepsilon B}(t/\varepsilon)(I-P_\varepsilon+e^{-t/\varepsilon}P_\varepsilon)-e^{-t/\varepsilon}T_{A+\varepsilon B}(t/\varepsilon)P_\varepsilon \\ &= T_{A+\varepsilon B-P_\varepsilon}(t/\varepsilon)-e^{-t/\varepsilon}T_{A+\varepsilon B}(t/\varepsilon)P_\varepsilon, \qquad t>0. \end{aligned}$$

We show that $T_{A+\varepsilon B-P_\varepsilon}(t/\varepsilon)$ decays exponentially as $\varepsilon\to 0+$ uniformly for $t\in(0,\infty)$. Since $T_A(t)\to P$ as $t\to\infty$, then $T_{A-P}(t)=T_A(t)(I-P+e^{-t}P)\to 0$, and according to [5, Theorem 2.3] there exist positive constants N,ν such that

$$\|T_{A-P}(t)\|\le Ne^{-\nu t}\quad \text{for all } t>0.$$

Applying [7, Theorem 3.1.1], we get

$$\|T_{A+\varepsilon B-P_\varepsilon}(t)\|=\|T_{(A-P)+\varepsilon B+P-P_\varepsilon}(t)\|\le Ne^{(-\nu+N\|\varepsilon B+P-P_\varepsilon\|)t},\quad t>0.$$

Select a positive constant $\beta<\nu$. Since $P_\varepsilon\to P$, there exists ε_1 such that for all $\varepsilon\in(0,\varepsilon_1)$,

$$\varepsilon\|B\|+\|P-P_\varepsilon\|<(\nu-\beta)N^{-1}.$$

Then $-\nu+N(\varepsilon\|B\|+\|P-P_\varepsilon\|)<-\beta$, and $\|T_{A+\varepsilon B-P_\varepsilon}(t)\|\le Ne^{-\beta t}$ for all $t>0$, that is,

$$\|T_{A+\varepsilon B-P_\varepsilon}(t/\varepsilon)\|\le Ne^{-\beta t/\varepsilon},\quad t>0,\quad 0<\varepsilon<\varepsilon_1.$$

Let $\tau>0$. We show that $e^{-t/\varepsilon}T_{A+\varepsilon B}(t/\varepsilon)P_\varepsilon$ decays exponentially as $\varepsilon\to 0+$ uniformly for $t\in(0,\tau)$. For this we note that by Theorem 2.3,

$$T_{A+\varepsilon B}(t/\varepsilon)P_\varepsilon=S_\varepsilon(t)P_\varepsilon$$

converges uniformly on $(0,\tau)$ as $\varepsilon\to 0+$. Then there exists positive constants C_τ,η_τ such that $\|T_{A+\varepsilon B}(t/\varepsilon)P_\varepsilon\|\le C_\tau$ for all $t\in(0,\tau)$ and all $\varepsilon\in(0,\eta_\tau)$.

For a given $\tau>0$ set $M_\tau=N+C_\tau$, $\mu=\min(\beta,1)$ and $\varepsilon_\tau=\min(\varepsilon_1,\eta_\tau)$. Combining the two inequalities obtained earlier, we have

$$\|S_\varepsilon(t)(I-P_\varepsilon)\|\le Ne^{-\beta t/\varepsilon}+C_\tau e^{-t/\varepsilon}\le M_\tau e^{-\mu t/\varepsilon},\quad t\in(0,\tau),\quad \varepsilon\in(0,\varepsilon_\tau). \quad \square$$

Combining the preceding two theorems, we get the following generalization of [3, Theorem 2.2].

Corollary 2.5. *Let $T_A(t)$ be an asymptotically convergent C_0-semigroup and let $B\in\mathcal{B}(X)$. Then*

$$\lim_{\varepsilon\to 0+}S_\varepsilon(t)=\lim_{\varepsilon\to 0+}S_\varepsilon(t)P_\varepsilon=\exp(tPB)P=P\exp(tBP), \tag{2.9}$$

uniformly on compact subsets of $(0,\infty)$.

