Opial type L^p-inequalities for fractional derivatives

G. A. Anastassiou, J. J. Koliha and J. Pečarić*

Abstract

This paper presents a class of L^p type Opial inequalities for generalized fractional derivatives for integrable functions based on the results obtained earlier by the first author for continuous functions (*Acta Appl. Math.* **54** (1998), 303–317)). The novelty of our approach is the use of the index law for fractional derivatives in lieu of Taylor's formula, which enables us to relax restrictions on the orders of fractional derivatives.

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1 Introduction and preliminaries

The Opial inequality, which appeared in [9], is of great interest in differential equations and other areas of mathematics, and has attracted a great deal of attention in the recent literature. For classical derivatives it has been generalized in several directions (see, for instance, [2, 3, 10]), and was a subject of a monograph by Agarwal and Pang [4]. Love gave a generalization for fractional integrals [7]. The present paper takes its inspiration in an earlier paper [1] by Anastassiou. In the present work we consider Lebesgue integrable functions, whereas [1] dealt with continuous functions using a different definition of fractional derivative.

Our brief survey of basic facts about fractional derivatives is based on the monograph [11] by Samko et al.; most of the results needed in the sequel are contained in Chapter 1 of [11]. The crucial result is Theorem 1.4, which replaces Taylor's formula in the derivation of various estimates.

Throughout the paper, x denotes a fixed positive number. By $C^m[0,x]$ we denote the space of all functions on $[0,x]$ which have continuous derivatives up to order m, and $AC[0,x]$ is the space of all absolutely continuous functions on $[0,x]$. By $AC^m[0,x]$

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we denote the space of all functions $g \in C^m[0,x]$ with $g^{(m-1)} \in AC[0,x]$. For any $\alpha \in \mathbb{R}$ we denote by $[\alpha]$ the integral part of α (the integer k satisfying $k \le \alpha < k+1$). If $p \in \mathbb{R}$, $p > 0$, by $L^p(0,x)$ we denote the space of all Lebesgue measurable functions f for which $|f|^p$ is Lebesgue integrable on the interval $(0,x)$, and by $L^\infty(0,x)$ the set of all functions measurable and essentially bounded on $(0,x)$. For any $f \in L^\infty(0,x)$ we write $\|f\|_\infty = \operatorname{ess\,sup}_{t\in[0,x]} |f(t)|$. We also write $L(0,x) = L^1(0,x)$. We observe that $L^\infty(0,x) \subset L^p(0,x)$ for all $p > 0$.

For any $a \in \mathbb{R}$ we write $a_+ = \max(a,0)$ and $a_- = (-a)_+$.

For the sake of completeness we give a proof of the following known result which provides a basis for the existence of fractional integrals and is needed in another context in the paper.

Lemma 1.1. *Let $f \in L(0,x)$ and let $\alpha > -1$ be a real number. Then*

$$F(s) = \int_0^s (s-t)^\alpha f(t)\,dt$$

exists for almost all $s \in [0,x]$, and $F \in L(0,x)$.

Proof. Define $k : \Omega := [0,x] \times [0,x] \to \mathbb{R}$ by $k(s,t) = (s-t)_+^\alpha$, that is,

$$k(s,t) = \begin{cases} (s-t)^\alpha & \text{if } 0 \le t < s \le x, \\ 0 & \text{if } 0 \le s \le t \le x. \end{cases}$$

Then k is measurable on Ω, and

$$\int_0^x k(s,t)\,ds = \int_0^t k(s,t)\,ds + \int_t^x k(s,t)\,ds = \int_t^x (s-t)^\alpha\,ds = (\alpha+1)^{-1}(x-t)^{\alpha+1}.$$

Since the repeated integral

$$\int_0^x dt \int_0^x k(s,t)|f(t)|\,ds = (\alpha+1)^{-1} \int_0^x (x-t)^{\alpha+1}|f(t)|\,dt$$

exists and is finite, the function $(s,t) \mapsto k(s,t)f(t)$ is integrable over Ω by Tonelli's theorem, and the conclusion follows from Fubini's theorem. □

Let $\alpha > 0$. For any $f \in L(0,x)$ the *Riemann–Liouville fractional integral of f* of order α is defined by

$$I^\alpha f(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} f(t)\,dt, \quad s \in [0,x]. \tag{1.1}$$

By Lemma 1.1 the integral on the right side of (1.1) exists for almost all $s \in [0,x]$, and $I^\alpha f \in L(0,x)$. The *Riemann–Liouville fractional derivative of* $f \in L(0,x)$ of order α is defined by

$$D^\alpha f(s) = \left(\frac{d}{ds}\right)^m I^{m-\alpha} f(s) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{ds}\right)^m \int_0^s (s-t)^{m-\alpha-1} f(t)\, dt \tag{1.2}$$

where $m = [\alpha] + 1$, provided that the derivative exists. In addition, we stipulate

$$D^0 f := f =: I^0 f, \quad I^{-\alpha} f := D^\alpha f \text{ if } \alpha > 0, \quad D^{-\alpha} f := I^\alpha f \text{ if } 0 < \alpha \le 1. \tag{1.3}$$

If α is a positive integer, then $D^\alpha f = (d/ds)^\alpha f$.

