LEBESGUE THROUGH NEWTON INTEGRAL

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In elementary calculus, most definite integrals over a finite interval $[a, b]$ are calculated from the fundamental theorem of calculus

$$\int_a^b f(t)\,dt = F(b) - F(a), \tag{1}$$

where F is a *primitive* of f, that is, a function F continuous on $[a, b]$ such that $F'(x) = f(x)$ for all $x \in (a, b)$.

In this classroom note we start from the right hand side of Equation (1), and show directly, without recourse to Lebesgue's theory of integrating derivatives, how the Lebesgue integral may be proved to exist, and evaluated, given the existence of primitives. We assume a basic knowledge of the Lebesgue integral on the real line, such as can be found in [4], pointing out the results needed in the proof as we go along.

Definition. Let $-\infty \le a < b \le \infty$. A function $f : (a, b) \to \mathbb{R}$ is said to be *Newton integrable* if f has a primitive F in (a, b), and if the one sided limits $F(a+)$ and $F(b-)$ exist and are finite. The real number

$$(\mathcal{N})\int_a^b f(t)\,dt = F(b-) - F(a+)$$

is the *Newton integral* of f over (a, b).

The Newton integral can be applied to unbounded functions and unbounded intervals, unlike the Riemann integral. In many situations we are able to find a primitive to a given function, so it is of great practical value to know the relation between the Lebesgue and Newton integral. The symbol $\int_a^b f(t)\,dt$ stands for the Lebesgue integral, and $m(E)$ for the Lebesgue measure of $E \subset \mathbb{R}$.

Theorem. *Let $-\infty \le a < b \le \infty$, and let f be a real valued function on (a, b) such that both f and $|f|$ are Newton integrable. Then f is also Lebesgue integrable, and*

$$\int_a^b f(t)\,dt = (\mathcal{N})\int_a^b f(t)\,dt. \tag{2}$$

Proof. (i) Assume first that f is nonnegative, and choose a compact interval $[a_0, b_0] \subset (a, b)$. First we show that f is Lebesgue integrable on $[a_0, b_0]$ and that

$$\int_{a_0}^{b_0} f(t)\,dt \le F(b_0) - F(a_0), \tag{3}$$

where F is a primitive of f on (a, b). For all sufficiently large n define

$$f_n(t) = \frac{F(t + 1/n) - F(t)}{1/n}, \quad a_0 \le t \le b_0.$$

Each f_n is Lebesgue integrable on $[a_0, b_0]$ being continuous on a compact interval. Taking into account that F is increasing, we get

$$\begin{aligned}
\int_{a_0}^{b_0} f_n(t)\,dt &= n \int_{a_0}^{b_0} (F(t + 1/n) - F(t))\,dt \\
&= n \int_{b_0}^{b_0+1/n} F(t)\,dt - n \int_{a_0}^{a_0+1/n} F(t)\,dt \\
&\le n \int_{b_0}^{b_0+1/n} F(b_0)\,dt - n \int_{a_0}^{a_0+1/n} F(a_0)\,dt \\
&= F(b_0) - F(a_0).
\end{aligned}$$

We observe that $f_n \to f$ pointwise on $[a_0, b_0]$. By Fatou's lemma [4, p. 86], f is Lebesgue integrable on $[a_0, b_0]$ and

$$\int_{a_0}^{b_0} f(t)\,dt = \int_{a_0}^{b_0} \lim_{n\to\infty} f_n(t)\,dt \le \liminf_{n\to\infty} \int_{a_0}^{b_0} f_n(t)\,dt \le F(b_0) - F(a_0).$$

This proves (3).