3 A POWER SERIES EXPANSION FOR $S_\varepsilon(t)$

In this section we adapt some of the techniques employed by Campbell [1] for matrices to obtain a power series expansion in ε for $S_\varepsilon(t)$ when $T_A(t)$ is an symptotically convergent C_0-semigroup. We will find such an expansion by considering separately $S_\varepsilon(t)P_\varepsilon$ (the so called inner solution of (1.1)) and $S_\varepsilon(t)(I-P_\varepsilon)$ (the so called outer solution of (1.1)).

3.1 A power series expansion for $S_\varepsilon(t)P_\varepsilon$

By Lemma 1.2, for all sufficiently small $\varepsilon > 0$ we have

$$S_\varepsilon(t)P_\varepsilon = \exp\left(t(\varepsilon^{-1}A+B)P_\varepsilon\right)P_\varepsilon, \quad t \geq 0,$$

and the substitution of the power series (2.5) into the power series for the exponential ensures that, for all sufficiently small ε, we have an expansion

$$U_\varepsilon(t) = S_\varepsilon(t)P_\varepsilon = \sum_{k=0}^{\infty} X_k(t)\varepsilon^k, \quad t \geq 0, \tag{3.1}$$

where $X_k : [0,\infty) \to \mathcal{B}(X)$ are continuous functions. We have already found (2.9) that

$$X_0(t) = \lim_{\varepsilon\to 0+} S_\varepsilon(t)P_\varepsilon = \exp\left(tPB\right)P. \tag{3.2}$$

Observe that $U_\varepsilon(t)$ satisfies the differential equation

$$\frac{\mathrm{d}U_\varepsilon(t)}{\mathrm{d}t} = (\varepsilon^{-1}A+B)U_\varepsilon(t), \quad t>0, \quad U_\varepsilon(0) = P_\varepsilon.$$

We calculate $X_k(t)$ for $k = 0,1,2\ldots,$ by substituting (3.1) into the preceding differential equation and equating the coefficients of ε^k:

$$AX_0 = 0 \quad \text{and} \quad \frac{\mathrm{d}X_k}{\mathrm{d}t} = AX_{k+1} + BX_k, \quad k \geq 0. \tag{3.3}$$

To determine $X_1(t)$, observe from (3.3) that

$$(I-P)X_1 = A^{\mathrm{D}}\frac{\mathrm{d}X_0}{\mathrm{d}t} - A^{\mathrm{D}}BX_0 = -A^{\mathrm{D}}BX_0 = -A^{\mathrm{D}}B\exp\left(tPB\right)P$$

taking into account that $A^{\mathrm{D}}P = 0$. Multiplying $\mathrm{d}X_1/\mathrm{d}t = AX_2 + BX_1$ from the left by P produces

$$P\frac{\mathrm{d}X_1}{\mathrm{d}t} = PBPX_1 + PB(I-P)X_1 = PBPX_1 - PBA^{\mathrm{D}}B\exp\left(tPB\right)P.$$

The differential equation

$$\frac{\mathrm{d}(PX_1)}{\mathrm{d}t} = PB(PX_1) + Q,$$

where $Q(t) = -PBA^{\mathrm{D}}B\exp(tPB)P$, has a solution

$$\begin{aligned} PX_1(t) &= \exp(tPB)\int_0^t \exp(-sPB)Q(s)\,\mathrm{d}s + \exp(tPB)PX_1(0) \\ &= -\exp(tPB)\int_0^t \exp(-sPB)PBA^{\mathrm{D}}B\exp(sPB)P\,\mathrm{d}s + \exp(tPB)PX_1(0). \end{aligned}$$

From $X_1(t) = PX_1(t) + (I-P)X_1(t)$ we get

$$\begin{aligned} X_1(t) = &-\exp(tPB)\int_0^t \exp(-sPB)PBA^{\mathrm{D}}B\exp(sPB)P\,\mathrm{d}s \\ &+ \exp(tPB)PX_1(0) - A^{\mathrm{D}}B\exp(tPB)P. \end{aligned} \tag{3.4}$$