A more general definition of fractional integrals and derivatives uses an anchor point other than 0: Let $f \in L(a,b)$, where $-\infty < a < b < \infty$. For any $s \in [a,b]$ set

$$I^\alpha_{a+} f(s) := \frac{1}{\Gamma(\alpha)} \int_a^s (s-t)^{\alpha-1} f(t)\, dt, \quad I^\alpha_{b-} f(s) := \frac{1}{\Gamma(\alpha)} \int_s^b (s-t)^{\alpha-1} f(t)\, dt.$$

The two fractional derivatives are then defined by an obvious modification of (1.2). All our results stated for the specialized fractional derivative (1.2) have an interpretation for the fractional derivatives with a general anchor point.

Let $\alpha > 0$ and $m = [\alpha] + 1$. A function $f \in L(0,x)$ is said to have an *integrable fractional derivative* $D^\alpha f$ (see Definition 2.4 in [11, p. 44]) if

$$I^{m-\alpha} f \in AC^m[0,x]. \tag{1.4}$$

We define the space $I^\alpha(L(0,x))$ as the set of all functions f on $[0,x]$ of the form $f = I^\alpha \varphi$ for some $\varphi \in L(0,x)$ (see Definition 2.3 in [11, p. 43]). We express these conditions in terms of fractional derivatives.

Lemma 1.2. *Let* $\alpha > 0$ *and* $m = [\alpha] + 1$. *A function* $f \in L(0,x)$ *has an integrable fractional derivative* $D^\alpha f$ *if and only if*

$$D^{\alpha-k} f \in C[0,x],\ k = 1, \dots, m, \quad \text{and} \quad D^{\alpha-1} f \in AC[0,x]. \tag{1.5}$$

Further, $f \in I^\alpha(L(0,x))$ *if and only if* f *has an integrable fractional derivative* $D^\alpha f$ *and satisfies the conditions*

$$D^{\alpha-k} f(0) = 0 \text{ for } k = 1, \dots, m. \tag{1.6}$$

Proof. Note that

$$\left(\frac{d}{ds}\right)^k I^{m-\alpha} f = \left(\frac{d}{ds}\right)^k I^{k-(\alpha-m+k)} f = D^{\alpha-m+k} f$$

in view of the definition of fractional derivative and the equation $[\alpha - m + k] + 1 = k$. Then (1.5) is equivalent to (1.4) and (1.6) is equivalent to condition (2.56) in [11, p. 43]. (For $k = m$ we use the stipulation $D^{\alpha-m} f = I^{m-\alpha} f$ in (1.5).) □

We will need the following result on the law of indices for fractional integration and differentiation using the unified notation (1.3).

Lemma 1.3. (Theorem 2.5 in [11, p. 46]) *The law of indices*

$$I^{\mu}I^{\nu}f = I^{\mu+\nu}f \tag{1.7}$$

is valid in the following cases:

(i) $\nu > 0$, $\mu + \nu > 0$ *and* $f \in L(0, x)$;

(ii) $\nu < 0$, $\mu > 0$ and $f \in I^{-\nu}(L(0, x))$;

(iii) $\mu < 0$, $\mu + \nu < 0$ and $f \in I^{-\mu-\nu}(L(0, x))$.

The following theorem is a powerful analogue of Taylor's formula with vanishing fractional derivatives of lower orders. In this paper it is used as the main tool for deriving inequalities. Observe that we do not require $\alpha \ge \beta + 1$ but merely $\alpha > \beta$.

Theorem 1.4. *Let* $\alpha > \beta \ge 0$, *let* $f \in L(0, x)$ *have an integrable fractional derivative* $D^{\alpha}f$, *and let* $D^{\alpha-k}f(0) = 0$ *for* $k = 1, \dots, [\alpha] + 1$. *Then*

$$D^{\beta}f(s) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^s (s-t)^{\alpha-\beta-1} D^{\alpha}f(t)\, dt, \quad s \in [0, x]. \tag{1.8}$$

Proof. Set $\mu = \alpha - \beta > 0$ and $\nu = -\alpha < 0$. According to Lemma 1.2, $f \in I^{-\nu}(L(0, x))$. Then case (ii) of Lemma 1.3 guarantees that the law of indices holds for this choice of μ, ν, namely

$$I^{\alpha-\beta}D^{\alpha}f = I^{\mu}I^{\nu}f = I^{\mu+\nu}f = I^{-\beta}f = D^{\beta}f;$$

this proves the result. Note that the existence of the integral on the right in (1.8) is guaranteed by Lemma 1.1. ☐

