Invertibility of the sum of idempotents

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Abstract

We study necessary and sufficient conditions for the invertibility of the sum $f+g$ when f and g are idempotents in a unital ring or bounded linear operators in Hilbert or Banach spaces. We describe the relation between the invertibility of $f+g$ and $f-g$.

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1 Introduction

In this paper we find several equivalent characterizations of the invertibility of $f+g$, where f, g are general idempotents in a unital ring. The question of the invertibility of $f-g$ was studied by the present authors [4] in the setting of rings and by Gross and Trenkler [3] for matrices. Buckholtz [1, 2], Wimmer [8, 9] and Rakočević [6] investigated the problem for orthogonal projections in Hilbert spaces.

In [3], Gross and Trenkler considered the invertibility of $P-Q$ and $P+Q$ for general matrix idempotents P, Q. Their methods relied strongly on finite dimensionality of the underlying space, and on the theory of the matrix rank developed in [5] and further extended in [3].

In the present paper we use simple algebraic methods independent of the spectral theory of operators or the theory of rank.

2 Preliminaries

By $\mathcal{R}$ we denote an associative ring with unit $1 \neq 0$; the set of all invertible elements in $\mathcal{R}$ is written as $\mathcal{R}^{-1}$. An *idempotent* in $\mathcal{R}$ is an element h satisfying $h^2 = h$. If

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$\mathcal{R}$ has an involution $x \mapsto x^*$, an element $a \in \mathcal{R}$ is called *hermitian* if $a^* = a$. If $f, g \in \mathcal{R}$ are idempotents, then

$$f\mathcal{R} = g\mathcal{R} \iff gf = f \text{ and } fg = g, \tag{2.1}$$

$$\mathcal{R}f = \mathcal{R}g \iff gf = g \text{ and } fg = f. \tag{2.2}$$

With each element a of a unital ring $\mathcal{R}$ we associate two image ideals

$$a\mathcal{R} = \{ax : x \in \mathcal{R}\}, \qquad \mathcal{R}a = \{xa : x \in \mathcal{R}\} \tag{2.3}$$

and two kernel ideals

$$a^0 = \{x \in \mathcal{R} : ax = 0\}, \qquad {}^0a = \{x \in \mathcal{R} : xa = 0\}. \tag{2.4}$$

We say that $a \in \mathcal{R}$ is *regular* (in the sense of von Neumann) if $a \in a\mathcal{R}a$.

For our future use and reference we state several known results in the form of a lemma. (Proofs can be found in [4].)

Lemma 2.1. *Let* $a, b \in \mathcal{R}$; *let* $f, g \in \mathcal{R}$ *be idempotents. Then*:

(i) $1 \in \mathcal{R}a \iff (a \in a\mathcal{R}a$ *and* $a^0 = \{0\})$, $1 \in a\mathcal{R} \iff (a \in a\mathcal{R}a$ *and* ${}^0a = \{0\})$.

(ii) $1 - ab \in \mathcal{R}^{-1} \iff 1 - ba \in \mathcal{R}^{-1}$.

(iii) *If* $a - aba$ *is regular, then so is* a.

(iv) $f - g \in \mathcal{R}^{-1} \implies 1 - fgf \in \mathcal{R}^{-1} \iff 1 - fg \in \mathcal{R}^{-1}$.

3 The invertibility of the sum of idempotents

The following two results on the invertibility of the difference of two idempotents in a ring were obtained by the present authors in [4].

Theorem 3.1. ([4, Theorem 3.2]) *Let* f, g *be idempotents in a unital ring* $\mathcal{R}$. *Then the following conditions are equivalent*:

(i) $f - g \in \mathcal{R}^{-1}$.

(ii) $\mathcal{R} = f\mathcal{R} \oplus g\mathcal{R}$ *and* $\mathcal{R} = \mathcal{R}f \oplus \mathcal{R}g$.

(iii) *There exist idempotents* $h, k \in \mathcal{R}$ *such that* $h\mathcal{R} = f\mathcal{R}$, $(1 - h)\mathcal{R} = g\mathcal{R}$, *and* $\mathcal{R}k = \mathcal{R}f$, $\mathcal{R}(1 - k) = \mathcal{R}g$; h *and* k *are unique if they exist.*

(iv) $1 - fg \in \mathcal{R}^{-1}$, $\mathcal{R} = f\mathcal{R} + g\mathcal{R}$ *and* $f^0 \cap g^0 = \{0\}$.