Using (3.3) again, we obtain $(I-P)X_k$ and PX_k, respectively. A similar calculation as for $X_1(t)$ yields

$$\begin{aligned} X_{k+1}(t) = &\exp(tPB)\int_0^t \exp(-sPB)PBA^{\mathrm{D}}\left(\frac{\mathrm{d}X_k(s)}{\mathrm{d}t} - BX_k(s)\right)\mathrm{d}s \\ &+ A^{\mathrm{D}}\left(\frac{\mathrm{d}X_k(t)}{\mathrm{d}t} - BX_k(t)\right) + \exp(tPB)PX_{k+1}(0). \end{aligned} \tag{3.5}$$

We complete the calculation of $X_k(t)$ by finding $X_k(0)$. From (3.1),

$$\sum_{k=0}^{\infty} X_k(0)\varepsilon^k = \frac{1}{2\pi i}\int_\Gamma R(\lambda; A+\varepsilon B)\,\mathrm{d}\lambda.$$

By (2.3), for $\varepsilon > 0$ sufficiently small

$$R(\lambda; A+\varepsilon B) = \sum_{k=0}^{\infty}(R(\lambda; A)B)^k R(\lambda; A)\varepsilon^k.$$

Hence

$$\sum_{k=0}^{\infty} X_k(0)\varepsilon^k = \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_\Gamma (R(\lambda; A)B)^k R(\lambda; A)\,\mathrm{d}\lambda\right)\varepsilon^k,$$

and

$$X_k(0) = \frac{1}{2\pi i}\int_\Gamma (R(\lambda; A)B)^k R(\lambda; A)\,\mathrm{d}\lambda. \tag{3.6}$$

In particular,

$$X_0(0) = \frac{1}{2\pi i}\int_\Gamma R(\lambda; A)\,\mathrm{d}\lambda = P,$$

which agrees with (3.2). In fact, we only need to calculate $PX_k(0)$ in (3.5) for $k \geq 1$. By (1.4), for $0 < |\lambda| < r$, $R(\lambda; A) = \lambda^{-1}P + H(\lambda) = \lambda^{-1}P - \sum_{n=0}^{\infty}\lambda^n (A^{\mathrm{D}})^{n+1}$, which implies $PR(\lambda; A) = \lambda^{-1}P$. Then

$$PX_k(0) = PB\frac{1}{2\pi i}\int_\Gamma \lambda^{-1}(R(\lambda; A)B)^{k-1}R(\lambda; A)\,\mathrm{d}\lambda.$$

For $k = 1$,

$$\begin{aligned} PX_1(0) &= PB\frac{1}{2\pi i}\int_\Gamma \lambda^{-1}R(\lambda; A)\,\mathrm{d}\lambda = PB\frac{1}{2\pi i}\int_\Gamma(\lambda^{-2}P + \lambda^{-1}H(\lambda))\,\mathrm{d}\lambda \\ &= PBH(0) = -PBA^{\mathrm{D}}. \end{aligned}$$

A similar calculation for $k = 2$ gives

$$PX_2(0) = PB\Big(A^{\mathrm{D}}BA^{\mathrm{D}} - (A^{\mathrm{D}})^2BP - PB(A^{\mathrm{D}})^2\Big).$$

We observe that according to (3.4),

$$\begin{aligned} X_1(t) = &-\exp(tPB)\int_0^t \exp(-sPB)PBA^{\mathrm{D}}B\exp(sPB)P\,\mathrm{d}s \\ &-\exp(tPB)PBA^{\mathrm{D}} - A^{\mathrm{D}}B\exp(tPB)P. \end{aligned} \tag{3.7}$$

3.2 A power series expansion for $S_\varepsilon(t)(I - P_\varepsilon)$

Since $S_\varepsilon(t) = T_{A+\varepsilon B}(t/\varepsilon)$, we will work with $T_{A+\varepsilon B}(\tau)$, where $\tau = t/\varepsilon$. For the perturbation of A by $\varepsilon B \in \mathcal{B}(X)$ we can use the series [7, p. 78],

$$T_{A+\varepsilon B}(\tau) = \sum_{n=0}^{\infty} S_n(\tau) = \sum_{n=0}^{\infty} W_n(\tau)\varepsilon^n,$$

where

$$S_0(\tau) = T_A(\tau), \quad S_{n+1}(\tau) = \varepsilon\int_0^\tau T_A(\tau - s)BS_n(s)\,\mathrm{d}s,$$

and where $W_n(\tau) = \varepsilon^{-n}S_n(\tau) \in \mathcal{B}(X)$ are free of ε for all n. Since

$$I - P_\varepsilon = I - \sum_{k=0}^{\infty}\Big(\frac{1}{2\pi i}\int_\Gamma (R(\lambda; A)B)^k R(\lambda; A)\,\mathrm{d}\lambda\Big)\varepsilon^k,$$