Now for the harder part: We need to show that

$$F(b_0) - F(a_0) \le \int_{a_0}^{b_0} f(t)\,dt. \tag{4}$$

Let $\varepsilon > 0$ be given. Choose a sequence of real numbers c_i such that $c_0 = 0$, $c_i \to \infty$, $0 < c_i - c_{i-1} < \varepsilon/(b_0 - a_0)$ for all $i \in \mathbb{N}$, and define

$$E_i = \{t \in [a_0, b_0] : c_{i-1} \le f(t) < c_i\} = f^{-1}([c_{i-1}, c_i)) \cap [a_0, b_0].$$

Since f is Lebesgue integrable on $[a_0, b_0]$, the sets E_i are Lebesgue measurable, and $[a_0, b_0]$ is the disjoint union of the E_i. Hence $b_0 - a_0 = \sum_{i=1}^{\infty} m(E_i)$, and

$$c_{i-1} m(E_i) \le \int_{E_i} f(t)\,dt \le c_i m(E_i), \quad i \in \mathbb{N},$$

which gives

$$0 \le c_i m(E_i) - \int_{E_i} f(t)\,dt \le \frac{\varepsilon}{b_0 - a_0}\, m(E_i).$$

From the countable additivity of Lebesgue integral we conclude that

$$\sum_{i=1}^{\infty} c_i m(E_i) \le \int_{a_0}^{b_0} f(t)\,dt + \varepsilon. \tag{5}$$

For each $i \in \mathbb{N}$ there exists a bounded open set $G_i \subset \mathbb{R}$ such that

$$G_i \supset E_i \text{ and } m(G_i) \le m(E_i) + c_i^{-1}(\tfrac{1}{2})^i \varepsilon, \quad i \in \mathbb{N}; \tag{6}$$

see [4, p. 63]. Define $H : [a_0, b_0] \to \mathbb{R}$ by

$$H(t) = \sum_{i=1}^{\infty} c_i m(G_i \cap [a_0, t]), \quad a_0 \le t \le b_0.$$

Let x be the supremum of all $t \in [a_0, b_0]$ such that $F(t) - F(a_0) \le H(t)$. For a proof by contradiction assume that $x < b_0$. Then $x \in E_k$ for some $k \in \mathbb{N}$, and from $F'(x) = f(x) < c_k$ it follows that there exists $x_0 \in (x, b_0]$ such that

$$[x, x_0] \subset G_k \text{ and } F(x_0) - F(x) < c_k(x_0 - x).$$

Starting with

$$F(x_0) - F(a_0) = [F(x_0) - F(x)] + [F(x) - F(a_0)] \le c_k(x_0 - x) + H(x),$$

we deduce that $F(x_0) - F(a_0) \le H(x_0)$. This contradicts the definition of x, and implies that $x = b_0$. Hence, by (5) and (6),

$$F(b_0) - F(a_0) \le H(b_0) \le \int_{a_0}^{b_0} f(t)\,dt + 2\varepsilon.$$

Since ε was arbitrary, (4) holds. Hence the theorem is proved in this case.

(ii) Towards the case of a nonnegative function f and a general interval (a, b), $-\infty \le a < b \le \infty$, we choose sequences (a_n), (b_n) in $\mathbb{R}$ such that $a < a_n < b_n < b$ and $a_n \searrow a$, $b_n \nearrow b$. The result of part (i) of this proof can be applied to f on each interval $[a_n, b_n]$. Since F is increasing,

$$0 \le \int_{a_n}^{b_n} f(t)\,dt = F(b_n) - F(a_n) \le F(b-) - F(a+) = \text{const}.$$

We observe that $[a_n, b_n] \nearrow (a, b)$ and that $\int_{a_n}^{b_n} f(t)\,dt = \int_a^b g_n(t)\,dt$, where g_n is equal to f on $[a_n, b_n]$ and to 0 elsewhere in (a, b). Thus we can apply the monotone convergence theorem [4, p. 87] to conclude that f is Lebesgue integrable on (a, b), and satisfies

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \int_{a_n}^{b_n} f(t)\,dt = \lim_{n\to\infty} (F(b_n) - F(a_n)) = F(b-) - F(a+).$$

Thus the equality of the Lebesgue and Newton integrals holds in this case.