(v) $f+g-fg \in \mathcal{R}^{-1}$, $f\mathcal{R} \cap g\mathcal{R} = \{0\}$ *and* $\mathcal{R} = f^0 + g^0$.

(vi) $1-fg \in \mathcal{R}^{-1}$ *and* $f+g-fg \in \mathcal{R}^{-1}$.

If $f-g \in \mathcal{R}^{-1}$, *then*

$$(f-g)^{-1} = h+k-1,$$
$$h = (f-g)^{-1}(1-g), \quad k = (1-g)(f-g)^{-1},$$

where h, k *are defined in* (iii).

Theorem 3.2. ([4, Corollary 3.6]) *Let* f, g *be idempotents in a unital ring* $\mathcal{R}$. *Then the following conditions are equivalent*:

(i) $f-g \in \mathcal{R}^{-1}$.

(ii) $f-g$ *is regular and*

$$f\mathcal{R} \cap g\mathcal{R} = \{0\} = f^0 \cap g^0, \quad \mathcal{R}f \cap \mathcal{R}g = \{0\} = {}^0f \cap {}^0g.$$

Our first result is a characterization of the invertibility of $f+g$ in terms of regularity and properties of image ideals.

Theorem 3.3. *Let* $2 \in \mathcal{R}^{-1}$ *and let* f, $g \in \mathcal{R}$ *be idempotents. Then the following conditions are equivalent*:

(i) $f+g \in \mathcal{R}^{-1}$.

(ii) $f+g$ *is regular and*

$$f\mathcal{R} \cap g(1-f)\mathcal{R} = \{0\} = f^0 \cap g^0, \tag{3.1}$$
$$\mathcal{R}f \cap \mathcal{R}(1-f)g = \{0\} = {}^0f \cap {}^0g. \tag{3.2}$$

Proof. We show that $f+g$ is left invertible in $\mathcal{R}$ if and only $f+g$ is regular and (3.1) holds, and that $f+g$ is right invertible in $\mathcal{R}$ if and only if $f+g$ is regular and (3.2) holds.

Suppose first that $f+g$ is left invertible in $\mathcal{R}$. This implies that $f+g$ is regular. Let $x \in f\mathcal{R} \cap g(1-f)\mathcal{R}$. Then $x = fx = gx = g(1-f)y$ for some $y \in \mathcal{R}$, and

$$(f+g)x = 2x = 2g(1-f)y = (f+g)(1-f)2y.$$

Hence $x = (1-f)2y \in (1-f)\mathcal{R}$. Since also $x \in f\mathcal{R}$, we conclude that $x=0$. Therefore $f\mathcal{R} \cap g(1-f)\mathcal{R} = \{0\}$. Next assume that $x \in f^0 \cap g^0$. Then $fx = gx = 0$, and from $(f+g)x = 0$ we get $x = 0$. Hence $f^0 \cap g^0 = \{0\}$, and (3.1) holds. A symmetrical argument applies to show that if $f+g$ is right invertible, then (3.2) holds.

Conversely, assume that $f+g$ is regular and that (3.1) is satisfied. Suppose that $(f+g)x = 0$. Then $fx = -gx = -gfx - g(1-f)x$, and $gfx = -gx = fx$. Hence

$$2fx = fx + gfx = -g(1-f)x,$$

and, since $2 \in \mathcal{R}^{-1}$, $fx \in f\mathcal{R} \cap g(1-f)\mathcal{R} = \{0\}$. Thus $fx = 0 = gx$, and $x \in f^0 \cap g^0 = \{0\}$. This proves that $(f+g)^0 = \{0\}$, and $f+g$ is then left invertible by Lemma 2.1 (i). Analogously we prove that the regularity of $f+g$ together with (3.2) implies that $f+g$ is right invertible. $\square$

The next result shows that if $f-g$ is invertible, then so is $f+g$, and we obtain an explicit expression for the inverse of $f+g$ in terms of the idempotents h, k from Theorem 3.1.