$T_{A+\varepsilon B}(\tau)(I-P_\varepsilon)$ is a product of two power series in ε, and

$$W(\tau) = T_{A+\varepsilon B}(\tau)(I-P_\varepsilon) = \sum_{k=0}^{\infty} Y_k(\tau)\varepsilon^k,$$

where $Y_k(\tau) \in \mathcal{B}(X)$ for each $k \geq 0$. On the other hand, $W(\tau)$ satisfies

$$\frac{\mathrm{d}W(\tau)}{\mathrm{d}\tau} = (A+\varepsilon B)W(\tau).$$

Substituting the series for $W(\tau)$ into this equation and equating the coefficients of ε^k gives

$$\frac{\mathrm{d}Y_0}{\mathrm{d}\tau} = AY_0 \quad \text{and} \quad \frac{\mathrm{d}Y_k}{\mathrm{d}\tau} = AY_k + BY_{k-1}, \quad k \geq 1.$$

This infinite system can be solved iteratively if the initial values $Y_k(0)$ are known:

$$Y_0(\tau) = T_A(\tau)Y_0(0), \tag{3.8}$$

$$Y_k(\tau) = T_A(\tau)Y_k(0) + \int_0^\tau T_A(\tau-s)BY_{k-1}(s)\,\mathrm{d}s, \quad k \geq 1. \tag{3.9}$$

We complete the expansion by finding $Y_k(0)$ for $k \geq 0$.

Since $\sum_{k=0}^\infty Y_k(\tau)\varepsilon^k$ is a product of two power series, we have

$$Y_k(\tau) = W_k(\tau) - \sum_{j=0}^{k} W_j(\tau)\frac{1}{2\pi i}\int_\Gamma (R(\lambda;A)B)^{k-j}R(\lambda;A)\,\mathrm{d}\lambda, \quad k \geq 0.$$

For $k=0$ we have

$$Y_0(\tau) = W_0(\tau) - W_0(\tau)\int_\Gamma R(\lambda;A)\,\mathrm{d}\lambda = T_A(\tau)(I-P).$$

For $k \geq 1$ we have

$$Y_k(0) = W_k(0) - \sum_{j=0}^{k} W_j(0)\frac{1}{2\pi i}\int_\Gamma (R(\lambda;A)B)^{k-j}R(\lambda;A)\,\mathrm{d}\lambda.$$

But $W_0(0) = I$ and $W_k(0) = 0$ for all $k \geq 1$, and so

$$Y_0(0) = I-P, \quad Y_k(0) = -\frac{1}{2\pi i}\int_\Gamma (R(\lambda;A)B)^k R(\lambda;A)\,\mathrm{d}\lambda, \quad k \geq 1. \tag{3.10}$$

When A, B are matrices, this agrees with Campbell [1, p. 99, Equation (19)].

Putting together the results of Sections 3.1 and 3.2 we obtain the following theorem which generalizes [1, Theorem 5.3.1].

Theorem 3.1. *Let $T_A(t)$ be an asymptotically convergent C_0-semigroup, let $B \in \mathcal{B}(X)$, and let $S_\varepsilon(t)$ be the C_0-semigroup generated by $\varepsilon^{-1}A + B$. Then, for all sufficiently small $\varepsilon > 0$,*

$$S_\varepsilon(t) = \sum_{k=0}^{\infty} X_k(t)\varepsilon^k + \sum_{j=0}^{\infty} Y_j(t/\varepsilon)\varepsilon^j, \quad t \geq 0, \tag{3.11}$$

where the X_k are given by (3.2), (3.5) *and* (3.6), *and the Y_j are given by* (3.8), (3.9) *and* (3.10).

