Opial inequalities for fractional derivatives

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Abstract

This paper proves Opial inequalities for generalized fractional derivatives analogous to inequalities established by the first author using a different definition of fractional derivative. The inequalities are proved for integrable functions with a minimal restriction on the order of the derivatives.

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1 Introduction and preliminaries

The original Opial inequality [7] (see also [6, p. 114]) states the following:

Theorem 1.1. *If $f \in C^1[0,a]$ with $f(0) = f(a) = 0$ and $f(x) > 0$ on $(0,a)$, then*

$$\int_0^a |f(x)f'(x)|\,dx \le \frac{a}{4}\int_0^a (f'(x))^2\,dx.$$

The constant $a/4$ is the best possible.

This result for classical derivatives has been generalized in several directions (see, for instance, [2, 3]). The present paper is motivated by an earlier paper [1] by Anastassiou. Using a different (nonequivalent) definition of fractional derivative, we obtain inequalities for integrable rather than continuous functions, while being able to relax the conditions on the order of fractional derivatives.

We give a brief survey of some facts about fractional derivatives needed in the sequel; for more details see the monograph [8, Chapter 1].

Let $x > 0$. By $C^m[0,x]$ we denote the space of all functions on $[0,x]$ which have continuous derivatives up to order m, and $AC[0,x]$ is the space of all absolutely continuous functions on $[0,x]$. By $AC^m[0,x]$ we denote the space of all functions $g \in C^m[0,x]$ with $g^{(m-1)} \in AC[0,x]$. For any $\nu \in \mathbb{R}$ we denote by $[\nu]$ the integral part of ν (the integer k satisfying $k \le \nu < k+1$).

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By $L(0,x)$ we denote the space of all functions integrable on the interval $(0,x)$, and by $L^\infty(0,x)$ the set of all functions measurable and essentially bounded on $(0,x)$. Clearly, $L^\infty(0,x) \subset L(0,x)$. Let $\nu > 0$. For any $f \in L(0,x)$ the *Riemann–Liouville fractional integral of* f of order ν is defined by

$$I^\nu f(s) = \frac{1}{\Gamma(\nu)} \int_0^s (s-t)^{\nu-1} f(t)\,dt, \quad s \in [0,x], \tag{1.1}$$

and the *Riemann–Liouville fractional derivative of* f of order ν by

$$D^\nu f(s) = \left(\frac{d}{ds}\right)^m I^{m-\nu} f(s) = \frac{1}{\Gamma(m-\nu)} \left(\frac{d}{ds}\right)^m \int_0^s (s-t)^{m-\nu-1} f(t)\,dt \tag{1.2}$$

where $m = [\nu] + 1$. In addition, we stipulate

$$D^0 f := f =: I^0 f, \quad I^{-\nu} f := D^\nu f \text{ if } \nu > 0, \quad D^{-\nu} f := I^\nu f \text{ if } 0 < \nu \le 1. \tag{1.3}$$

If ν is a positive integer, then $D^\nu f = (d/ds)^\nu f$. Let us remark that a somewhat more general definition of the Riemann–Liouville fractional derivative is used in the literature with an anchor point a other that 0: Let $a \in \mathbb{R}$ be fixed, $s \ge a$, and let $f_a(t) = f(a+t)$ be a translation of f. Then set

$$D_a^\nu f(s) := D^\nu f_a(s-a).$$

All our results stated for the fractional derivative defined by (1.2) have an interpretation for the fractional derivative with a general anchor point.

Let $\nu > 0$ and $m = [\nu] + 1$. We define the space $I^\nu(L(0,x))$ as the set of all functions f on $[0,x]$ of the form $f = I^\nu \varphi$ for some $\varphi \in L(0,x)$ (see Definition 1.2.3 in [8, p. 43]). According to Theorem 1.2.3 in [8, p. 43], the latter characterization is equivalent to the condition

$$I^{m-\nu} f \in AC^m[0,x], \tag{1.4}$$

$$\left(\frac{d}{ds}\right)^j I^{m-\nu} f(0) = 0, \quad j = 0, 1, \ldots, m-1. \tag{1.5}$$

A function $f \in L(0,x)$ satisfying (1.4) is said to have an *integrable fractional derivative* $D^\nu f$ (see Definition 1.2.4 in [8, p. 44]). We express these conditions in terms of fractional derivatives.

