The Mapping $\Psi^p_{x,y}$ in Normed Linear Spaces and its Applications in the Theory of Inequalities

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In this paper we introduce the mapping $\Psi^p_{x,y}(t) = (x, x+ty)_p \|x+ty\|^{-1}$, which is derived from the lower and upper semi-inner products $(\cdot,\cdot)_i$ and $(\cdot,\cdot)_s$, and study its properties of monotonicity, boundedness and convexity. We give applications to height functions and to inequalities in analysis, including a refinement of the Schwarz inequality.

Keywords: Normed linear spaces; the lower and upper semi-inner products; inner product spaces; the Schwarz inequality.

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1 Introduction

This paper continues the investigation started by the present authors in [6, 7]. In [6], we studied the mapping

$$v_{x,y}(t) = \frac{\|x+ty\| - \|x\|}{t}$$

defined on $\mathbb{R}\backslash\{0\}$ in order to obtain a refinement of the norm in X as an instrument of measurement, and obtained refinements of inequalities important in analysis as a result.

In [7] the mapping

$$\gamma_{x,y}(t) = \frac{\|x+2ty\| - \|x+ty\|}{t}, \quad t \in \mathbb{R}\backslash\{0\},$$

was introduced to study the finer points of geometry of the normed linear spaces; applications included new characterizations of the Birkhoff orthogonality and best approximants.

Indispensable tools in this process are the *lower* and *upper semi-inner product* in X, defined by

$$(y,x)_i = \lim_{t\to 0-} \frac{\|x+ty\|^2 - \|x\|^2}{2t},$$

and

$$(y,x)_s = \lim_{t\to 0+} \frac{\|x+ty\|^2 - \|x\|^2}{2t},$$

respectively. These limits are well defined for every pair $x, y \in X$, and play a rôle in normed spaces similar to that of the inner product in an inner product space. For reference we list some of the main properties of these products that will be used in the sequel (see [2, 3, 4, 5]).

(I) $(x,x)_p = \|x\|^2$ for all $x \in X$;

(II) $(\alpha x, \beta y)_p = \alpha\beta(x,y)_p$ if $\alpha\beta \geq 0$ and $x, y \in X$;

(III) $|(x,y)_p| \leq \|x\| \, \|y\|$ for all $x, y \in X$;

(IV) $(\alpha x + y, x)_p = \alpha(x,x)_p + (y,x)_p$ if x, y belong to X and α is a real number;

(V) $(-x, y)_p = -(x,y)_q$ for all $x, y \in X$;

(VI) $(x+y, z)_p \leq \|x\| \, \|z\| + (y,z)_p$ for all $x, y, z \in X$;

(VII) The mapping $(\cdot,\cdot)_p$ is continuous and subadditive (superadditive) in the first variable for $p = s$ (or $p = i$);

(VIII) We have the inequality

$$(y,x)_i \leq (y,x)_s \text{ for all } x, y \in X;$$

(IX) If the norm $\|\cdot\|$ is induced by an inner product $(\cdot,\cdot)$, then

$$(y,x)_i = (y,x) = (y,x)_s \text{ for all } x, y \in X.$$

In the present paper we introduce the mappings $\Psi^p_{x,y}$ by (2.1), and start investigating their monotonicity, boundedness and convexity. These properties are then applied to height functions, and to obtaining refinements of the Schwarz inequality, and other inequalities useful in analysis.

2 Properties of the mapping $\Psi^p_{x,y}$

Let x, y be two fixed linearly independent vectors in the normed linear space $(X, \|\cdot\|)$. We consider the two mappings

$$\Psi^p_{x,y}(t) := \frac{(x, x+ty)_p}{\|x+ty\|}, \quad p \in \{s, i\}, \tag{2.1}$$

well defined for all $t \in \mathbb{R}$, which often arise in analytic and geometric considerations of normed linear spaces employing the lower and upper semi-inner product. (See, for instance [7], where the authors used $\Psi^p_{x,y}$ to investigate the mapping $\gamma_{x,y}(t) = (\|x+2ty\| - \|x+ty\|)/t$.) This paper is devoted to a systemic study of the mapping $\Psi^p_{x,y}$, which plays an important role in geometry of normed linear spaces and the theory of inequalities in analysis.

The following theorem describes the main properties of this mapping.

Theorem 2.1 *Let $(X, \|\cdot\|)$ be a real normed linear space and x, y two linearly independent vectors in X. Then*

(i) *The mapping $\Psi^p_{x,y}$ is bounded on $\mathbb{R}$ with*

$$\left|\Psi^p_{x,y}(t)\right| \leq \|x\| \text{ for all } t \in \mathbb{R}; \tag{2.2}$$

(ii) *We have the inequality*

$$\delta_{x,y}(t) \leq \Psi^i_{x,y}(t) \leq \Psi^s_{x,y}(t) \leq \|x\| \text{ for all } t \in \mathbb{R} \tag{2.3}$$

and

$$\delta_{x,y}(t) \geq \Psi^s_{x,y}(2t) \geq \Psi^i_{x,y}(2t) \geq \|x+2ty\| - 2|t|\,\|y\|$$

$$\geq \begin{cases} \dfrac{(x,y)_s}{\|y\|} & \textit{if } t \geq 0 \\ -\dfrac{(x,y)_i}{\|y\|} & \textit{if } t < 0, \end{cases} \tag{2.4}$$

where $\delta_{x,y}(t) = 2\|x+ty\| - \|x+2ty\|$.