4 NONHOMOGENEOUS SINGULARLY PERTURBED DIFFERENTIAL EQUATION

In this section, as an application of the preceding results, we describe the asymptotic behaviour of the solutions to the singularly perturbed nonhomogeneous differential equation. In the special case when the operators are matrices, we recover [1, Theorem 5.8.3]. By assuming the boundedness of the forcing term, we are able to express the limit of the solution explicitly in terms of the g-Drazin inverse.

Theorem 4.1. *Let $T_A(t)$ be an asymptotically convergent C_0-semigroup, let $B \in \mathcal{B}(X)$ and let $f : [0, \infty) \to X$ be continuous and bounded. Then the mild solution $u_\varepsilon(t)$ of the singularly perturbed differential equation*

$$\begin{aligned} \varepsilon \frac{\mathrm{d}u_\varepsilon(t)}{\mathrm{d}t} &= (A + \varepsilon B)u_\varepsilon(t) + f(t), \\ u_\varepsilon(0) &= x, \quad \varepsilon > 0, \end{aligned} \tag{4.1}$$

converges as $\varepsilon \to 0+$ if and only if $Pf(t) = 0$ for all $t \geq 0$. If this is the case, then

$$u(t) = \lim_{\varepsilon \to 0+} u_\varepsilon(t) = \exp(tPB)Px + \int_0^t X_1(t-s)f(s)\,\mathrm{d}s - A^{\mathrm{D}}f(t) \tag{4.2}$$

uniformly on compact subsets of $(0, \infty)$, where $X_1(t)$ is given by (3.7). *The limit u is the solution of the reduced equation*

$$0 = Au(t) + f(t), \qquad u(0) = Px - A^{\mathrm{D}}f(0). \tag{4.3}$$

Proof. The mild solution of (4.1) (see [7, p. 106]) is given by

$$u_\varepsilon(t) = S_\varepsilon(t)x + \varepsilon^{-1}\int_0^t S_\varepsilon(t-s)f(s)\,\mathrm{d}s, \quad t \geq 0. \tag{4.4}$$

By Corollary 2.5, $S_\varepsilon(t) \to \exp(tPB)P$ as $\varepsilon \to 0+$ uniformly on compact subsets of $(0,\infty)$. Let P_ε be the spectral projection of $A+\varepsilon B$. The second term in (4.4) can be expressed as the sum

$$\varepsilon^{-1}\int_0^t S_\varepsilon(t-s)P_\varepsilon f(s)\,\mathrm{d}s + \varepsilon^{-1}\int_0^t S_\varepsilon(t-s)(I-P_\varepsilon)f(s)\,\mathrm{d}s. \tag{4.5}$$

Let $\tau > 0$. By Theorem 2.4 there exist positive constants $\mu, M_\tau, \varepsilon_\tau$ such that

$$\|S_\varepsilon(t)(I-P_\varepsilon)\| \le M_\tau e^{-\mu t/\varepsilon}, \quad t\in(0,\tau), \quad \varepsilon\in(0,\varepsilon_\tau).$$

Given $\nu > 0$, there exists $\delta > 0$ such that for any $t_1, t_2 \in (0,\tau)$ satisfying $|t_1 - t_2| < \delta$ we have $\|f(t_1) - f(t_2)\| < \nu$ (uniform continuity of f on $(0,\tau)$).

Let $t \in (0,\tau)$. Choose $t_0 \in (0,t)$ such that $t_0 > t-\delta$. Then

$$\begin{aligned}\varepsilon^{-1}\int_0^t S_\varepsilon(t-s)(I-P_\varepsilon)f(s)\,\mathrm{d}s &= \varepsilon^{-1}\int_0^t S_\varepsilon(t-s)(I-P_\varepsilon)(f(s)-f(t))\,\mathrm{d}s\\ &\quad + \varepsilon^{-1}\int_0^t S_\varepsilon(t-s)(I-P_\varepsilon)f(t)\,\mathrm{d}s\\ &= v_\varepsilon^1(t) + v_\varepsilon^2(t).\end{aligned}$$