Lemma 1.2. *Let $\nu > 0$ and $m = [\nu] + 1$. A function $f \in L(0,x)$ has an integrable fractional derivative $D^\nu f$ if and only if*

$$D^{\nu-k} f \in C[0,x],\ k = 1, \ldots, m, \quad \textit{and} \quad D^{\nu-1} f \in AC[0,x]. \tag{1.6}$$

Further, $f \in I^{\nu}(L(0,x))$ if and only if f has an integrable fractional derivative $D^{\nu}f$ and satisfies the conditions

$$D^{\nu-k}f(0) = 0 \text{ for } k = 1, \dots, m. \tag{1.7}$$

Proof. Note that

$$\left(\frac{d}{ds}\right)^k I^{m-\nu}f = \left(\frac{d}{ds}\right)^k I^{k-(\nu-m+k)}f = D^{\nu-m+k}f$$

in view of the definition of fractional derivative and the equation $[\nu - m + k] + 1 = k$. Then (1.6) is equivalent to (1.4) and (1.7) is equivalent to (1.5). (For $k = m$ we use the stipulation $D^{\nu-m}f = I^{m-\nu}f$ in (1.6).) □

We will need the following result on the law of indices for fractional integration and differentiation using the unified notation (1.3).

Lemma 1.3. (Theorem 1.2.5 [8, p. 46]) *The law of indices*

$$I^u I^v f = I^{u+v} f \tag{1.8}$$

is valid in the following cases:

(i) $v > 0$, $u + v > 0$ *and* $f \in L(0,x)$;

(ii) $v < 0$, $u > 0$ and $f \in I^{-v}(L(0,x))$;

(iii) $u < 0$, $u + v < 0$ and $f \in I^{-u-v}(L(0,x))$.

Finally we give an integral representation of the fractional derivative $D^{\gamma}f$ (see also [5]).

Theorem 1.4. *Let $\nu > \gamma \geq 0$, let $f \in L(0,x)$ have an integrable fractional derivative $D^{\nu}f \in L^{\infty}(0,x)$, and let $D^{\nu-k}f(0) = 0$ for $k = 1, \dots, [\nu] + 1$. Then*

$$D^{\gamma}f(s) = \frac{1}{\Gamma(\nu - \gamma)} \int_0^s (s-t)^{\nu-\gamma-1} D^{\nu}f(t)\, dt, \quad s \in [0,x]. \tag{1.9}$$

Proof. Set $u = \nu - \gamma > 0$ and $v = -\nu < 0$. According to Lemma 1.2, $f \in I^{-v}(L(0,x))$. Then case (ii) of Lemma 1.3 guarantees that the law of indices holds for this choice of u, v, namely

$$I^{\nu-\gamma}D^{\nu}f = I^u I^v f = I^{u+v}f = I^{-\gamma}f = D^{\gamma}f;$$

this is (1.9). □

2 The main results

Our first result is an Opial type inequality involving fractional derivatives. We assume throughout that x, ν, γ are real numbers, $x > 0$, ν, $\gamma \geq 0$, and that $f \in L(0,x)$. The standard assumption on f is that $f \in I^\nu(L(0,x))$; we prefer to spell this out in the formulation of each theorem by specifying that f has an integrable fractional derivative $D^\nu f$ satisfying (1.6).

Theorem 2.1. *Let $1/p + 1/q = 1$ with p, $q > 1$, let $\gamma \geq 0$, $\nu \geq \gamma + 1 - 1/p$, and let $f \in L(0,x)$ have an integrable fractional derivative $D^\nu f \in L^\infty(0,x)$ such that $D^{\nu-j}f(0) = 0$ for $j = 1, \ldots, [\nu]+1$. Then*

$$\int_0^x |D^\gamma f(s)\, D^\nu f(s)|\, ds \leq \Omega(x) \cdot \left(\int_0^x |D^\nu f(s)|^q\, ds \right)^{2/q}, \tag{2.1}$$

where

$$\Omega(x) = \frac{x^{(rp+2)/p}}{2^{1/q}\Gamma(r+1)((rp+1)(rp+2))^{1/p}}, \quad r = \nu - \gamma - 1. \tag{2.2}$$