(iii) $\Psi^p_{x,y}$ *is continuous at* 0*, and we have the limits*

$$\lim_{t\to+\infty} \Psi^p_{x,y}(t) = \frac{(x,y)_s}{\|y\|} \tag{2.5}$$

and

$$\lim_{t\to-\infty} \Psi^p_{x,y}(t) = -\frac{(x,y)_i}{\|y\|}; \tag{2.6}$$

(iv) $\Psi^p_{x,y}$ *is increasing on* $(-\infty, 0]$ *and decreasing on* $[0,+\infty)$.

Proof (i) follows from the Schwarz inequality.

(ii) Using the basic properties of the lower and upper semi-inner products as set out in the Introduction,we have

$$\begin{aligned} \|x+2ty\|\,\|2x+2ty\| &\geq (x+2ty, 2x+2ty)_s \\ &= (2x+2ty-x, 2x+2ty)_s = \|2x+2ty\|^2 - (x, 2x+2ty)_i \end{aligned}$$

which yields

$$\|x+2ty\| - \|2x+2ty\| \geq -(x, 2x+2ty)_i \, \|2x+2ty\|^{-1};$$

this is equivalent to

$$2\|x+ty\| - \|x+2ty\| \leq \frac{(x, x+ty)_i}{\|x+ty\|}$$

for all $t \in \mathbb{R}$. This proves the first inequality in (2.3). The second inequality is clear. To obtain the third inequality we apply (III): $(x, x+ty)_s \leq \|x+ty\|\,\|x\|$.

The Schwarz inequality is used to prove the first inequality in (2.4):

$$\|2x+2ty\|\,\|x+2ty\| \geq (2x+2ty, x+2ty)_s$$

$$= (x + x + 2ty, x + 2ty)_s = \|x + 2ty\|^2 + (x, x + 2ty)_s,$$

hence

$$2\|x + ty\| - \|x + 2ty\| \geq (x, x + 2ty)_s \|x + 2ty\|^{-1} = \Psi_{x,2y}^{s}(t)$$

for all $t \in \mathbb{R}$.

Now suppose that $t \geq 0$. Then $\|x + 2ty\| - 2|t|\|y\| = \|x + 2ty\| - 2t\|y\|$, and by the Schwarz inequality we have

$$\|x + 2ty\|\|y\| \geq (x + 2ty, y)_s = (x, y)_s + 2t\|y\|^2,$$

which implies

$$\|x + 2ty\| - 2t\|y\| \geq \frac{(x, y)_s}{\|y\|}.$$

For $t < 0$ we replace t by $-t$ and y by $-y$ in the preceding argument, and the last inequality in (2.3) follows.

(iii) We observe that

$$\lim_{t \to 0} \delta_{x,y}(t) = \|x\|;$$

then by inequality (2.3) the limit $\lim_{t\to 0} \Psi_{x,y}^{p}(t)$ exists and equals $\|x\|$. Hence $\Psi_{x,y}^{p}$ is continuous at 0.

To establish limits (2.5) and (2.6) we observe that

$$\lim_{\alpha \to 0+} \frac{\|y + \alpha x\| - \|y\|}{\alpha} = \frac{(y, x)_s}{\|y\|}.$$

After substitution $\alpha = t^{-1}$ for $t > 0$ and a short calculation we obtain

$$2\|x + ty\| - \|x + 2ty\| = 2\frac{\|y + \alpha x\| - \|x\|}{\alpha} - \frac{\left\|y + \frac{1}{2}\alpha\right\| - \|y\|}{\frac{1}{2}\alpha} =: F_{x,y}(\alpha).$$

Consequently,

$$\lim_{t \to +\infty} \delta_{x,y}(t) = \lim_{\alpha \to 0+} F_{x,y}(\alpha) = \frac{(y, x)_s}{\|y\|}.$$

(2.5) then follows by applying inequality (2.3). Using this result, we have

$$\lim_{t\to-\infty}\Psi^p_{x,y}(t)=\lim_{u\to+\infty}\Psi^p_{x,-y}(u)=\frac{(-y,x)_s}{\|-y\|}=-\frac{(y,x)_i}{\|y\|},$$

and (2.6) obtains.

(iv) This was already proved in [7] as an intermediate step in the proof of monotonicity of $\gamma_{x,y}(t)=(\|x+2ty\|-\|x+ty\|)/t$. For the sake of completeness, we include a simplified proof here.