2 Main results

We assume throughout that x, ν are positive real numbers, and that $f \in L(0, x)$. The standard assumption on f is that $f \in I^{\nu}(L(0, x))$; this is equivalent to f having an integrable fractional derivative $D^{\nu}f$ satisfying (1.5). In addition we require that $D^{\nu}f$ is essentially bounded to guarantee that $D^{\nu}f \in L^p(0, x)$ for $p > 0$. We tabulate the notation used in this section. The inequalities between ν and μ_i are assumed throughout.

l	a positive integer
x, ν, r_i	positive real numbers, $i = 1, \dots, l$
r	$r = \sum_{i=1}^{l} r_i$
μ_i	real numbers satisfying $0 \le \mu_i < \nu$, $i = 1, \dots, l$
α_i	$\alpha_i = \nu - \mu_i - 1$, $i = 1, \dots, l$
α	$\alpha = \max\{(\alpha_i)_- : i = 1, \dots, l\}$
β	$\beta = \max\{(\alpha_i)_+ : i = 1, \dots, l\}$
ω_1, ω_2	continuous positive weight functions on $[0, x]$
ω	continuous nonnegative weight function on $[0, x]$
s_k, s_k'	$s_k > 0$ and $1/s_k + 1/s_k' = 1$, $k = 1, 2$

For brevity we write $\boldsymbol{\mu} = (\mu_1, \dots, \mu_l)$ for a selection of the orders μ_i of fractional derivatives, and $\boldsymbol{r} = (r_1, \dots, r_l)$ for a selection of the constants r_i.

We derive a very general Opial type inequality involving fractional derivatives of an integrable function f, which is analogous to [10, Theorem 1.3] for ordinary derivatives and to [1, Theorem 2] for fractional derivatives.

Theorem 2.1. *Let $f \in L(0, x)$ have an integrable fractional derivative $D^\nu f \in L^\infty(0, x)$ such that $D^{\nu-j} f(0) = 0$ for $j = 1, \dots, [\nu] + 1$. For $k = 1, 2$, let $s_k > 1$, let $p \in \mathbb{R}$ satisfy*

$$\alpha s_2 < 1, \qquad p > \frac{s_2}{1 - \alpha s_2}, \tag{2.1}$$

and let $\sigma = 1/s_2 - 1/p$. Finally let

$$Q_1 = \left(\int_0^x \omega_1(\tau)^{s_1'} \, d\tau \right)^{1/s_1'} \quad \text{and} \quad Q_2 = \left(\int_0^x \omega_2(\tau)^{-s_2'/p} \, d\tau \right)^{r/s_2'}. \tag{2.2}$$

Then

$$\int_0^x \omega_1(\tau) \prod_{i=1}^{l} |D^{\mu_i} f(\tau)|^{r_i} \, d\tau \le Q_1 Q_2 \, C_1 x^{\rho + 1/s_1} \left(\int_0^x \omega_2(\tau) |D^\nu f(\tau)|^p \, d\tau \right)^{r/p}, \tag{2.3}$$

where $\rho := \sum_{i=1}^{l} \alpha_i r_i + \sigma r$ and

$$C_1 = C_1(\nu, \boldsymbol{\mu}, \boldsymbol{r}, p, s_1, s_2) := \frac{\sigma^{r\sigma}}{\prod_{i=1}^{l} \Gamma(\nu - \mu_i)^{r_i} (\alpha_i + \sigma)^{r_i \sigma} (\rho s_1 + 1)^{1/s_1}}. \tag{2.4}$$

Proof. First we show that the conditions on s_2 and p guarantee that, for $i = 1, \dots l$,

$$\text{(a) } p > s_2 > 1, \quad \text{(b) } \alpha_i s_2 > -1, \quad \text{(c) } \alpha_i + \sigma > 0. \tag{2.5}$$

This is clear if $\alpha = 0$. If $\alpha > 0$, then $0 < 1 - \alpha s_2 < 1$ and $p > s_2/(1 - \alpha s_2) > s_2 > 1$. For each $i \in \{1, \dots, l\}$, $\alpha_i \ge -\alpha$, and $\alpha_i s_2 \ge -\alpha s_2 > -1$; further

$$\alpha_i + \sigma = \alpha_i + \frac{1}{s_2} - \frac{1}{p} = \frac{1 + \alpha_i s_2}{s_2} - \frac{1}{p} \ge \frac{1 - \alpha s_2}{s_2} - \frac{1}{p} > 0.$$

For brevity we write

$$k_i(\tau,t)=(\tau-t)_+^{\alpha_i},\ i=1,\dots,l,\qquad \Phi(t)=|D^\nu f(t)|,\quad 0\le\tau,\,t\le x.$$

From (2.5) it follows that

$$k_i(\tau,\cdot)\in L^{s_2}(0,x)\quad\text{and}\quad k_i(\tau,\cdot)\in L^{1/\sigma}(0,x). \tag{2.6}$$

