Range projections of idempotents in C^* algebras

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Abstract

In this paper we study range projections of idempotents in C^*-algebras, and use them to obtain a Schur type decomposition that leads to simple proofs of results on Moore–Penrose inverse and norms of idempotents. We analyze the continuity of range projections, obtain a general result on their approximation, and recover a result of Vidav on two projections in a Hilbert space. Several representations of range projections are given.

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1 Range projections

Basic facts about C^*-algebras needed in this paper can be found, for example, in Davidson's monograph [3]. In this paper, $\mathfrak{A}$ is a unital C^*-algebra with unit I. By $\mathfrak{A}^{-1}$ we denote the set of all invertible elements of $\mathfrak{A}$. We recall that $I + A^*A \in \mathfrak{A}^{-1}$ for all $A \in \mathfrak{A}$ and that $\|A^*A\| = \|A\|^2$ (the C^*-identity). An element $A \in \mathfrak{A}$ is *polar* if 0 is at most a pole of the resolvent of A, and *quasipolar* if 0 is an isolated singularity of the resolvent of A. We will need the following characterization of quasipolar elements of $\mathfrak{A}$.

LEMMA 1.1. [9, Theorem 4.2] *An element $A \in \mathfrak{A}$ is quasipolar if and only if there exists an idempotent $P \in \mathfrak{A}$ commuting with A such that AP is quasinilpotent and $A+P \in \mathfrak{A}^{-1}$. Such idempotent is unique, and is called the* spectral idempotent *of A corresponding to 0, written A^π.*

From the preceding lemma it follows that A is polar if and only it is quasipolar and $A^kA^\pi = 0$ for some integer k. In particular, A is simply polar if and only if A is quasipolar and $AA^\pi = 0$.

The word 'projection' will be reserved for an element Q of a C^*-algebra $\mathfrak{A}$ which is self-adjoint and idempotent, that is, $Q^* = Q = Q^2$. A motivation for the concept of the range projection of an idempotent P comes from Hilbert space operator theory, where the range projection of an oblique projection P is the self-adjoint (orthogonal) projection Q onto the range of P. It is easily checked that $QP = P$ and $PQ = Q$. In the context of C^*-algebras, the existence of the range projection of an idempotent P was originally proved by Kaplansky. The formula (1.2) was given in [8] for integral operators on L^2.

DEFINITION 1.2. Let $P \in \mathfrak{A}$ be an idempotent. A *range projection* of P is an element $Q \in \mathfrak{A}$ such that

$$Q^2 = Q = Q^*, \quad PQ = Q, \quad QP = P. \tag{1.1}$$

A range projection of P, if it exists, will be denoted by $P^\perp$.

THEOREM 1.3. *For every idempotent $P \in \mathfrak{A}$ there exists a unique range projection $P^\perp \in \mathfrak{A}$ given explicitly by the formula*

$$P^\perp = P(P + P^* - I)^{-1}, \tag{1.2}$$

and satisfying

$$P^\perp\mathfrak{A} = P\mathfrak{A}. \tag{1.3}$$

Proof. Uniqueness. Suppose that there exists $Q \in \mathfrak{A}$ which satisfies (1.1). Then

$$Q(P + P^* - I) = QP + (PQ)^* - Q = P + Q - Q = P.$$

We observe that

$$(P + P^* - I)^2 = I + (P - P^*)(P - P^*)^*;$$

hence $P + P^* - I$ is invertible, and $Q = P(P + P^* - I)^{-1}$. The equation (1.3) follows from (1.1). This result motivates the following construction.

Existence. Set $S = P + P^* - I$; then $S \in \mathfrak{A}^{-1}$ as above. We verify directly that

$$PS^{-1} = S^{-1}P^*, \quad P^*S^{-1} = S^{-1}P, \quad S^{-1}P^*P = P, \quad PP^*S^{-1} = P;$$

this enables us to show that $Q = PS^{-1}$ is a range projection of P:

$$\begin{aligned} Q^* &= (PS^{-1})^* = S^{-1}P^* = PS^{-1} = Q, \\ Q^2 &= (PS^{-1})(PS^{-1}) = S^{-1}P^*PS^{-1} = PS^{-1} = Q, \\ PQ &= P^2S^{-1} = PS^{-1} = Q, \\ QP &= PS^{-1}P = PP^*S^{-1} = P. \end{aligned}$$

□

In the following proposition we list without proof some useful properties of range projections.