To take advantage of the properties of the upper and lower semi-inner products we introduce the mapping

$$\Phi^p_{x,y}(t):=\frac{(y,x+ty)_p}{\|x+ty\|},\quad p\in\{i,s\}.$$

We have

$$\Psi^p_{x,y}(t)=\Phi^p_{y,x}\left(\frac{1}{t}\right)\quad\text{if } t>0,$$
$$\Psi^p_{x,y}(u)=\Phi^q_{y,x}\left(\frac{1}{u}\right)\quad\text{if } u<0,$$

where $q\in\{s,i\}$, $q\neq p$. The proof will be completed when we show that $\Phi_{y,x}$ is increasing on $\mathbb{R}$.

Suppose that $p\in\{i,s\}$ and $t_2>t_1$. Then, by the Schwarz inequality,

$$\|y+t_2x\|\,\|y+t_1x\|\geq(y+t_2x,y+t_1x)_p$$

for all $x,y\in X$. Using properties of the norm derivatives, we obtain

$$\begin{aligned}(y+t_2x,y+t_1x)_p&=((t_2-t_1)x+y+t_1x,y+t_1x)_p\\&=\|y+t_1x\|^2+(t_2-t_1)(x,y+t_1x)_p,\end{aligned}$$

and the above inequality yields

$$\|y+t_2x\|\,\|y+t_1x\|\geq\|y+t_1x\|^2+(t_2-t_1)(x,y+t_1x)_p.$$

Hence

$$\Phi_{y,x}^{p}(t_1) = \frac{(x, y + t_1 x)_p}{\|y + t_1 x\|} \le \frac{\|y + t_2 x\| - \|y + t_1 x\|}{t_2 - t_1}.$$

Let $t := t_2 - t_1 > 0$ and let $q \in \{s, i\}$, $q \neq p$. Then

$$\begin{aligned}\|y\| \, \|y + tx\| &\ge (y, y + tx)_q = (y + tx - tx, y + tx)_q \\ &= \|y + tx\|^2 + (-tx, y + tx)_q = \|y + tx\|^2 - t(x, y + tx)_p,\end{aligned}$$

and

$$(\|y + tx\| - \|y\|)t^{-1} \le \Phi_{y,x}^{p}(t).$$

Consequently,

$$\frac{\|y + t_2 x\| - \|y + t_1 x\|}{t_2 - t_1} = \frac{\|y + t_1 x + tx\| - \|y + t_1 x\|}{t} \le \Phi_{y+t_1 x, x}^{p}(t) = \Phi_{y,x}^{p}(t_2).$$

This proves $\Psi_{y,x}^{p}(t_1) \le \Psi_{y,x}^{p}(t_2)$.

Remark 2.2 The graphs of $\Psi_{x,y}^{p}$, $p \in \{i, s\}$, in the case of a normed linear space are depicted in Fig. 1. They are drawn in a dashed line to suggest that there is no information about the convexity of $\Psi_{x,y}^{p}$, $p \in \{i, s\}$. The absolute maximum is $\|x\|$, attained at $t = 0$. The graph has two horizontal asymptotes, $(x, y)_s / \|y\|$ as $t \to \infty$ and $-(x, y)_i / \|y\|$ as $t \to -\infty$.

Fig. 1 depicts the graph in the case that $(x, y)_s > 0$. Then $-(x, y)_i \ge -(x, y)_s$, but $-(x, y)_i$ may be positive, negative or zero.

We now turn our attention to the case when X is an inner product space. While we have no information about the convexity properties of $\Psi_{x,y}^{p}$ in a general normed linear space, we are able to determine the convexity and concavity of $\Psi_{x,y}$ in an inner product space from its second derivative.

Proposition 2.3 *Let $(X, (\cdot, \cdot))$ be a real inner product space and x, y two linearly independent vectors in X. The mapping $\Psi_{x,y} : \mathbb{R} \to \mathbb{R}$ defined by*

$$\Psi_{x,y}(t) = \frac{\|x\|^2 + t(x,y)}{\|x + ty\|}$$

is twice differentiable on $\mathbb{R}$ with

$$\frac{d\Psi_{x,y}(t)}{dt} = t\,\frac{(x,y)^2 - \|x\|^2\|y\|^2}{\|x+ty\|^3} \tag{2.7}$$

and

$$\frac{d^2\Psi_{x,y}(t)}{dt^2} = \frac{\|x\|^2\|y\|^2 - (x,y)^2}{\|x+ty\|^5}\,(2t^2\|x\|^2 + t(y,x) - \|x\|^2). \tag{2.8}$$

Moreover, the mapping $\Psi_{x,y}$ is strictly convex on $(-\infty, t_1) \cup (t_2, \infty)$ and strictly concave on (t_1, t_2), where

$$t_1 = \frac{-(x,y) - \sqrt{\Delta_{x,y}}}{4\|y\|^2}, \qquad t_2 = \frac{-(x,y) + \sqrt{\Delta_{x,y}}}{4\|y\|^2} \tag{2.9}$$

and $\Delta_{x,y} = 8\|x\|^2\|y\|^2 + (x,y)^2 > 0$.