Let $i\in\{1,\dots,l\}$ and $\tau\in[0,x]$. We then apply Hölder's inequality twice (with the conjugate indices s_2', s_2 and p/s_2, $p/(p-s_2)$) taking into account (2.6) and the fact that ω_2^{-1}, ω_2 and Φ are (essentially) bounded:

$$\begin{aligned}\int_0^x k_i(\tau,t)\Phi(t)\,dt&=\int_0^x \omega_2(t)^{-1/p}\omega_2(t)^{1/p}\Phi(t)k_i(\tau,t)\,dt\\&\le\left(\int_0^x\omega_2(t)^{-s_2'/p}\,dt\right)^{1/s_2'}\left(\int_0^x\omega_2(t)^{s_2/p}\Phi(t)^{s_2}k_i(\tau,t)^{s_2}\,dt\right)^{1/s_2}\\&\le Q_2^{1/r}\left(\int_0^x\omega_2(t)\Phi(t)^p\,dt\right)^{1/p}\left(\int_0^x k_i(\tau,t)^{1/\sigma}\,dt\right)^{\sigma}\\&=Q_2^{1/r}\left(\int_0^x\omega_2(t)\Phi(t)^p\,dt\right)^{1/p}\frac{\sigma^\sigma\tau^{\alpha_i+\sigma}}{(\alpha_i+\sigma)^\sigma}.\end{aligned}$$

By Theorem 1.4,

$$\Gamma(\nu-\mu_i)|D^{\mu_i}f(\tau)|\le\int_0^\tau(\tau-t)^{\alpha_i}\Phi(t)\,dt=\int_0^x k_i(\tau,t)\Phi(t)\,dt. \tag{2.7}$$

Therefore

$$\begin{aligned}&\int_0^x\omega_1(\tau)\prod_{i=1}^l|D^{\mu_i}f(\tau)|^{r_i}\,d\tau\\&\quad\le\int_0^x\omega_1(\tau)\prod_{i=1}^l\frac{1}{\Gamma(\nu-\mu_i)^{r_i}}\left(\int_0^x k_i(\tau,t)\Phi(t)\,dt\right)^{r_i}d\tau\\&\quad\le\int_0^x\omega_1(\tau)\prod_{i=1}^l\frac{1}{\Gamma(\nu-\mu_i)^{r_i}}\,Q_2^{r_i/r}\left(\int_0^x\omega_2(t)\Phi(t)^p\,dt\right)^{r_i/p}\\&\qquad\cdot\frac{\sigma^{r_i\sigma}}{(\alpha_i+\sigma)^{r_i\sigma}}\,\tau^{(\alpha_i+\sigma)r_i}\,d\tau\\&\quad=\frac{\sigma^{r\sigma}}{\prod_{i=1}^l\Gamma(\nu-\mu_i)^{r_i}(\alpha_i+\sigma)^{r_i\sigma}}\,Q_2\left(\int_0^x\omega_2(t)\Phi(t)^p\,dt\right)^{r/p}\\&\qquad\cdot\int_0^x\omega_1(\tau)\left(\prod_{i=1}^l\tau^{(\alpha_i+\sigma)r_i}\right)d\tau\end{aligned}$$

$$
\begin{aligned}
&= \Delta Q_2 \left(\int_0^x \omega_2(t)\Phi(t)^p \, dt \right)^{r/p} \int_0^x \omega_1(\tau)\tau^\rho \, d\tau \\
&\le \Delta Q_2 \left(\int_0^x \omega_2(t)\Phi(t)^p \, dt \right)^{r/p} \left(\int_0^x \omega_1(\tau)^{s_1'} \, d\tau \right)^{1/s_1'} \left(\int_0^x \tau^{\rho s_1} \, d\tau \right)^{1/s_1} \\
&= \frac{\Delta}{(\rho s_1 + 1)^{1/s_1}} Q_2 \left(\int_0^x \omega_2(t)\Phi(t)^p \, dt \right)^{r/p} Q_1 \, x^{\rho + 1/s_1}
\end{aligned}
$$

where $\Delta := \sigma^{r\sigma}/(\prod_{i=1}^l \Gamma(\nu-\mu_i)^{r_i}(\alpha_i+\sigma)^{r_i\sigma})$. This completes the proof. ☐

Next we consider the extreme case $p=\infty$ in analogy with [1, Proposition 1].

Theorem 2.2. *Let $f \in L(0,x)$ have an integrable fractional derivative $D^\nu f \in L^\infty(0,x)$ such that $D^{\nu-j}f(0)=0$ for $j=1,\dots,[\nu]+1$. Then*

$$
\int_0^x \omega(\tau) \prod_{i=1}^l |D^{\mu_i} f(\tau)|^{r_i} \, d\tau \le \frac{\|\omega\|_\infty \, x^\rho}{\rho \prod_{i=1}^l \Gamma(\nu-\mu_i+1)^{r_i}} \, \|D^\nu f\|_\infty^r, \tag{2.8}
$$

where $\rho = \sum_{i=1}^l (\nu-\mu_i)r_i + 1$.