Proof. We write $\Phi(t) = |D^\nu f(t)|$ and $r = \nu - \gamma - 1$. Since $1 - 1/p > 0$, we have $\nu > \gamma$, and Theorem 1.4 applies. Further, $r > -1$, and $t \mapsto (s-t)^r \in L(0,s)$ for any $s \in [0,x]$. Let $0 < s \leq x$. Applying Hölder's inequality to (1.9), we get

$$|D^\gamma f(s)| \leq \frac{1}{\Gamma(r+1)} \frac{s^{(rp+1)/p}}{(rp+1)^{1/p}} \left(\int_0^s \Phi(t)^q\, dt \right)^{1/q}. \tag{2.3}$$

Write $z(s) = \int_0^s \Phi(t)^q\, dt$. Then $z'(t) = \Phi(t)^q$ almost everywhere in $(0,s)$,

$$|D^\nu f(s)| = (z'(s))^{1/q} \quad \text{a.e. in } (0,x),$$

and

$$|D^\gamma f(s)\, D^\nu f(s)| \leq \frac{s^{(rp+1)/p}(z(s)z'(s))^{1/q}}{\Gamma(r+1)(rp+1)^{1/p}} \quad \text{a.e. in } (0,x).$$

The function $s^{rp+1}(z(s)z'(s))^{1/q}$ is integrable over $(0,x)$ as $rp+1 \geq 0$ and $z(s)z'(s)$ is measurable and essentially bounded on $(0,x)$. Applying Hölder's inequality, we obtain

$$\begin{aligned} \int_0^x s^{rp+1}(z(s)z'(s))^{1/q}\, ds &\leq \left(\int_0^x s^{rp+1}\, ds \right)^{1/p} \left(\int_0^x z(s)z'(s)\, ds \right)^{1/q} \\ &\leq \frac{x^{(rp+2)/p}}{(rp+2)^{1/p}} \frac{(z(s))^{2/q}}{2^{1/q}}. \end{aligned}$$

The result then follows when we observe that $|D^\gamma f(s)\, D^\nu f(s)|$ is integrable as $D^\gamma f \in AC[0,x]$ and $D^\nu f \in L^\infty(0,x)$. □

The following result deals with the extreme case of the preceding theorem when $p = 1$ and $q = \infty$.

Theorem 2.2. *Let $\nu > \gamma \geq 0$, and let $f \in L(0, x)$ have an integrable fractional derivative $D^\nu f \in L^\infty(0, x)$ such that $D^{\nu-j} f(0) = 0$ for $j = 1, \ldots, [\nu] + 1$. Then*

$$\int_0^x |D^\gamma f(s)\, D^\nu f(s)|\, ds \leq \Omega_1(x) \cdot \operatorname*{ess\,sup}_{s\in[0,x]} |D^\nu f(s)|^2, \tag{2.4}$$

where

$$\Omega_1(x) = \frac{x^{r+2}}{\Gamma(r+3)}, \quad r = \nu - \gamma - 1.$$

Proof. A straightforward application of Theorem 1.4. □

Theorem 2.1 has the following counterpart for the case $0 < p < 1$.

Theorem 2.3. *Let $1/p + 1/q = 1$ with $0 < p < 1$, let $\nu > \gamma \geq 0$, and let $f \in L(0, x)$ have an integrable fractional derivative $D^\nu f \in L^\infty(0, x)$ with $(D^\nu f)^{-1} \in L^\infty(0, x)$ such that $D^{\nu-j} f(0) = 0$ for $j = 1, \ldots, [\nu] + 1$. Then*

$$\int_0^x |D^\gamma f(s)\, D^\nu f(s)|\, ds \geq \Omega(x) \cdot \left(\int_0^x |D^\nu f(s)|^q\, ds \right)^{2/q}, \tag{2.5}$$

where $\Omega(x)$ is defined by (2.2).