Proof For notational convenience we introduce the mapping $n_{x,y}(t) = \|x + ty\|$, and find that

$$\frac{d}{dt}n_{x,y}(t) = \frac{(x,y) + t\|y\|^2}{n_{x,y}(t)}.$$

Then

$$\begin{aligned}
\frac{d\Psi_{x,y}(t)}{dt} &= \frac{1}{n_{x,y}^2(t)}\left[\frac{d}{dt}(\|x\|^2 + t(x,y))n_{x,y}(t) - (\|x\|^2 + t(x,y))\frac{dn_{x,y}(t)}{dt}\right] \\
&= \frac{1}{n_{x,y}^2(t)}\left[(x,y)n_{x,y}(t) - (\|x\|^2 + t(x,y))\,\frac{(x,y) + t\|y\|^2}{n_{x,y}(t)}\right] \\
&= \frac{1}{n_{x,y}^3(t)}\left[(x,y)n_{x,y}^2(t) - (\|x\|^2 + t(x,y))((x,y) + t\|y\|^2)\right] \\
&= t\,\frac{(x,y)^2 - \|x\|^2\|y\|^2}{n_{x,y}^3(t)},
\end{aligned}$$

and equation (2.7) is proved. Further,

$$\frac{d^2\Psi_{x,y}(t)}{dt^2} = \frac{(x,y)^2 - \|x\|^2\|y\|^2}{n_{x,y}^6(t)}(n_{x,y}^3(t) - 3tn_{x,y}^2(t)n'_{x,y}(t))$$

$$= \frac{(x,y)^2 - \|x\|^2 \|y\|^2}{n_{x,y}^4(t)} \left[n_{x,y}(t) - \frac{3t((y,x) + t\|y\|^2)}{n_{x,y}(t)} \right]$$
$$= \frac{(x,y)^2 - \|x\|^2 \|y\|^2}{n_{x,y}^5(t)} (n_{x,y}^2(t) - 3t(y,x) - 3t^2 \|y\|^2)$$
$$= \frac{\|x\|^2 \|y\|^2 - (x,y)^2}{n_{x,y}^5(t)} (2t^2 \|y\|^2 + t(y,x) - \|x\|^2),$$

which proves (2.8).

The quadratic equation

$$2t^2 \|y\|^2 + t(y,x) - \|x\|^2 = 0$$

has two distinct solutions t_1, t_2 given in the statement of the proposition. Then $\Psi_{x,y}''(t) > 0$ if $t \in (-\infty, t_1) \cup (t_2, \infty)$ and $\Psi_{x,y}''(t) < 0$ if $t \in (t_1, t_2)$. The proposition is now proved.

Remark 2.4 The graph of $\Psi_{x,y}$ in an inner product space is given in Fig. 2, this time for the case $(x,y) < 0$. The t-intercept t_0 is obtained from the equation $\|x\|^2 + t(x,y) = 0$. In the case of an inner product space, the convexity behaviour of $\Psi_{x,y}$ is known from Proposition 2.3; in particular, t_1 and t_2 defined by (2.9) are the inflection points of $\Psi_{x,y}$.

The graph of $\Psi_{x,y}$ for the case of two orthogonal vectors x, y is given in Fig. 3; then $\Psi_{x,y}$ is even and the graph is asymptotic to the t-axis.

3 Applications to height functions

Alsina, Guijarro and Tomas [1] considered the following so-called height functions:

$$h_1(x) = y + \frac{\|y\|^2 - (y,x)_s}{\|x-y\|^2}(x-y),$$
$$h_2(x) = y + \frac{\|y\|^2 - (x,y)_s}{\|x-y\|^2}(x-y),$$
$$h_3(x) = y + \frac{(y, y-x)_s}{\|x-y\|^2}(x-y),$$

where x, y are two distinct vectors in a real normed linear space X, and applied them in characterizing inner product spaces in the class of normed spaces. For their interesting results see paper [1].

We observe that the function h_3 is related to the mappings introduced in this paper, namely,

$$\|y - h_3(-tx, y)\| = \left|\Psi^s_{y,x}(t)\right| \quad \text{for all } t \in \mathbb{R}.$$

For two given linearly independent vectors x, y in a normed linear space $(X, \|\cdot\|)$ we define the following objects:

$$\begin{aligned}
\cos_p \widehat{(x,y)} &:= \frac{(y,x)_p}{\|y\|\,\|x\|} && \text{where } p = s \text{ or } p = i, \\
A_p(x,y) &:= \|x\|\,\|y\| \sin_p \widehat{(x,y)} && \text{where } p = s \text{ or } p = i, \\
A_h^{[k]}(x,y) &:= \tfrac{1}{2}\,\|h_k(x,y)\|\,\|x-y\| && \text{where } k = 1, 2 \text{ or } 3.
\end{aligned}$$

If X is an inner product space, then $h_1 = h_2 = h_3 = h$, and the following proposition is true.