Proof. By Theorem 1.4,

$$
|D^{\mu_i} f(\tau)| \le \frac{1}{\Gamma(\nu-\mu_i)} \int_0^\tau (\tau-t)^{\alpha_i} |D^\nu f(t)| \, dt,
$$

which implies

$$
|D^{\mu_i} f(\tau)| \le \frac{\|D^\nu f\|_\infty}{\Gamma(\nu-\mu_i)} \, \frac{\tau^{\nu-\mu_i}}{\nu-\mu_i} = \frac{\|D^\nu f\|_\infty \tau^{\nu-\mu_i}}{\Gamma(\nu-\mu_i+1)}. \tag{2.9}
$$

The result then follows when we raise (2.9) to the power r_i, take the product from $i=1$ to l, multiply by $\omega(\tau)$, and integrate with respect to τ from 0 to x. ☐

We have the following counterpart of Theorem 2.1 with s_1, $s_2 \in (0,1)$ and p negative.

Theorem 2.3. *Let $f \in L(0,x)$ have an integrable fractional derivative $D^\nu f \in L^\infty(0,x)$ which is of the same sign a. e. in $(0,x)$ and satisfies $D^{\nu-j}f(0)=0$, $j=1,\dots,[\nu]+1$. For $k=1,2$, let $0<s_k<1$, let $p<0$ and let $\sigma = 1/s_2 - 1/p$. Then*

$$
\int_0^x \omega_1(\tau) \prod_{i=1}^l |D^{\mu_i} f(\tau)|^{r_i} \, d\tau \ge Q_1 Q_2 C_1 x^{\rho+1/s_1} \left(\int_0^x \omega_2(\tau) |D^\nu f(\tau)|^p \, d\tau \right)^{r/p}, \tag{2.10}
$$

where $\rho = \sum_{i=1}^l \alpha_i r_i + \sigma r$, Q_1 and Q_2 are defined by (2.2), *and C_1 is defined by* (2.4).

Proof. Combining Theorem 1.4 with the hypotheses on $D^\nu f$, we have

$$\Gamma(\nu-\mu_i)|D^{\mu_i}f(\tau)| = \int_0^\tau (\tau-t)^{\alpha_i}\Phi(t)\,dt = \int_0^x k_i(\tau,t)\Phi(t)\,dt, \tag{2.11}$$

where $\Phi(t)=D^\nu f$ or $\Phi(t)=-D^\nu f$ (depending on the sign of $D^\nu f$ in $(0,x)$).

Since $\alpha_i>-1$ and $0<s_2<1$, we have $\alpha_i s_2>-1$. Further, $\sigma=1/s_2-1/p>0$. Writing $k_i(\tau,t)=(\tau-t)_+^{\alpha_i}$ $(i=1,\dots,l)$, we have

$$k_i(\tau,\cdot)\in L^{s_2}(0,x) \quad \text{and} \quad k_i(\tau,\cdot)\in L^{1/\sigma}(0,x). \tag{2.12}$$

We can now retrace the proof of Theorem 2.1, relying on (2.12) and using the reverse Hölder's inequality in place of Hölder's inequality proper (as $0<s_k<1$ for $k=1,2$ and $p<0$). □

A possible choice of p in this theorem is $p=(s_1s_2^2)/(s_1s_2-1)$. This results in an inequality similar to the one obtained earlier by Anastassiou [1, Theorem 3].

We obtain yet another counterpart of Theorem 2.1 if we assume that s_1, s_2 and p lie in the interval $(0,1)$. In this case the hypotheses on s_1, s_2 and p are of necessity more restrictive.

Theorem 2.4. *Let $f\in L(0,x)$ have an integrable fractional derivative $D^\nu f\in L^\infty(0,x)$ which is of the same sign a. e. in $(0,x)$ and satisfies $D^{\nu-j}f(0)=0$, $j=1,\dots,[\nu]+1$. For $k=1,2$, let $0<s_k<1$, let $rs_1\le 1$, $p\in\mathbb{R}$,*

$$\frac{s_2}{1-\alpha s_2+s_2}<p<\frac{s_2}{1+\beta s_2}, \tag{2.13}$$

and let $\sigma=1/s_2-1/p$. Then the inequality (2.10) holds where $\rho=\sum_{i=1}^l\alpha_i r_i+\sigma r$, Q_1 and Q_2 are defined by (2.2), and C_1 is defined by (2.4).