Proof. The proof follows a similar pattern as the proof of Theorem 2.1. Since $0 < p < 1$, we need to apply reverse Hölder's inequality [6, p. 135]

$$\int_0^x |u(s)v(s)|\, ds \geq \left(\int_0^x |u(s)|^p \right)^{1/p} \left(\int_0^x |v(s)|^q \right)^{1/q}$$

valid for any $u \in L^p(0, x)$ and $v \in L^q(0, x)$. Secondly, the assumption that $(D^\nu f)^{-1} \in L^\infty(0, x)$ is needed since $q < 0$. The details of the proof are omitted. □

Under slightly strengthened hypotheses of Theorem 2.1 we obtain the following inequality involving fractional derivatives of three orders.

Theorem 2.4. *Let $1/p + 1/q = 1$ with $p, q > 1$, let $\gamma \geq 0$, $\nu \geq \gamma + 2 - 1/p$, and let $f \in L(0, x)$ have an integrable fractional derivative $D^\nu f \in L^\infty(0, x)$ such that $D^{\nu-j} f(0) = 0$ for $j = 1, \ldots, [\nu] + 1$. Then*

$$\int_0^x |D^\gamma f(s)\, D^{\gamma+1} f(s)|\, ds \leq \Omega_2(x) \cdot \left(\int_0^x |D^\nu f(s)|^q\, ds \right)^{2/q}, \tag{2.6}$$

where

$$\Omega_2(x) = \frac{x^{2(rp+1)/p}}{2(\Gamma(r+1))^2 (rp+1)^{2/p}}, \quad r = \nu - \gamma - 1. \tag{2.7}$$

Proof. Write $\Phi(t) = |D^\nu f(t)|$ and $r = \nu - \gamma - 1$. From Theorem 1.4 and from the definition of the fractional integral we obtain

$$|D^\gamma f(x)| \le U(x) := I^{r+1}\Phi(x), \quad |D^{\gamma+1} f(x)| \le I^r\Phi(x).$$

Observing that $U'(x) = (I^{r+1}\Phi(x))' = I^r\Phi(x)$ and using Hölder's inequality, we get

$$\begin{aligned}
\int_0^x |D^\gamma f(t)\, D^{\gamma+1} f(t)|\, dt &\le \int_0^x U(t)U'(t)\, dt = \tfrac{1}{2}U^2(x) \\
&= \frac{1}{2(\Gamma(r+1))^2}\left(\int_0^x (x-t)^r \Phi(t)\, dt\right)^2 \\
&\le \frac{1}{2(\Gamma(r+1))^2}\left(\int_0^x (x-t)^{rp}\, dt\right)^{2/p}\left(\int_0^x \Phi(t)^q\, dt\right)^{2/q} \\
&= \frac{1}{2(\Gamma(r+1))^2}\,\frac{x^{2(rp+1)/p}}{(rp+1)^{2/p}}\left(\int_0^x \Phi(t)^q\, dt\right)^{2/q}. \qquad \square
\end{aligned}$$

The following result is concerned with the case when $p = 1$ an $q = \infty$ in the preceding theorem. The proof is straightforward, and we omit it.

Theorem 2.5. *Let $\gamma \ge 0$, $\nu > \gamma+1$, and let $f \in L(0,x)$ have an integrable fractional derivative $D^\nu f \in L^\infty(0,x)$ such that $D^{\nu-j} f(0) = 0$ for $j = 1, \dots, [\nu]+1$. Then*

$$\int_0^x |D^\gamma f(s)\, D^{\gamma+1} f(s)|\, ds \le \Omega_3(x) \cdot \operatorname*{ess\,sup}_{t\in(0,x)} |D^\nu f(t)|^2, \tag{2.8}$$

where

$$\Omega_3(x) = \frac{x^{2(\nu-\gamma)}}{2(\Gamma(\nu-\gamma+1))^2}. \tag{2.9}$$

Remark 2.6. We show that inequality (2.8) is sharp, attained for the function $f(t) = t^\nu$. From the known properties of the gamma function,

$$\int_0^s (s-t)^{u-1} t^{v-1}\, dt = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}\, s^{u+v-1}, \quad u,\, v > 0.$$