Proposition 3.1 *Let $(X, \|\cdot\|)$ be an inner product space. Then for all $x, y \in X$ with $x \neq y$ we have the identity*

$$A_h(x,y) = \tfrac{1}{2}\,\|y\|\,\|y-x\| \sin \widehat{(y, y-x)}. \tag{3.1}$$

Proof A simple calculation in inner product spaces gives

$$\begin{aligned}
\|h(x,y)\|^2 &= \|y\|^2 - 2\frac{(y, y-x)^2}{\|y-x\|^2} + \frac{(y,y-x)^2}{\|y-x\|^4}\,\|y-x\|^2 \\
&= \frac{\|y\|^2\,\|y-x\|^2 - (y,y-x)^2}{\|y-x\|^2}.
\end{aligned}$$

As $\cos \widehat{(y, y-x)} = (y, y-x)\,\|y\|^{-1}\,\|y-x\|^{-1}$, we have

$$\|h(x,y)\|^2 = \frac{\|y\|^2\,\|y-x\|^2 - \|y\|^2\,\|y-x\|^2 \cos^2 \widehat{(y, y-x)}}{\|y-x\|^2}$$

$$= \|y\|^2 \sin^2 (\widehat{y, y-x})$$

which implies

$$\|h(x,y)\| = \|y\| \sin (\widehat{y, y-x}),$$

and (3.1) is obtained.

The following result reveals the geometric nature of the height function h_3 in a normed linear space.

Proposition 3.2 *Let $(X, \|\cdot\|)$ be a normed linear space. Then, for all $x, y \in X$ with $x \neq y$, we have the representation*

$$h_3(x,y) = \cos_s (\widehat{y, y-x}) \frac{\|y\|}{\|y-x\|}\, x - \cos_s (\widehat{x, y-x}) \frac{\|x\|}{\|y-x\|}\, y, \tag{3.2}$$

and the inequality

$$A_h^{[3]} \leq \tfrac{1}{2} \|x\| \, \|y\| \left(|\cos_s (\widehat{y, y-x})| + |\cos_s (\widehat{x, y-x})| \right) \leq \|x\| \, \|y\| \,. \tag{3.3}$$

Proof Using the properties of the upper and lower semi-inner products we have in succession

$$\begin{aligned}
h_3(x,y) &= y + \frac{(y, y-x)_s}{\|y-x\|^2} (x-y) \\
&= \Big(1 - \frac{(y, y-x)_s}{\|y-x\|^2}\Big) y + \frac{(y, y-x)_s}{\|y-x\|^2}\, x \\
&= \frac{\|y-x\|^2 - (y, y-x)_s}{\|y-x\|^2}\, y + \frac{(y, y-x)_s}{\|y-x\|^2}\, x \\
&= \frac{(y-x-y, y-x)_s}{\|y-x\|^2}\, y + \frac{(y, y-x)_s}{\|y-x\|^2}\, x \\
&= -\frac{(x, y-x)_s}{\|y-x\|^2}\, y + \frac{(y, y-x)_s}{\|y-x\|^2}\, x \\
&= \frac{(y, y-x)_s}{\|y\| \, \|y-x\|} \frac{\|y\|}{\|y-x\|}\, x - \frac{(x, y-x)_s}{\|x\| \, \|y-x\|} \frac{\|x\|}{\|y-x\|}\, y
\end{aligned}$$

and the identity (3.2) is obtained.

We have

$$\begin{aligned} A_h^{[3]}(x,y) &= \tfrac{1}{2}\left\|h_3(x,y)\right\|\left\|x-y\right\| \\ &= \frac{1}{2}\left\|\frac{(y,y-x)_s}{\|y-x\|^2}\,x-\frac{(x,y-x)_s}{\|y-x\|^2}\,y\right\|\|y-x\| \\ &\le \frac{|(y,y-x)_s|}{\|y\|\,\|y-x\|}\,\|x\|\,\|y\|+\frac{|(x,y-x)_s|}{\|y-x\|\,\|y\|}\,\|x\|\,\|y\|\,, \end{aligned}$$

and the first part of the inequality (3.3) is proved. The second part is obvious.

4 Applications to inequalities

The inequalities obtained in Section 2 can be applied in concrete Banach spaces to obtain improvements of classical inequalities.

Example 4.1 Theorem 2.1 can be applied to the space $L^1(\Omega)$ of all μ-integrable real valued functions on the measure space $(\Omega,\mathcal{A},\mu)$, where $\mathcal{A}$ is a σ-algebra of subsets of Ω, and μ a complete positive measure on $\mathcal{A}$. The norm on $L(\Omega)$ is given by $\|x\|=\int_\Omega |x|\,d\mu$. A modification of the argument in [9] gives

$$(x,y)_p=\|y\|\left(\int_{\Omega_1(y)}\operatorname{sgn}(y)x\,d\mu+\varepsilon(p)\int_{\Omega_0(y)}|x|\,d\mu\right),$$

where $\varepsilon(i)=-1$ and $\varepsilon(s)=+1$, $\Omega_0(y)=\{t\in\Omega : y(t)=0\}$, and $\Omega_1(y)=\Omega\backslash\Omega_0(y)$. If x,y are linearly independent vectors in $L(\Omega)$ and $t>0$, we get