Proof. We show that condition (2.13) guarantees that, for $i=1,\dots l$,

$$\text{(a) } 0<p<s_2<1, \qquad \text{(b) } -1<\alpha_i+\sigma<0. \tag{2.14}$$

Since $1-\alpha s_2+s_2>0$ and $1+\beta s_2\ge 1$, inequality (2.14) (a) follows directly from (2.13). Further, we have $\alpha_i+\sigma=(1+\alpha_i s_2)/s_2-1/p$, and

$$-1<\frac{1-\alpha s_2}{s_2}-\frac{1}{p}\le\frac{1+\alpha_i s_2}{s_2}-\frac{1}{p}<\frac{1+\beta s_2}{s_2}-\frac{1}{p}<0.$$

This proves (2.14) (b).

Since $\alpha_i>-1$ and $0<s_2<1$, we have $\alpha_i s_2>-1$. Further, $\sigma<0$, and $\alpha_i/\sigma>-1$. Writing $k_i(\tau,t)=(\tau-t)_+^{\alpha_i}$ $(i=1,\dots,l)$, we have

$$k_i(\tau,\cdot)\in L^{s_2}(0,x) \quad \text{and} \quad k_i(\tau,\cdot)\in L^{1/\sigma}(0,x). \tag{2.15}$$

As in the proof of the preceding theorem we have

$$\Gamma(\nu-\mu_i)|D^{\mu_i}f(\tau)| = \int_0^\tau (\tau-t)^{\alpha_i}\Phi(t)\,dt = \int_0^x k_i(\tau,t)\Phi(t)\,dt, \tag{2.16}$$

where $\Phi(t) = D^\nu f$ or $\Phi(t) = -D^\nu f$ (depending on the sign of $D^\nu f$ in $(0,x)$).

We can now retrace the proof of Theorem 2.1, relying on (2.15) and using the reverse Hölder's inequality in place of Hölder's inequality proper (as $0 < s_k < 1$ for $k = 1, 2$ and $0 < p < 1$). For the last application of Hölder's inequality we need $\tau^\rho \in L^{s_1}(0,x)$. This follows from

$$\rho s_1 = \sum_{i=1}^{l}(\alpha_i+\sigma)r_i s_1 > -rs_1 \geq -1$$

taking into account the assumption $rs_1 \leq 1$. ☐

We present a version of Opial's inequality with $l = 2$ motivated by Pang and Agarwal's extension [10, Theorem 1.1] of an inequality due to Fink [5] for classical derivatives. This was further extended in [1, Theorem 4] to fractional derivatives. Our proof is similar to the one given in [10]. In view of the auxiliary inequalities used, in particular of (2.19), the theorem does not extend easily to $l > 2$.

Theorem 2.5. *Let $f \in L(0,x)$ have an integrable fractional derivative $D^\nu f \in L^\infty(0,x)$ such that $D^{\nu-j}f(0) = 0$ for $j = 1,\dots,[\nu]+1$. Let $\nu > \mu_2 \geq \mu_1 + 1 \geq 1$. If p, $q > 1$ are such that $1/p + 1/q = 1$, then*

$$\int_0^x |D^{\mu_1}f(\tau)||D^{\mu_2}f(\tau)|\,d\tau \leq C_2 x^{2\nu-\mu_1-\mu_2-1+2/q}\left(\int_0^x |D^\nu f(\tau)|^p\,d\tau\right)^{2/p}, \tag{2.17}$$

where $C_2 = C_2(\nu,\mu_1,\mu_2,p)$ is given by

$$C_2 := \frac{(1/2)^{1/p}}{\Gamma(\nu-\mu_1)\Gamma(\nu-\mu_2+1)((\nu-\mu_1)q+1)^{1/q}((2\nu-\mu_1-\mu_2-1)q+2)^{1/q}}. \tag{2.18}$$

Proof. First an auxiliary inequality. Write $\alpha_i = \nu - \mu_i - 1$ for $i = 1, 2$; in view of the hypothesis $\mu_2 \geq \mu_1 + 1$ we have $\alpha_1 - \alpha_2 - 1 \geq 0$. Let $0 \leq t \leq s \leq x$. Then

$$\int_0^x \left[(\tau-t)_+^{\alpha_1}(\tau-s)_+^{\alpha_2} + (\tau-s)_+^{\alpha_1}(\tau-t)_+^{\alpha_2}\right]\,d\tau \leq \frac{1}{(\nu-\mu_2)}(x-t)^{\alpha_1}(x-s)^{\alpha_2+1}. \tag{2.19}$$

This is verified by estimating the integrand in (2.19) (with $\tau \geq s \geq t$):

$$(\tau-t)^{\alpha_1}(\tau-s)^{\alpha_2} + (\tau-s)^{\alpha_1}(\tau-t)^{\alpha_2}$$

$$
\begin{aligned}
&= (\tau-t)^{\alpha_1-\alpha_2-1}(\tau-t)^{\alpha_2+1}(\tau-s)^{\alpha_2}+(\tau-s)^{\alpha_1-\alpha_2-1}(\tau-s)^{\alpha_2+1}(\tau-t)^{\alpha_2}\\
&\le (x-t)^{\alpha_1-\alpha_2-1}\left[(\tau-t)^{\alpha_2+1}(\tau-s)^{\alpha_2}+(\tau-s)^{\alpha_2+1}(\tau-t)^{\alpha_2}\right]
\end{aligned}
$$