Theorem 3.4. *Let $f, g \in \mathcal{R}$ be idempotents. If $f-g \in \mathcal{R}^{-1}$, then also $f+g \in \mathcal{R}^{-1}$, and*

$$(f+g)^{-1} = 1 - h - k + 2kh, \tag{3.3}$$

where h and k are the idempotents satisfying $h\mathcal{R} = f\mathcal{R}$, $(1-h)\mathcal{R} = g\mathcal{R}$, $\mathcal{R}k = \mathcal{R}f$, $\mathcal{R}(1-k) = \mathcal{R}g$.

Proof. If $f-g \in \mathcal{R}^{-1}$, the existence of h, k satisfying the stated conditions is guaranteed by Theorem 3.1. We observe that

$$hf = f,\ fh = h,\ (1-h)g = g,\ g(1-h) = 1-h,$$
$$kf = k,\ fk = f,\ (1-k)g = 1-k,\ g(1-k) = g.$$

Using these equations, we can verify that

$$(f+g)(1-h-k+2kh) = 1 = (1-h-k+2kh)(f+g). \qquad \square$$

We now address the relation between the invertibility of $f+g$ and $f-g$.

Theorem 3.5. *Let $2 \in \mathcal{R}^{-1}$ and let $f, g \in \mathcal{R}$ be idempotents. Then the following conditions are equivalent:*

(i) $f - g \in \mathcal{R}^{-1}$.

(ii) $f + g \in \mathcal{R}^{-1}$ *and* $1 - fg \in \mathcal{R}^{-1}$.

Proof. (i) $\Longrightarrow$ (ii). Assume that $f-g$ is invertible in $\mathcal{R}$. Then $f+g\in\mathcal{R}^{-1}$ by the preceding theorem, and $1-fg\in\mathcal{R}^{-1}$ by Lemma 2.1 (iv).

(ii) $\Longrightarrow$ (i) First we prove that $f-g$ is regular. From $(f+g)(1-g)=f(1-g)$ we get $1-g=w(1-g)$ with $w=(f+g)^{-1}f$. Hence $(f-g)(1-g)=(f-g)w(f-g)$ and

$$f-g=(f-g)w(f-g)+(f-g)g.$$

We show that $(f-g)g$ is regular and then apply Lemma 2.1 (iii): Since $(f-g)g=(fg-1)g$, we have $g=g(fg-1)^{-1}(f-g)g$, and

$$(f-g)g=(f-g)g(fg-1)^{-1}(f-g)g.$$

Next we show that $(f-g)^0=\{0\}$. Let $(f-g)x=0$ for some $x\in\mathcal{R}$. Then $fx=gx=gfx$, and

$$(1-f)x=(f+g)^{-1}(f+g)(1-f)x=(f+g)^{-1}(gx-gfx)=0,$$
$$fx=(1-gf)^{-1}(1-gf)fx=(1-gf)^{-1}(fx-gfx)=0.$$

Hence $x=(1-f)x+fx=0$. Symmetrically we show that ${}^0(f-g)=\{0\}$. Then $f-g\in\mathcal{R}^{-1}$ by Lemma 2.1 (i). □

Theorem 3.6. *Let $2\in\mathcal{R}^{-1}$ and let $f,\,g\in\mathcal{R}$ be idempotents. Then the following conditions are equivalent*:

(i) $fg+gf\in\mathcal{R}^{-1}$.

(ii) $f+g\in\mathcal{R}^{-1}$ *and* $1-f-g\in\mathcal{R}^{-1}$.

(iii) $f+g$ *is regular and*

$$\mathcal{R}=f^0\oplus g\mathcal{R}={}^0f\oplus\mathcal{R}g, \tag{3.4}$$
$$f\mathcal{R}\cap g(1-f)\mathcal{R}=\{0\}=f^0\cap g^0, \tag{3.5}$$
$$\mathcal{R}f\cap\mathcal{R}(1-f)g=\{0\}={}^0f\cap{}^0g. \tag{3.6}$$

Proof. The equivalence of (i) and (ii) is a consequence of the equation

$$(f+g-1)(f+g)=fg+gf=(f+g)(f+g-1).$$

The equivalence of (ii) and (iii) follows from Theorem 3.3 (the invertibility of $f+g$) and parts (i) and (ii) of Theorem 3.1 (the invertibility of $(1-f)-g$). □

Theorem 3.3 takes the following form in involutory rings.