$$\begin{aligned} \int_{\Omega_1(y)}\operatorname{sgn}(y)x\,d\mu-\int_{\Omega_0(y)}|x|\,d\mu &\le \int_{\Omega_1(y)}\operatorname{sgn}(y)x\,d\mu+\int_{\Omega_0(y)}|x|\,d\mu \\ &\le \int_\Omega |x+2ty|\,d\mu-2t\int_\Omega |y|\,d\mu \\ &\le \int_{\Omega_1(x+2ty)}\operatorname{sgn}(x+2ty)x\,d\mu-\int_{\Omega_0(x+2ty)}|x|\,d\mu \\ &\le \int_{\Omega_1(x+2ty)}\operatorname{sgn}(x+2ty)x\,d\mu+\int_{\Omega_0(x+2ty)}|x|\,d\mu \\ &\le \int_\Omega 2|x+ty|\,d\mu-\int_\Omega |x+2ty|\,d\mu \end{aligned}$$

$$\begin{aligned}
&\le \int_{\Omega_1(x+ty)} \operatorname{sgn}(x+ty)x\,d\mu - \int_{\Omega_0(x+ty)} |x|\,d\mu \\
&\le \int_{\Omega_1(x+ty)} \operatorname{sgn}(x+ty)x\,d\mu + \int_{\Omega_0(x+ty)} |x|\,d\mu \\
&\le \int_{\Omega} |x|\,d\mu.
\end{aligned}$$

If $t < 0$, the only change in the above inequalities occurs in the first two lines, which become

$$\begin{aligned}
-\int_{\Omega_1(y)} \operatorname{sgn}(y)x\,d\mu - \int_{\Omega_0(y)} |x|\,d\mu &\le -\int_{\Omega_1(y)} \operatorname{sgn}(y)x\,d\mu + \int_{\Omega_0(y)} |x|\,d\mu \\
&\le \int_{\Omega} |x+2ty|\,d\mu + 2t\int_{\Omega} |y|\,d\mu.
\end{aligned}$$

Example 4.2 Let $C(\Omega)$ be the space of all continuous real valued functions on the compact metric space Ω equipped with the norm $\|x\| = \sup\{|x(u)| : u \in \Omega\}$. Then

$$(x,y)_s = \sup\{x(u)y(u) : |y(u)| = \|y\|\}, \tag{4.1}$$

$$(x,y)_i = \inf\{x(u)y(u) : |y(u)| = \|y\|\}. \tag{4.2}$$

This result can be found in [2, Example 12.2] in the case that Ω is a compact subset of $\mathbb{R}^m$. The proof in [2] depends on the Riesz representation theorem. We give an elementary proof of (4.1); (4.2) then follows from $(x,y)_i = -(-x,y)_s$.

Let x, y be two elements of $C(\Omega)$. The set $\Omega(y) = \{u \in \Omega : |y(u)| = \|y\|\}$ is compact and nonempty. If $u \in \Omega(y)$, then

$$\lim_{t\to 0+} \frac{(x(u)+ty(u))^2 - \|y\|^2}{2t} \le \lim_{t\to 0+} \frac{\|y+tx\|^2 - \|y\|^2}{2t},$$

which implies $x(u)y(u) \le (x,y)_s$ for all $u \in \Omega(y)$.

For each $t > 0$ there is $u_t \in \Omega$ with $\|y+tx\| = |y(u_t) + tx(u_t)|$. From the inequality

$$\|y\| - t\|x\| \le \|y+tx\| = |y(u_t)+tx(u_t)| \le |y(u_t)| + t\|x\|$$

we conclude that $0 \le \|y\| - |y(u_t)| \le 2t\,\|x\|$, and that $|y(u_t)| \to \|y\|$ as $t \to 0+$. Since Ω is compact, there exists a sequence (t_n), $t_n > 0$, $t_n \to 0$, such that $w_n = u_{t_n} \to w$ (as $n \to \infty$) for some $w \in \Omega$. As $|y(w)| = \lim_{n\to\infty} |y(w_n)| = \|y\|$, we have $w \in \Omega(y)$.

Let β be an upper bound for the set $\{x(u)y(u) : u \in \Omega(y)\}$. Then

$$\begin{aligned}\frac{\|x+t_n y\|^2 - \|y\|^2}{2t_n} &= \frac{(y(w_n)+t_n x(w_n))^2 - \|y\|^2}{2t_n} \\ &= x(w_n)y(w_n) + \tfrac{1}{2}t_n|x(w_n)|^2 + \frac{|y(w_n)|^2 - \|y\|^2}{2t_n} \\ &\le x(w_n)y(w_n) + \tfrac{1}{2}t_n\,\|x\|^2 .\end{aligned}$$