(where the last inequality requires $\alpha_1-\alpha_2-1\ge 0$); (2.19) follows from

$$\int_0^x \left[(\tau-t)_+^{\alpha_2+1}(\tau-s)_+^{\alpha_2}+(\tau-s)_+^{\alpha_2+1}(\tau-t)_+^{\alpha_2}\right]d\tau=\frac{1}{\alpha_2+1}\left[(x-t)(x-s)\right]^{\alpha_2+1}.$$

In the following calculation we abbreviate

$$
\begin{aligned}
&c_1:=(\Gamma(\nu-\mu_2)\Gamma(\nu-\mu_1))^{-1}, && c_2:=(\Gamma(\nu-\mu_2+1)\Gamma(\nu-\mu_1))^{-1},\\
&c_3:=(\nu-\mu_2)q+1, && \varepsilon:=2\nu-\mu_1-\mu_2-1+1/q.
\end{aligned}
$$

By Theorem 1.4,

$$D^{\mu_i}f(\tau)=\frac{1}{\Gamma(\nu-\mu_i)}\int_0^x(\tau-t)_+^{\alpha_i}D^\nu f(t)\,dt,\quad i=1,2.$$

Using this representation, the auxiliary inequality (2.19) and Hölder's inequality, we obtain

$$
\begin{aligned}
&\int_0^x |D^{\mu_1}f(\tau)||D^{\mu_2}f(\tau)|\,d\tau\\
&\le c_1\int_0^x\left(\int_0^x|D^\nu f(t)|(\tau-t)_+^{\alpha_1}\,dt\right)\left(\int_0^x|D^\nu f(s)|(\tau-s)_+^{\alpha_2}\,ds\right)d\tau\\
&= c_1\int_0^x|D^\nu f(t)|\left(\int_t^x|D^\nu f(s)|\left(\int_0^x(\tau-t)_+^{\alpha_1}(\tau-s)_+^{\alpha_2}\,d\tau\right)ds\right)dt\\
&= c_1\int_0^x|D^\nu f(t)|\left(\int_t^x|D^\nu f(s)|\cdot\right.\\
&\qquad\left.\cdot\left(\int_0^x\left[(\tau-t)_+^{\alpha_1}(\tau-s)_+^{\alpha_2}+(\tau-s)_+^{\alpha_1}(\tau-t)_+^{\alpha_2}\right]d\tau\right)ds\right)dt\\
&\le c_2\int_0^x|D^\nu f(t)|\left(\int_t^x|D^\nu f(s)|(x-t)^{\alpha_1}(x-s)^{\alpha_2+1}\,ds\right)dt\\
&= c_2\int_0^x|D^\nu f(t)|(x-t)^{\alpha_1}\left(\int_t^x|D^\nu f(s)|(x-s)^{\alpha_2+1}\,ds\right)dt\\
&\le c_2\int_0^x|D^\nu f(t)|(x-t)^{\alpha_1}\left(\int_t^x|D^\nu f(s)|^p\,ds\right)^{1/p}\left(\int_t^x(x-s)^{q(\alpha_2+1)}ds\right)^{1/q}dt\\
&= c_2c_3^{-1/q}\int_0^x|D^\nu f(t)|(x-t)^{\varepsilon q}\left(\int_t^x|D^\nu f(s)|^p\,ds\right)^{1/p}dt\\
&\le c_2c_3^{-1/q}\left(\int_0^x|D^\nu f(t)|^p\left(\int_t^x|D^\nu f(s)|^p\,ds\right)dt\right)^{1/p}\left(\int_0^x(x-t)^{\varepsilon q}\,dt\right)^{1/q}\\
&\le c_2c_3^{-1/q}(\varepsilon q+1)^{-1/q}x^{(\varepsilon q+1)/q}\left(\tfrac12\left(\int_0^x|D^\nu f(t)|^p\,dt\right)^2\right)^{1/p}.
\end{aligned}
$$

This implies (2.17). □

In the following theorem we address the case when the function $|D^\nu f|$ is monotonic.