PROPOSITION 1.4. *Let $P \in \mathfrak{A}$ be idempotent. Then*

(i) $I - P^\perp = (I - P^*)^\perp$;

(ii) $I - (P^*)^\perp = (I - P)^\perp$;

(iii) *for any* $A, B \in \mathfrak{A}$, $PA = B \implies P^\perp B = B$;

(iv) *for any* $A, B \in \mathfrak{A}$, $AP = B \implies B(P^*)^\perp = B$.

PROPOSITION 1.5. *Let P, $Q \in \mathfrak{A}$ be idempotents. Then*

$$\|Q^\perp - P^\perp\| \le \max\{\|(I - Q)P\|, \|(I - P)Q\|\} \le \|P\|\|Q\|\|Q^\perp - P^\perp\|, \tag{1.4}$$

$$\|(Q^*)^\perp - (P^*)^\perp\| \le \max\{\|P(I - Q)\|, \|Q(I - P)\|\} \le \|P\|\|Q\|\|(Q^*)^\perp - (P^*)^\perp\|. \tag{1.5}$$

Proof. In view of the Gelfand–Naimark theorem [3, Theorem I.9.12] we assume without a loss of generality that $\mathfrak{A}$ is a norm closed self-adjoint subalgebra of $\mathcal{B}(H)$, the full algebra of bounded linear operators on a Hilbert space H. If $P \in \mathfrak{A}$ is idempotent, then $P^\perp \in \mathcal{B}(H)$ exists and $R(P) = R(P^\perp)$; the crucial fact is that $P^\perp \in \mathfrak{A}$.

By gap (M, N) we denote the gap between two closed subspaces M, N of H [6, Chapter IV]. Then gap $(M, N) = \|E_M - E_N\|$, where E_K denotes the (self-adjoint) projection whose range is the closed subspace K of H. The result then follows from [12, Lemma 3.1] and Proposition 1.4 (ii) when we observe that

$$\begin{aligned}\operatorname{gap}(N(Q), N(P)) &= \operatorname{gap}(R(I-Q), R(I-P)) \\ &= \|(I-Q)^\perp - (I-P)^\perp\| = \|(Q^*)^\perp - (P^*)^\perp\|.\end{aligned}$$

(We have also used the equality $\|I - S\| = \|S\|$ valid for any idempotent $S \in \mathfrak{A}$; see Proposition 4.2.) □

The range projection $P \mapsto P^\perp$ is a continuous function of P in the operator norm. The following theorem gives a complete description.

Theorem 1.6. *Let P_n, $n = 1, 2, \dots$ and P be idempotents in $\mathfrak{A}$. Then*

$$P_n \to P \iff \big[P_n^\perp \to P^\perp \text{ and } (P_n^*)^\perp \to (P^*)^\perp\big]. \tag{1.6}$$

Proof. The forward implication follows from formula (1.2) and the continuity of the mapping $A \mapsto A^{-1}$ in $\mathfrak{A}$. The converse is a consequence of Proposition 1.5 when we observe that $\|P_n - P\| \le \|P_n(I-P)\| + \|(I-P_n)P\|$. □

The following result is an analogue of the Schur decomposition extended to idempotents in a C^*-algebra.

Theorem 1.7. *Let $P \in \mathfrak{A}$ be idempotent. Then there exist Q, $N \in \mathfrak{A}$ such that*

$$P = Q + N, \quad Q^2 = Q = Q^*, \quad NQ = 0, \quad QN = N. \tag{1.7}$$

Such a decomposition is unique, Q is the range projection of P, and N is nilpotent with $N^2 = 0$.

Proof. Suppose a decomposition (1.7) exists. Then

$$PQ = (Q+N)Q = Q, \qquad QP = Q(Q+N) = Q + N = P,$$

that is, Q is the range projection of P. Further, $N^2 = (P-Q)^2 = P-PQ-QP+Q = 0$.