We have

$$\begin{aligned}\|v_\varepsilon^1(t)\| &\le \varepsilon^{-1}\int_0^t M_\tau e^{-\mu(t-s)/\varepsilon}\|f(s)-f(t)\|\,\mathrm{d}s\\ &= \varepsilon^{-1}\int_0^{t_0} M_\tau e^{-\mu(t-s)/\varepsilon}2\|f\|_\infty\,\mathrm{d}s + \varepsilon^{-1}\int_{t_0}^t M_\tau e^{-\mu(t-s)/\varepsilon}\nu\,\mathrm{d}s\\ &\le 2M_\tau\mu^{-1}\|f\|_\infty(e^{-\mu(t-t_0)/\varepsilon} - e^{-\mu t/\varepsilon}) + \nu M_\tau\mu^{-1}(1-e^{-\mu(t-t_0)/\varepsilon}).\end{aligned}$$

Hence $\limsup_{\varepsilon\to 0+}\|v_\varepsilon^1(t)\| \le \nu M_\tau \mu^{-1}$. Since $\nu > 0$ is arbitrary, we conclude that $\lim_{\varepsilon\to 0+}\|v_\varepsilon^1(t)\| = 0$ uniformly for $t \in (0,\tau)$. Further,

$$\begin{aligned}\lim_{\varepsilon\to 0+} v_\varepsilon^2(t) &= \lim_{\varepsilon\to 0+}\varepsilon^{-1}\int_0^t S_\varepsilon(t-s)(I-P_\varepsilon)f(t)\,\mathrm{d}s\\ &= \lim_{\varepsilon\to 0+}\int_0^{t/\varepsilon} T_{A+\varepsilon B}(z)(I-P_\varepsilon)f(t)\,\mathrm{d}z\\ &= \int_0^\infty T_A(z)(I-P)f(t)\,\mathrm{d}z, \end{aligned} \tag{4.6}$$

provided we show that

$$\lim_{\varepsilon\to 0+}\int_0^{t/\varepsilon}[T_{A+\varepsilon B}(z)(I-P_\varepsilon)-T_A(z)(I-P)]f(t)\,\mathrm{d}z=0 \tag{4.7}$$

uniformly on $(0,\tau)$. We have

$$\begin{aligned} I(\varepsilon,z)&=T_{A+\varepsilon B}(z)(I-P_\varepsilon)-T_A(z)(I-P)\\ &=[T_{A+\varepsilon B-P_\varepsilon}(z)-T_{A+\varepsilon B}(z)e^{-z}P_\varepsilon]-[T_{A-P}(z)-T_A(z)e^{-z}P]\\ &=[T_{A+\varepsilon B-P_\varepsilon}(z)-T_{A-P}(z)]-[T_{A+\varepsilon B}(z)P_\varepsilon-T_A(z)P]e^{-z}\\ &=I_1(\varepsilon,z)-I_2(\varepsilon,z). \end{aligned} \tag{4.8}$$

We recall [5, Theorem 3.4] that, for an asymptotically convergent semigroup $T_A(t)$ with $P=A^\pi$, we have $\|T_{A-P}(t)\|\le Ne^{-\nu t}$ for some positive constants N,ν and for all $t>0$. Applying [7, Corollary 3.1.3], we obtain

$$\begin{aligned} \|I_1(\varepsilon,z)\|&=\|T_{A+\varepsilon B-P_\varepsilon}(z)-T_{A-P}(z)\|\\ &\le Ne^{-\nu z}(e^{N\|\varepsilon B+P-P_\varepsilon\|z}-1)=Ne^{-\nu z}(e^{\rho(\varepsilon)z}-1), \end{aligned}$$

where $\rho(\varepsilon)\to 0$ as $\varepsilon\to 0+$. Then

$$\int_0^{t/\varepsilon}e^{-\nu z}(e^{\rho(\varepsilon)z}-1)\,\mathrm{d}z=\frac{1}{\nu-\rho(\varepsilon)}(1-e^{(-\nu+\rho(\varepsilon))t/\varepsilon})+\frac{1}{\nu}(e^{-\nu t/\varepsilon}-1),$$

and, proceeding similarly as in the proof of Theorem 2.4, we conclude that

$$\Big\|\int_0^{t/\varepsilon}I_1(\varepsilon,z)f(t)\,\mathrm{d}z\Big\|\le N\|f\|_\infty\int_0^{t/\varepsilon}e^{-\nu z}(e^{\rho(\varepsilon)z}-1)\,\mathrm{d}z\to 0 \text{ as } \varepsilon\to 0+$$

uniformly on $(0,\tau)$.