Let $0 \le j \le [\nu]+1$, $m = [\nu]-j+1$ and $\alpha = \nu - [\nu]$. Then $1-\alpha > 0$, $\nu+1 > 0$, and

$$\begin{aligned}
D^{\nu-j} f(s) &= \frac{1}{\Gamma(1-\alpha)}\left(\frac{d}{ds}\right)^m \int_0^s (s-t)^{(1-\alpha)-1} t^{(\nu+1)-1}\, dt \\
&= \frac{1}{\Gamma(1-\alpha)}\,\frac{\Gamma(1-\alpha)\Gamma(\nu+1)}{\Gamma(m+j+1)}\left(\frac{d}{ds}\right)^m s^{m+j} \\
&= \frac{\Gamma(\nu+1)}{j!} s^j.
\end{aligned}$$

Then $D^{\nu-j}f(0)=0$ for $j=1,\ldots,[\nu]+1$, while $D^{\nu}f(s)=\Gamma(\nu+1)$. Using Theorem 1.4, we obtain

$$D^{\gamma}f(s)=\frac{\Gamma(\nu+1)}{\Gamma(\nu-\gamma+1)}\,s^{\nu-\gamma},\quad D^{\gamma+1}f(s)=\frac{\Gamma(\nu+1)}{\Gamma(\nu-\gamma)}\,s^{\nu-\gamma-1}.$$

Hence

$$\begin{aligned}\int_0^x |D^{\gamma}f(s)\,D^{\gamma+1}f(s)|\,ds &= \frac{(\Gamma(\nu+1))^2}{\Gamma(\nu-\gamma+1)\Gamma(\nu-\gamma)}\int_0^x s^{2(\nu-\gamma)-1}\,ds\\ &= \frac{1}{2}\left(\frac{\Gamma(\nu+1)}{\Gamma(\nu-\gamma+1)}\right)^2 x^{2(\nu-\gamma)}.\end{aligned}$$

On the other hand, with $\Omega_3(x)$ given by (2.9) and

$$\operatorname{ess\,sup}|D^{\nu}f(s)|^2=(\Gamma(\nu+1))^2,$$

the left side of (2.8) is equal to the right side of (2.8) for $f(t)=t^{\nu}$.

We give a counterpart of Theorem 2.4 for the case $0<p<1$. The proof again depends on the reverse Hölder's inequality, and is omitted.

Theorem 2.7. *Let $1/p+1/q=1$ with $0<p<1$, let $\gamma\ge 0$, $\nu>\gamma+1$, and let $f\in L(0,x)$ have an integrable fractional derivative $D^{\nu}f\in L^{\infty}(0,x)$ with $(D^{\nu}f)^{-1}\in L^{\infty}(0,x)$ such that $D^{\nu-j}f(0)=0$ for $j=1,\ldots,[\nu]+1$. Then*

$$\int_0^x |D^{\gamma}f(s)\,D^{\gamma+1}f(s)|\,ds \ge \Omega_2(x)\cdot\left(\int_0^x |D^{\nu}f(s)|^q\,ds\right)^{2/q}, \tag{2.10}$$

where $\Omega_2(x)$ is given by (2.7).

We derive yet another useful variant of Opial type inequality.

Theorem 2.8. *Let $1/p+1/q=1$ with $p,\,q>1$, let $\gamma\ge 0$, $\nu\ge\gamma+1-1/p$, and let $f\in L(0,x)$ have an integrable fractional derivative $D^{\nu}f\in L^{\infty}(0,x)$ such that $D^{\nu-j}f(0)=0$ for $j=1,\ldots,[\nu]+1$. Then, for any $m>0$,*

$$\int_0^x |D^{\gamma}f(s)|^m\,ds \le \Omega_4(x)\cdot\left(\int_0^x |D^{\nu}f(s)|^q\,ds\right)^{m/q}, \tag{2.11}$$

where

$$\Omega_4(x)=\frac{x^{(rm+1+m/p)}}{(\Gamma(r+1))^m(rm+1+m/p)(rp+1))^{m/p}},\quad r=\nu-\gamma-1. \tag{2.12}$$

Proof. Inequality (2.3) holds under the hypotheses of the theorem. Raising both sides of (2.3) to the power of m and integrating from 0 to x, we get the result. □

The extreme case of Theorem 2.8 with $p = 1$ and $q = \infty$ follows. The proof is omitted, as it is again a straightforward application of Theorem 1.4.