Theorem 3.7. *Let $2 \in \mathcal{R}^{-1}$ and let f, g be idempotents in a ring $\mathcal{R}$ with involution. Then the following conditions are equivalent*:

(i) $f + g \in \mathcal{R}^{-1}$.

(ii) *$f + g$ is regular and*

$$f\mathcal{R} \cap g(1-f)\mathcal{R} = \{0\} = f^0 \cap g^0, \tag{3.7}$$

$$f^*\mathcal{R} \cap g^*(1-f^*)\mathcal{R} = \{0\} = (f^*)^0 \cap (g^*)^0. \tag{3.8}$$

Proof. From the proof of Theorem 3.3 it follows that the regularity of $f+g$ together with condition (3.7) implies that $f + g$ is left invertible. Since $f^* + g^*$ is regular if $f + g$ is, we conclude by the same argument that the regularity of $f + g$ together with (3.8) implies that $f^* + g^*$ is left invertible. Hence $f + g$ is right invertible, and the result follows. □

Corollary 3.8. *Let $2 \in \mathcal{R}^{-1}$ and let f, g be hermitian idempotents in a ring $\mathcal{R}$ with involution. Then the following conditions are equivalent*:

(i) $f + g \in \mathcal{R}^{-1}$.

(ii) *$f + g$ is regular and $f\mathcal{R} \cap g(1-f)\mathcal{R} = \{0\} = f^0 \cap g^0$.*

4 Applications to Hilbert and Banach space operators

We start by considering the invertibility of $F + G$, where F, G are idempotent elements of $B(H)$, the C^*-algebra of all bounded linear operators on a Hilbert space H. Such F, G will be called *oblique projections*, or *orthogonal projections* if they are also hermitian.

In the following lemma, $FB(H)$ and F^0 denote the image and kernel ideals of an oblique projection F in $B(H)$ defined in (2.3) and (2.4).

Lemma 4.1. *Let F, G be oblique projections on a Hilbert space H. Then*

(i) $FB(H) \cap G(I-F)B(H) = \{0\} \iff R(F) \cap R(G(I-F)) = \{0\}$.

(ii) $F^0 \cap G^0 = \{0\} \iff N(F) \cap N(G) = \{0\}$.

Proof. (i) Let $FB(H) \cap G(I-F)B(H) = \{0\}$ and let $x \in R(F) \cap R(G(I-F))$. Then $x = Fx = G(I-F)w$ for some $w \in H$. For a proof by contradiction assume that $x \neq 0$ and write $z = x\|x\|^{-2}$. Define $T : H \to H$ and $S : H \to H$ by

$$Tu = \langle u, z\rangle x, \quad Su = \langle u, z\rangle w \quad \text{for all } u \in H.$$

Then T, $S \in B(H)$, and $(FT - G(I-F)S)u = \langle u, z\rangle (Fx - G(I-F)w) = 0$ for all $u \in H$. Therefore $FT = G(I-F)S \in FB(H) \cap G(I-F)B(H)$, and $FT = 0$ by hypothesis. Hence $0 = FTx = \langle Fx, z\rangle x = \langle x, z\rangle x = x$, which is a desired contradiction. This proves $R(F) \cap R(G(I-F)) = \{0\}$.

Conversely, suppose that $R(F) \cap R(G(I-F)) = \{0\}$. Let $A \in FB(H) \cap G(I-F)B(H)$. Then, for each $x \in H$, $Ax \in R(F) \cap R(G(I-F))$, which implies $Ax = 0$, that is, $A = 0$.

(ii) Assume that $F^0 \cap G^0 = \{0\}$, and that $x \in N(F) \cap N(G)$. Assume that $x \neq 0$, and set $z = x\|x\|^{-2}$. Then $FTu = \langle u, z\rangle Fx = 0$ and $GTu = \langle u, z\rangle Gx = 0$ for all $u \in H$, which implies $T \in F^0 \cap G^0$ and $T = 0$. Hence $x = Tx = 0$ contrary to our assumption $x \neq 0$.

Conversely, let $N(F) \cap N(G) = \{0\}$ and let $A \in F^0 \cap G^0$. For each $x \in H$, $Ax \in N(F) \cap N(G)$, so that $Ax = 0$ and $A = 0$. □

For Hilbert space operators Theorem 3.7 and Corollary 3.8 take the following form.