Next we have

$$\begin{aligned} I_2(\varepsilon,z)&=[T_{A+\varepsilon B}(z)P_\varepsilon-T_A(z)P]e^{-z}\\ &=[T_{A+\varepsilon B}(z)-T_A(z)]P_\varepsilon e^{-z}+T_A(z)(P-P_\varepsilon)e^{-z}\\ &=I_{2,1}(\varepsilon,z)+I_{2,2}(\varepsilon,z). \end{aligned}$$

A similar argument as for $I_1(\varepsilon,z)$ shows that

$$\lim_{\varepsilon\to 0+}\int_0^{t/\varepsilon}I_{2,1}(\varepsilon,z)f(t)\,\mathrm{d}z=0$$

uniformly on $(0,\tau)$. Finally,

$$\lim_{\varepsilon\to 0+}\int_0^{t/\varepsilon} I_{2,2}(\varepsilon,z)f(t)\,\mathrm{d}z=0$$

uniformly on $(0,\tau)$ since $T_A(z)$ is bounded and $P-P_\varepsilon$ converges to 0. This proves (4.7). Return to (4.6) and observe that, by [4, Theorem 4.2 (iii)],

$$\int_0^\infty T_A(z)(I-P)f(t)\,\mathrm{d}z=-A^{\mathrm{D}}f(t).$$

Using the expansion (3.11), we can write the first term in (4.5) as

$$\varepsilon^{-1}\int_0^t \exp((t-s)PB)Pf(s)\,\mathrm{d}s+\int_0^t X_1(t-s)f(s)\,\mathrm{d}s+\int_0^t K_\varepsilon(t-s)f(s)\,\mathrm{d}s, \quad (4.9)$$

where K_ε is continuous on $[0,\infty)$ and $\lim_{\varepsilon\to 0+}K_\varepsilon(s)=0$ uniformly on compact subsets of $(0,\infty)$. Thus the limit of (4.9) as $\varepsilon\to 0+$ exists if and only it the limit

$$\lim_{\varepsilon\to 0+}\varepsilon^{-1}\int_0^t \exp((t-s)PB)Pf(s)\,\mathrm{d}s \quad (4.10)$$

exists for each $t\ge 0$. Since $\exp(sPB)$ is invertible for each $s\ge 0$, the limit (4.10) exists (and is equal to 0) if and only if $Pf(t)=0$ for all $t\ge 0$.

Combining all parts of the proof, we conclude that the solution $u_\varepsilon(t)$ converges to $u(t)=\exp{(tPB)}Px+\int_0^t X_1(t-s)f(s)\,\mathrm{d}s-A^{\mathrm{D}}f(t)$ as $\varepsilon\to 0+$, uniformly on any compact subset of $(0,\infty)$.

Finally we confirm by a direct substitution that $u(t)$ is the solution of the reduced equation (4.3). □

Setting $B=0$ in the preceding theorem we recover [4, Theorem 5.2].

References

[1] S. L. Campbell, *Singular Systems of Differential Equations*, Research Notes in Mathematics 40, Pitman, London, 1980.

[2] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin, 1966.

[3] J. J. Koliha and Trung Dinh Tran, Semistable operators and singularly perturbed differential equations, *J. Math. Anal. Appl.* **231** (1999), 446–458.

[4] J. J. Koliha and Trung Dinh Tran, The Drazin inverse for closed linear operators and the asymptotic convergence of C_0-semigroups *J. Operator Theory* **46** (2001), 323–336.

[5] J. J. Koliha and Trung Dinh Tran, Closed semistable operators and singular differential equations, *Czechoslovak Math. J.* **53 (128)** (2003), 605–620.

[6] J. van Neerven, *The Asymptotic Behaviour of Semigroups of Linear Operators*, Birkhäuser, Basel, 1996.

[7] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, Berlin, 1983.

[8] A. E. Taylor and D. C. Lay, *Introduction to Functional Analysis*, 2nd edition, Wiley, New York, 1980.

[9] G. F. Webb, An operator theoretic formulation of asynchronous exponential growth, *Trans. Amer. Math. Soc.* **303** (1987), 751-763.

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