Theorem 2.6. *Let $f \in L(0,x)$ have an integrable fractional derivative $D^\nu f \in L^\infty(0,x)$ such that $D^{\nu-j}f(0) = 0$ for $j = 1, \dots, [\nu]+1$ and that $|D^\nu f|$ is decreasing on $[0,x]$. Let $l \geq 2$. If p, $q > 1$ are such that $1/p + 1/q = 1$ and $\sum_{i=1}^l \alpha_i p > -1$, then*

$$\int_0^x \prod_{i=1}^l |D^{\mu_i} f(\tau)| \, d\tau \leq C_3 x^{(\gamma p + lp + 1)/p} \left(\int_0^x |D^\nu f(t)|^{lq} \, dt \right)^{1/q}, \tag{2.20}$$

where $\gamma := \sum_{i=1}^l \alpha_i$ and

$$C_3 = C_3(\nu, \boldsymbol{\mu}, p) := \frac{p}{(\gamma p + 1)^{1/p} (\gamma p + p + 1) \prod_{i=1}^l \Gamma(\nu - \mu_i)}. \tag{2.21}$$

Proof. By Theorem 1.4,

$$|D^{\mu_i} f(\tau)| \leq \frac{1}{\Gamma(\nu - \mu_i)} \int_0^x (\tau - t)_+^{\alpha_i} |D^\nu f(t)| \, dt.$$

The integrand $t \mapsto (\tau - t)_+^{\alpha_i} |D^\nu f(t)|$ is decreasing (and integrable) on $[0,x]$ for all $\tau \in [0,x]$. By Chebyshev's inequality for the product of integrals [6, p. 1099],

$$\begin{aligned}
\prod_{i=1}^l |D^{\mu_i} f(\tau)| &\leq \frac{x^{l-1}}{\prod_{i=1}^l \Gamma(\nu - \mu_i)} \int_0^x \prod_{i=1}^l (\tau - t)_+^{\alpha_i} |D^\nu f(t)| \, dt \\
&\leq \frac{x^{l-1}}{\prod_{i=1}^l \Gamma(\nu - \mu_i)} \int_0^x (\tau - t)_+^{\gamma} |D^\nu f(t)|^l \, dt \\
&\leq \frac{x^{l-1}}{\prod_{i=1}^l \Gamma(\nu - \mu_i)} \left(\int_0^\tau (\tau - t)^{\gamma p} \, dt \right)^{1/p} \left(\int_0^x |D^\nu f(t)|^{lq} \, dt \right)^{1/q} \\
&\leq \frac{x^{l-1}}{\prod_{i=1}^l \Gamma(\nu - \mu_i)} \left(\frac{\tau^{\gamma p + 1}}{\gamma p + 1} \right)^{1/p} \left(\int_0^x |D^\nu f(t)|^{lq} \, dt \right)^{1/q} \\
&= \frac{x^{l-1} \tau^{(\gamma p + 1)/p}}{(\gamma p + 1)^{1/p} \prod_{i=1}^l \Gamma(\nu - \mu_i)} \left(\int_0^x |D^\nu f(t)|^{lq} \, dt \right)^{1/q}.
\end{aligned}$$

Integrating with respect to τ from 0 to x, we get the result. Condition $\sum_{i=1}^l \alpha_i p > -1$ was needed in order to apply Hölder's inequality to $\int_0^x (\tau - t)_+^\gamma |D^\nu f(t)|^l \, dt$. □

The following extreme case of the theorem resembles [1, Proposition 4].

Theorem 2.7. *Let the hypotheses of Theorem 2.6 be satisfied, but let $p = 1$ and $q = \infty$. Then*

$$\int_0^x \prod_{i=1}^l |D^{\mu_i} f(\tau)| \, d\tau \leq C_4 x^{\gamma + l + 1} \|D^\nu f\|_\infty^l, \tag{2.22}$$

where $\gamma := \sum_{i=1}^{l} \alpha_i$ *and*

$$C_4 = C_4(\nu, \boldsymbol{\mu}) := \frac{1}{(\gamma+1)(\gamma+l+1)\prod_{i=1}^{l}\Gamma(\nu-\mu_i)}. \tag{2.23}$$

Proof. As in the proof of Theorem 2.6 we have

$$\begin{aligned}\prod_{i=1}^{l} |D^{\mu_i} f(\tau)| &\le \frac{1}{\prod_{i=1}^{l}\Gamma(\nu-\mu_i)} \prod_{i=1}^{l} \int_0^{\tau} |D^{\nu} f(t)|(\tau-t)^{\alpha_i}\,dt \\ &\le \frac{\tau^{l-1}}{\prod_{i=1}^{l}\Gamma(\nu-\mu_i)}\, \|D^{\nu} f\|_{\infty}^{l} \int_0^{\tau} (\tau-t)^{\gamma}\,dt \\ &\le \frac{\tau^{\gamma+l}\|D^{\nu} f\|_{\infty}^{l}}{(\gamma+1)\prod_{i=1}^{l}\Gamma(\nu-\mu_i)}.\end{aligned}$$

Integrating over $[0, x]$ with respect to τ we obtain (2.22). □

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G. A. Anastassiou, Department of Mathematical Sciences, University of Memphis, TN 38152, USA

J. J. Koliha, Department of Mathematics and Statistics, University of Melbourne, VIC 3010, Australia

J. Pečarić, Faculty of Textile Technology, University of Zagreb, 10000 Zagreb, Croatia