The proof of (4.1) is completed when we observe that

$$\begin{aligned}(x,y)_s = \lim_{n\to\infty}\frac{\|x+t_n y\|^2 - \|y\|^2}{2t_n} &\le \lim_{n\to\infty}(x(w_n)y(w_n) + \tfrac{1}{2}t_n\,\|x\|^2) \\ &= x(w)y(w) \le \beta.\end{aligned}$$

Suppose that x, y are two linearly independent vectors in $C(\Omega)$. When we apply Theorem 2.1 with $t > 0$, we get

$$\begin{aligned}\frac{\inf_{u\in\Omega(y)} x(u)y(u)}{\sup_{v\in\Omega}|y(v)|} &\le \frac{\sup_{u\in\Omega(y)} x(u)y(u)}{\sup_{v\in\Omega}|y(v)|} \\ &\le \sup_{v\in\Omega}|x(v)+2ty(v)| - 2t\sup_{v\in\Omega}|y(v)| \\ &\le \frac{\inf_{u\in\Omega(x+2ty)} x(u)(x(u)+2ty(u))}{\sup_{v\in\Omega}|x(v)+2ty(v)|} \\ &\le \frac{\sup_{u\in\Omega(x+2ty)} x(u)(x(u)+2ty(u))}{\sup_{v\in\Omega}|x(v)+2ty(v)|} \\ &\le 2\sup_{v\in\Omega}|x(v)+ty(v)| - \sup_{v\in\Omega}|x(v)+2ty(v)| \\ &\le \frac{\inf_{u\in\Omega(x+ty)} x(u)(x(u)+ty(u))}{\sup_{v\in\Omega}|x(v)+ty(v)|} \\ &\le \frac{\sup_{u\in\Omega(x+ty)} x(u)(x(u)+ty(u))}{\sup_{v\in\Omega}|x(v)+ty(v)|} \\ &\le \sup_{v\in\Omega}|x(v)|.\end{aligned}$$

If $t < 0$, the only change in the above inequalities occurs in the first two lines, which become

$$-\frac{\sup_{u\in\Omega(y)} x(u)y(u)}{\sup_{v\in\Omega}|y(v)|} \le -\frac{\inf_{u\in\Omega(y)} x(u)y(u)}{\sup_{v\in\Omega}|y(v)|} \le \sup_{v\in\Omega}|x(v)+2ty(v)| + 2t\sup_{v\in\Omega}|y(v)|.$$

Example 4.3 For $1 < p < \infty$ let ℓ^p be the Banach space of all vectors $x = (x_j)_{j\in\mathbb{N}}$ with real coordinates such that $= \sum_{j=1}^\infty |x_j|^p$ converges, equipped with the norm $\|x\| = (\sum_{j=1}^\infty |x_j|^p)^{1/p}$. Since ℓ^p is a smooth space, the upper and lower semi-inner products coincide for any pair $x, y \in \ell^p$ of nonzero vectors with the semi-inner product

$$[x,y]_p = \|y\| \left.\frac{d}{dt}\right|_0 \|y+tx\| = \|y\| \sum_{j=1}^\infty \left(\frac{|y_j|}{\|y\|}\right)^{p-1} \operatorname{sgn}(y_j)x_j$$

(see [2, Example 12.1]); $[\cdot,\cdot]_p$ is known to induce the norm of ℓ^p.

Let x, y be two linearly independent vectors in ℓ^p. When we apply Theorem 2.1 with $t > 0$, we obtain the following string of inequalities:

$$\begin{aligned}
\frac{\sum_{j=1}^\infty |y_j|^{p-1}\operatorname{sgn}(y_j)x_j}{\left(\sum_{j=1}^\infty |y_j|^p\right)^{1/q}} &\le \Big(\sum_{j=1}^\infty |x_j+2ty_j|^p\Big)^{1/p} - 2t\Big(\sum_{j=1}^\infty |y_j|^p\Big)^{1/p} \\
&\le \frac{\sum_{j=1}^\infty |x_j+2ty_j|^{p-1}\operatorname{sgn}(x_j+2ty_j)x_j}{\left(\sum_{j=1}^\infty |x_j+2ty_j|^p\right)^{1/q}} \\
&\le 2\Big(\sum_{j=1}^\infty |x_j+ty_j|^p\Big)^{1/p} - \Big(\sum_{j=1}^\infty |x_j+2ty_j|^p\Big)^{1/p} \\
&\le \frac{\sum_{j=1}^\infty |x_j+ty_j|^{p-1}\operatorname{sgn}(x_j+ty_j)x_j}{\left(\sum_{j=1}^\infty |x_j+ty_j|^p\right)^{1/q}} \\
&\le \Big(\sum_{j=1}^\infty |x_j|^p\Big)^{1/p},
\end{aligned}$$

where q is the conjugate index $q = p/(p-1)$ for p. Suppose that $t < 0$. Then the only change in the above inequalities occurs in the first line, which becomes

$$-\frac{\sum_{j=1}^{\infty} |y_j|^{p-1}\operatorname{sgn}(y_j)x_j}{\left(\sum_{j=1}^{\infty}|y_j|^p\right)^{1/q}} \le \Big(\sum_{j=1}^{\infty}|x_j+2ty_j|^p\Big)^{1/p} + 2t\Big(\sum_{j=1}^{\infty}|y_j|^p\Big)^{1/p}.$$

Let us consider the case when X is a general inner product space. For any given pair x, y of linearly independent vectors, Proposition 2.3 describes the convexity and concavity of the mapping $\Psi^p_{x,y}$. For this situation we are able to obtain a refinement of the Schwarz inequality.