(iii) Finally assume that f is a function of either sign, that F is a primitive for f, and G a primitive for $|f|$ on (a, b), while the limits $F(a+)$, $F(b-)$, $G(a+)$ and $G(b-)$ exist. Then the functions

$$\tfrac{1}{2}(G + F) \text{ and } \tfrac{1}{2}(G - F)$$

are primitives to

$$f^+ = \tfrac{1}{2}(|f| + f) \text{ and } f^- = \tfrac{1}{2}(|f| - f),$$

respectively. The functions f^+, f^- are nonnegative and $f = f^+ - f^-$. The result then follows on applying part (ii) of the proof separately to f^+ and f^-. □

The preceding theorem is not new, but we have endeavoured to present a new direct proof which avoids wading through extra material. See the Note at the end of this article for a suggestion how to prove the theorem using considerably deeper results.

Example 1. Proving the existence of the Lebesgue integral $\int_0^1 (1/\sqrt{t})\, dt$ and evaluating it becomes very easy if we use our theorem. This is left to the reader as an exercise.

The condition of our theorem that both f and $|f|$ have primitives cannot be dropped. The Lebesgue integral is often described as an *absolute integral* in view of the fact that the integrability of f implies the integrability of $|f|$. If f is Newton integrable, but changes sign very often, then $|f|$ need not be Lebesgue integrable. A classical example follows.

Example 2. The function

$$f(t) = 2t \sin \frac{1}{t^2} - \frac{2}{t} \cos \frac{1}{t^2}, \quad 0 < t < 1,$$

has a primitive $F(t) = t^2 \sin t^{-2}$ on $(0, 1)$, and the limits $F(0+) = 0$, $F(1-) = \sin 1$ exist. Hence f is Newton integrable on $(0, 1)$ with $(\mathcal{N})\int_0^1 f(t)\, dt = \sin 1$.

We note that f is Lebesgue integrable on each interval $[s, 1]$, $s > 0$, being continuous on a compact interval, and that

$$\lim_{s \to 0+} \int_s^1 f(t)\, dt = F(1-) - F(0+) = \sin 1.$$

However, f is not Lebesgue integrable on $(0, 1)$. For this it is enough to show that $\lim_{s \to 0+} \int_s^1 |f(t)|\, dt = \infty$. This is left to the reader.

Note. There are integration processes which include both the Lebesgue and Newton integrals. One such integral was developed by Arnaud Denjoy (1884–1974) in 1912 and by Oskar Perron (1880–1975) in 1914. (Their definitions are different, but lead to the same integral.) Unlike the Lebegue integral, the *Denjoy–Perron* ($\mathcal{DP}$) *integral* has the property that if $\lim_{t \to b} (\mathcal{DP})\int_a^t f$ exists, then f is Denjoy–Perron integrable on $[a, b]$, and $\lim_{t \to b} (\mathcal{DP})\int_a^t f = (\mathcal{DP})\int_a^b f$.

The $\mathcal{DP}$ integral is a nonabsolute integral, that is, f may be $\mathcal{DP}$-integrable without $|f|$ being $\mathcal{DP}$-integrable. Every Newton integrable function is $\mathcal{DP}$-integrable, and the integrals are consistent. A function is Lebesgue integrable if and only if it is absolutely $\mathcal{DP}$-integrable; the two integrals are then equal. This provides an alternative proof of the preceding theorem. A discussion of these and other properties of the Denjoy and Perron integrals can be found in [2].

In 1957 Jaroslav Kurzweil (b. 1926) introduced an innocent looking change in the definition of the Riemann integral and obtained a nonabsolute integral with the range and power of the Lebesgue integral. A similar definition was already given independently in 1955 by Ralph Henstock (b. 1923) who developed the integral and showed its relation to the Lebesgue integral. A detailed account of this integral, known as the *generalized Riemann integral* or the *Kurzweil–Henstock integral*, is presented, for instance, in [1], [2] and [3]. Surprisingly, the Kurzweil–Henstock integral coincides with the Denjoy–Perron integral.

References

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