Theorem 2.9. *Let $\nu > \gamma \geq 0$, and let $f \in L(0,x)$ have an integrable fractional derivative $D^\nu f \in L^\infty(0,x)$ such that $D^{\nu-j}f(0) = 0$ for $j = 1, \ldots, [\nu]+1$. Then, for any $m > 0$,*

$$\int_0^x |D^\gamma f(s)|^m \, ds \leq \Omega_5(x) \cdot \operatorname*{ess\,sup}_{t\in[0,x]} |D^\nu f(t)|^m, \tag{2.13}$$

where

$$\Omega_5(x) = \frac{x^{(r+1)m+1}}{(\Gamma(r))^m (rm+1+m/p)((r+1)m+1)}, \quad r = \nu - \gamma - 1. \tag{2.14}$$

3 Applications

(i) **Uniqueness of solution to fractional initial value problem**

$$\left\{ \begin{array}{l} \text{Let } \gamma_i \geq 0,\ \nu \geq \gamma_i + 1/2,\ i = 1, \ldots, r \in \mathbb{N}. \\ \text{Let } f \in L(0,x) \text{ have an integrable fractional derivative } D^\nu f \in L^\infty(0,x) \\ \text{such that } D^{\nu-j}f(0) = \alpha_j \in \mathbb{R},\ j = 1, \ldots, [\nu]+1. \\ \text{Furthermore, let} \\ D^\nu f(t) = F(t, \{D^{\gamma_i} f(t)\}_{i=1}^r) \text{ for all } t \in [0,x]. \end{array} \right. \tag{3.1}$$

Here $F(t, x_1, \ldots, x_r)$ is continuous for $(x_1, \ldots, x_r) \in \mathbb{R}^r$, bounded for $t \in [0,x]$, and fulfills the Lipschitz condition

$$|F(t, z_1, \ldots, z_r) - F(t, z_1', \ldots, z_r')| \leq \sum_{i=1}^r q_i(t)\, |z_i - z_i'|, \tag{3.2}$$

where $q_i(t) \geq 0$ are bounded on $[0,x]$, $i = 1, \ldots, r$. For $i = 1, \ldots, r$ and $0 \leq s \leq x$ we define

$$\Delta_i(s) := \frac{s^{\nu-\gamma_i}}{2\Gamma(\nu-\gamma_i)\sqrt{(\nu-\gamma_i)(2\nu-2\gamma_i-1)}}, \quad \psi(s) = \sum_{i=1}^r \|q_i\|_\infty \Delta_i(s), \tag{3.3}$$

where $\|q_i\|_\infty = \sup_{t\in[0,x]} |q_i(t)|$. We shall assume that

$$\psi(x) := \sum_{i=1}^r \|q_i\|_\infty \Delta_i(x) < 1. \tag{3.4}$$

Let $g \in L(0,x)$ have an integrable fractional derivative $D^\nu f \in L^\infty(0,x)$ such that $D^{\nu-j}f(0) = 0$, $j = 1,\ldots,[\nu]+1$. Then, by Theorem 1.4, we have

$$D^{\gamma_i}g(s) = \frac{1}{\Gamma(\nu-\gamma_i)} \int_0^s (s-t)^{\nu-\gamma_i-1} D^\nu g(t)\,dt, \quad s \in [0,x],\ i = 1,\ldots,r. \tag{3.5}$$

When $p = q = 2$, from (2.1) we get for $i = 1,\ldots,r$,

$$\int_0^x |(D^{\gamma_i}g)(w)|\,|(D^\nu g)(w)|\,dw \le \Delta_i(x) \int_0^x |(D^\nu g)(w)|^2\,dw. \tag{3.6}$$