Proposition 4.4 *Let $(X, (\cdot,\cdot))$ be an inner product space, x, y a pair of linearly independent vectors in X, and let t_1 and t_2 be defined by equations* (2.9)*.*

(i) *If $t_2 \le a < b$ or $a < b \le t_1$, and if $\eta = \operatorname{sgn}(\frac{1}{2}(a+b))$, then the following inequalities hold:*

$$\begin{aligned}\eta\,(x,y) &\le \int_a^b \frac{\|x\|^2 + t(x,y)}{\|x+ty\|}\,dt\\ &\le \frac{\|y\|}{2}\left\{\frac{2\,\|x\|^2+(a+b)(x,y)}{\|2x+(a+b)y\|} + \frac{1}{2}\left[\frac{\|x\|^2+a(x,y)}{\|x+ay\|} + \frac{\|x\|^2+b(x,y)}{\|x+by\|}\right]\right\}\\ &\le \frac{\|y\|}{2}\left[\frac{\|x\|^2+a(x,y)}{\|x+ay\|} + \frac{\|x\|^2+b(x,y)}{\|x+by\|}\right] \le \|x\|\,\|y\|\,; \qquad (4.3)\end{aligned}$$

(ii) *If $0 < a < b \le t_2$ or $t_1 \le a < b \le 0$, and if η is as above, then*

$$\begin{aligned}\eta\,(x,y) &\le \frac{\|y\|}{b-a}\int_a^b \frac{\|x\|^2+t(x,y)}{\|x+ty\|}\,dt\\ &\le \|y\|\left[\frac{2\,\|x\|^2+(a+b)(x,y)}{\|2x+(a+b)y\|}\right] \le \|x\|\,\|y\|\,. \qquad (4.4)\end{aligned}$$

Proof We note that $t_1 < 0 < t_2$ and that $|\Psi_{x,y}(t)| \le \|x\|$ for all $t \in \mathbb{R}$.

(i) By (2.4) we have $\Psi_{x,y}(t) \ge (x,y)\,\|y\|^{-1}$ for all $t \ge 0$, from which we deduce

$$(x,y) \le \frac{\|y\|}{2}\int_a^b \Psi_{x,y}(t)\,dt \quad \text{if} \quad 0 \le a < b.$$

If we choose a, b in the interval $[t_2, +\infty)$, then the function $\Psi(t) := \Psi_{x,y}(t)$ is convex on $[a, b]$ and Hermite-Hadamard's inequality [10, p.10] can be applied to Ψ:

$$\frac{1}{b-a}\int_a^b \Psi(t)\,dt \le \frac{1}{2}\left[\Psi\Big(\frac{a+b}{2}\Big) + \frac{\Psi(a)+\Psi(b)}{2}\right] \le \frac{\Psi(a)+\Psi(b)}{2} \le \|x\|\,. \quad (4.5)$$

This gives (4.3) with $\eta = 1$. The result for $a, b \in (-\infty, t_1]$ follows from the inequality $\Psi(t) \ge -(x, y)\,\|y\|^{-1}$ (valid for $t \le 0$) gleaned from (2.4).

(ii) follows from Hermite-Hadamard's inequality

$$\frac{1}{b-a}\int_a^b \Psi(t)\,dt \le \Psi\left(\frac{a+b}{2}\right)$$

for concave mappings; we omit the details.

Corollary 4.5 *If $x \perp y$ with x, y nonzero, then we have the inequality*

$$\frac{1}{b-a}\int_a^b \frac{dt}{\|x+ty\|} \le \frac{1}{2}\left\{\frac{1}{\left\|x+\frac{1}{2}(a+b)y\right\|} + \frac{1}{2}\left[\frac{1}{\|x+ay\|} + \frac{1}{\|x+by\|}\right]\right\}$$
$$\le \frac{1}{2}\left[\frac{1}{\|x+ay\|} + \frac{1}{\|x+by\|}\right] \le \frac{1}{\|x\|}$$

if $s_2 \le a < b$ or $a < b \le s_1$, respectively the inequality

$$\frac{1}{b-a}\int_a^b \frac{dt}{\|x+ty\|} \le \frac{1}{\left\|x+\frac{1}{2}(a+b)y\right\|} \le \frac{1}{\|x\|}$$

if $0 \le a < b \le s_2$ or $s_1 \le a < b \le 0$, where

$$s_1 = -\frac{\|x\|}{\sqrt{2}\,\|y\|}, \qquad s_2 = \frac{\|x\|}{\sqrt{2}\,\|y\|}.$$

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