Theorem 4.2. *Let F, G be oblique projections on a Hilbert space H. Then the following conditions are equivalent*:

(i) *$F + G$ is invertible.*

(ii) *The range of $F + G$ is closed and*

$$R(F) \cap R(G(I-F)) = \{0\} = N(F) \cap N(G),$$
$$R(F^*) \cap R(G^*(I-F^*) = \{0\} = N(F^*) \cap N(G^*).$$

Proof. It is well known that an operator $A \in B(H)$ is regular if and only it it has closed range. The rest follows from Theorem 3.7 and the preceding lemma. □

Corollary 4.3. *Let F, G be orthogonal projections on a Hilbert space H. Then the following conditions are equivalent*:

(i) *$F + G$ is invertible.*

(ii) *The range of* $F+G$ *is closed and* $R(F)\cap R(G(I-F))=\{0\}=N(F)\cap N(G)$.

If H is a finite dimensional Hilbert space, then every $A\in B(H)$ has a closed range, and invertibility of A is equivalent to its one sided invertibility. We then obtain the following corollary to Theorem 4.2.

Corollary 4.4. *Let* F, G *be oblique projections in a finite dimensional Hilbert space* H. *Then the following conditions are equivalent*:

(i) $F+G$ *is invertible.*

(ii) $R(F)\cap R(G(I-F))=\{0\}=N(F)\cap N(G)$.

The preceding result was proved by Gross and Trenkler [3, Corollary 2] for matrices by methods based on the theory of rank. Other conditions equivalent to the invertibility of $F+G$ involving the rank of a matrix were obtained by Tian and Styan [7].

For the final result of this section we turn our attention to bounded linear operators on a Banach space X. We use the following notation. If X is the direct sum $X=M\oplus N$ of closed subspaces M, N, then there exists a unique oblique projection $U\in B(X)$ with $R(U)=M$ and $N(U)=N$—the so-called projection of X onto M along N. We will write

$$U=P_{M,N}.$$

Theorem 4.5. *Let* F, G *be oblique projections on a Banach space* X *such that* $F-G$ *is invertible. Then also* $F+G$ *is invertible, and*

$$(F+G)^{-1}=I-P_{R(F),R(G)}P_{N(F),N(G)}-P_{R(G),R(F)}P_{N(G),N(F)}. \tag{4.1}$$

Proof. If $F-G$ is invertible, then so is $F+G$ by Theorem 3.5. According to Theorem 3.1 (iii) and equation (2.1), there are oblique projections U, $V\in B(X)$ such that $UF=F$, $FU=U$, $(I-U)G=G$, $G(I-U)=I-U$; $VF=V$, $FV=F$, $(I-V)G=I-V$, $G(I-V)=G$. From these relations we deduce that

$$U=P_{R(F),R(G)} \quad \text{and} \quad V=P_{N(G),N(F)}.$$

By Theorem 3.4, we obtain

$$\begin{aligned}(F+G)^{-1}&=I-U-V+2UV\\&=I-U(I-V)-(I-U)V\\&=I-P_{R(F),R(G)}P_{N(F),N(G)}-P_{R(G),R(F)}P_{N(G),N(F)},\end{aligned}$$

observing that $I - V = I - P_{N(G),N(F)} = P_{N(F),N(G)}$ and $I - U = I - P_{R(F),R(G)} = P_{R(G),R(F)}$. □

Specializing Theorem 3.6 to $d \times d$ matrices, we recover a result of Tian and Styan [7, Corollary 2.13]:

Corollary 4.6. *If F, G are idempotent $d \times d$ matrices, then the following conditions are equaivalent*:

(i) $FG + GF$ *is invertible.*

(ii) $\mathbb{C}^d = N(F) \oplus R(G) = R(F) \oplus N(G)$ *and* $R(F) \cap R(G(I - F)) = \{0\} = N(F) \cap N(G) = \{0\}$.

Proof. The regularity condition of Theorem 3.6 (iii) is fulfilled automatically and we need to assume only (3.4) and (3.5) (transcribed for ranges and nullspaces of matrices) as the one sided invertibility of a matrix is equivalent to its invertibility. □

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