Let f_1, f_2 solve (3.1), that is, let for $k = 1,2$,

$$(D^\nu f_k)(t) = F(t, \{(D^{\gamma_i}f_k)(t)\}_{i=1}^r), \quad t \in [0,x],$$

and

$$D^{\nu-j}f_k(0) = \alpha_j \in \mathbb{R}, \quad j = 1,\ldots[\nu]+1.$$

If $g := f_1 - f_2$, then

$$D^\nu f(t) = F(t, \{(D^{\gamma_i}f_1)(t)\}_{i=1}^r) - F(t, \{(D^{\gamma_i}f_2)(t)\}_{i=1}^r), \tag{3.7}$$

and

$$D^{\nu-j}g(0) = 0, \quad j = 1,\ldots,[\nu]+1.$$

By (3.2),

$$\begin{aligned} |F(t, D^{\gamma_1}f_1(t),\ldots,D^{\gamma_r}f_1(t)) &- F(t, D^{\gamma_1}f_2(t),\ldots,D^{\gamma_r}f_2(t))| \\ &\le \sum_{i=1}^r q_i(t)|D^{\gamma_i}f_1(t) - D^{\gamma_i}f_2(t)| \\ &= \sum_{i=1}^r q_i(t)|D^{\gamma_i}g(t)| \\ &\le \sum_{i=1}^r \|q_i\|_\infty |D^{\gamma_i}g(t)|. \end{aligned} \tag{3.8}$$

Thus

$$\begin{aligned} |D^\nu g(t)|^2 &= |D^\nu g(t)|\,|F(t, \{D^{\gamma_i}f_1(t)\}_{i=1}^r) - F(t, \{D^{\gamma_i}f_2(t)\}_{i=1}^r)| \\ &\le |D^\nu g(t)| \sum_{i=1}^r \|q_i\|_\infty |D^{\gamma_i}g(t)| \\ &= \sum_{i=1}^r \|q_i\|_\infty |D^{\gamma_i}g(t)|\,|D^\nu g(t)|. \end{aligned}$$

Consequently,

$$\begin{aligned}\int_0^x |D^\nu g(t)|^2\,dt &\le \sum_{i=1}^r \|q_i\|_\infty \int_0^x |D^{\gamma_i} g(t)|\,|D^\nu g(t)|\,dt\\ &\overset{(3.6)}{\le} \sum_{i=1}^r \|q_i\|_\infty \Delta_i(x) \int_0^x |D^\nu g(t)|^2\,dt\\ &\overset{(3.4)}{\le} \psi(x)\int_0^x |D^\nu g(t)|^2\,dt,\end{aligned}$$

that is,

$$\int_0^x |D^\nu g(t)|^2\,dt \le \psi(x)\int_0^x |D^\nu g(t)|^2\,dt. \tag{3.9}$$

If $\int_0^x |D^\nu g(t)|^2\,dt \ne 0$, then by (3.9) we obtain $\psi(x) \ge 1$, a contradiction to the assumption $\psi(x) < 1$. Hence $\int_0^x |D^\nu g(t)|^2\,dt = 0$, that is, $D^\nu g(t) = 0$ *a. e.* in $[0, x]$. But $D^{\nu-j} g(0) = 0$, $j = 1, \dots, [\nu] + 1$. Then from (1.9) for $\gamma = 0$ we find $g(t) \equiv 0$ in $[0, x]$. This implies $f_1 = f_2$ on $[0, x]$, proving the uniqueness of solution to the initial value problem (3.1).

(ii) **Upper bounds on $D^\nu f$, solution f, etc.**

$$\left\{\begin{array}{l}\text{Consider the initial value problem for } 0 \le t \le x\text{:}\\ (D^\nu f)'(t) = F(t, \{D^{\gamma_i} f(t)\}_{i=1}^r, D^\nu f);\\ \gamma_i \ge 0,\ \nu \ge \gamma_i + 1/2,\ i = 1, \dots, r \in \mathbb{N};\\ \text{here } f \in L(0,x) \text{ has an integrable fractional derivative } D^\nu f \in L^\infty(0,x),\\ \text{we assume that } D^{\nu-j} f(0) = 0,\ j = 1, \dots, [\nu]+1 \text{ and } D^\nu f(0) = A \in \mathbb{R}.\end{array}\right. \tag{3.10}$$

Here F is Lebesgue measurable on $[0, x] \times \mathbb{R}^{r+1}$, and fulfills the condition

$$|F(t, x_1, \dots, x_r, x_{r+1})| \le \sum_{i=1}^r q_i(t)|x_i|, \tag{3.11}$$

where $q_i(t) \ge 0$ are bounded on $[0, x]$, $i = 1, \dots, r$. We see that

$$D^\nu f(t)(D^\nu f)'(t) = D^\nu f(t) F(t, \{D^{\gamma_i} f(t)\}_{i=1}^r, D^\nu f(t)),$$

and for $0 \le s \le x$ we have

$$\int_0^s D^\nu f(t)(D^\nu f)'(t)\,dt = \int_0^s D^\nu f(t) F(t, \{D^{\gamma_i} f(t)\}_{i=1}^r, D^\nu f(t))\,dt.$$

Hence

$$\begin{aligned}
\tfrac{1}{2}|D^\nu f(t)|^2\Big|_0^s &\le \int_0^s |D^\nu f(t)||F(t,\{D^{\gamma_i}f(t)\}_{i=1}^r, D^\nu f(t))|\,dt \\
&\overset{(3.11)}{\le} \int_0^s |D^\nu f(t)|\left(\sum_{i=1}^r \|q_i\|_\infty |D^{\gamma_i}f(t)|\right)dt \\
&= \sum_{i=1}^r \|q_i\|_\infty \left(\int_0^s |D^{\gamma_i}f(t)|\,|D^\nu f(t)|\,dt\right).
\end{aligned}$$

Recall the notation (3.3) for $\Delta_i(s)$ and $\psi(s)$. Then

$$\begin{aligned}
|D^\nu f(s)|^2 &\le A^2 + 2\sum_{i=1}^r \|q_i\|_\infty \left(\int_0^s |D^{\gamma_i}f(t)|\,|D^\nu f(t)|\,dt\right) \\
&\overset{(2.1)}{\le} A^2 + \left(\sum_{i=1}^r \|q_i\|_\infty \Delta_i(s)\right)\left(\int_0^s |D^\nu f(t)|^2\,dt\right) \\
&= A^2 + \psi(s)\int_0^s |D^\nu f(t)|^2\,dt,
\end{aligned}$$

that is,

$$|D^\nu f(s)|^2 \le A^2 + \psi(s)\int_0^s |D^\nu f(t)|^2\,dt. \tag{3.12}$$

Set $\theta(s) := |D^\nu f(s)|^2$ for $0 \le s \le x$ and $\rho := A^2$. Then

$$\theta(s) \le \rho + \psi(s)\int_0^s \theta(t)\,dt,$$

where $\rho \ge 0$, $\psi(s) \ge 0$, $\psi(0) = 0$, $\theta(s) \ge 0$ for all $0 \le s \le x$. We can apply the generalized Gronwall lemma [4, Corollary 1.1.2] (with $H(x) = x$) to obtain

$$\theta(s) \le \rho\left(1 + \psi(s)\exp(\Psi(s))\int_0^s \exp(-\Psi(t))\,dt\right), \quad \Psi(t) = \int_0^t \psi(u)\,du. \tag{3.13}$$

We have shown that

$$|D^\nu f(s)| \le |A|\left(1 + \psi(s)\exp(\Psi(s))\int_0^s \exp(-\Psi(t))\,dt\right)^{1/2} =: K(s) \tag{3.14}$$

for all $0 \le s \le x$. From (1.9) with $\gamma = 0$ we get

$$|f(s)| \le \frac{1}{\Gamma(\nu)}\int_0^s (s-t)^{\nu-1}|D^\nu f(t)|\,dt$$

for all $0 \le s \le x$. Applying (3.14), we get

$$|f(s)| \le \frac{1}{\Gamma(\nu)}\int_0^s (s-t)^{\nu-1}K(t)\,dt. \tag{3.15}$$

Also from (1.9) we have

$$|D^{\gamma_i} f(s)| \le \frac{1}{\Gamma(\nu-\gamma_i)} \int_0^s (s-t)^{\nu-\gamma_i-1} |D^\nu f(t)|\, dt$$

for all $0 \le s \le x$, $i = 1, \ldots, r$. Finally, by (3.14) we get

$$|D^{\gamma_i} f(s)| \le \frac{1}{\Gamma(\nu-\gamma_i)} \int_0^s (s-t)^{\nu-\gamma_i-1} K(t)\, dt \tag{3.16}$$

for all $0 \le s \le x$ and $i = 1, \ldots, r$.

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