Conversely, let $Q = P^\perp$ be the range projection of P, and let $N = P-Q$. Then (1.7) holds. □

To justify the description 'Schur decomposition' we illustrate the theorem for matrices.

EXAMPLE 1.8. Let P be an idempotent $d\times d$ matrix with complex entries. By the Schur decomposition theorem for matrices, there exists a unitary matrix U such that

$$P = U^{\mathrm{H}} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} U,$$

where A and C are upper triangular, A has unit diagonal and C has zero diagonal. From $P^2 = P$ we obtain $A^2 = A$ and $C^2 = C$. Since C is nilpotent, $C = C^d = 0$. From

$$I - P = U^{\mathrm{H}} \begin{bmatrix} I-A & -B \\ 0 & I \end{bmatrix} U$$

we get $I - A = 0$ by a similar argument. Hence

$$P = U^{\mathrm{H}} \begin{bmatrix} I & B \\ 0 & 0 \end{bmatrix} U = U^{\mathrm{H}} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U + U^{\mathrm{H}} \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} U = Q + N.$$

2 Range projections to G and $I - G$

In this section we consider the problem of determining when two projections in $\mathfrak{A}$ are the range projection of an idempotent G and its complement $I - G$, respectively, and generalize to C^*-algebras the result obtained for bounded linear operators on Hilbert spaces by Vidav [16, Theorem 1].

THEOREM 2.1. *Let P, $Q \in \mathfrak{A}$ be projections. There exists an idempotent $G \in \mathfrak{A}$ such that*

$$P = G^\perp \quad \textit{and} \quad Q = (I-G)^\perp \tag{2.1}$$

if and only if

$$I - PQP \in \mathfrak{A}^{-1} \quad \textit{and} \quad \mathfrak{A} = P\mathfrak{A} + Q\mathfrak{A}. \tag{2.2}$$

The idempotent G is given by the formula

$$G = (I - PQP)^{-1}(P - PQ). \tag{2.3}$$

Proof. Let $G \in \mathfrak{A}$ be an idempotent, and let $P = G^\perp$, $Q = (I-G)^\perp = I - (G^*)^\perp$. From equation (1.2) we find

$$I - PQ = (I + G^*G - G^*)(G + G^* - I)^{-2}.$$

Note that $(G + G^* - I)^2 = I + (G^* - G)(G - G^*)$ commutes with G and G^*. A short calculation then gives

$$(I - PQ)(I + GG^* - G) = I = (I + GG^* - G)(I - PQ);$$

hence $I - PQ \in \mathfrak{A}^{-1}$. From the decomposition

$$I - PQ = P(I - PQP)P - PQ(I - P) + (I - P)$$

with $I - PQ \in \mathfrak{A}^{-1}$ and $I - P$ invertible in the corner algebra $(I-P)\mathfrak{A}(I-P)$ it follows that the element $P(I - PQP)P$ is invertible in the corner algebra $P\mathfrak{A}P$, and from

$$I - PQP = P(I - PQP)P + (I - P)$$

we conclude that $I - PQP \in \mathfrak{A}^{-1}$. The decomposition $\mathfrak{A} = G\mathfrak{A} \oplus (I-G)\mathfrak{A}$ and equation (1.3) imply the second equation in (2.2).

Conversely, assume that $P, Q \in \mathfrak{A}$ are projections satisfying (2.2). Define G by equation (2.3) and observe that P commutes with $I - PQP$ and $(I - PQP)^{-1}$. Then

$$\begin{aligned} G^2 &= (I - PQP)^{-1}(P - PQ)P(I - PQP)^{-1}(P - PQ) \\ &= (I - PQP)^{-1}(I - PQP)P(I - PQP)^{-1}(P - PQ) = G. \end{aligned}$$

Hence $G \in \mathfrak{A}$ is idempotent, and from the definition of G we deduce that

$$PG = G, \qquad GP = P, \qquad GQ = 0. \tag{2.4}$$

By the second equation in (2.2) there exist elements A and B in $\mathfrak{A}$ such that $I - G = PA + QB$. Multiplying this equation by G from the left we get $0 = GPA + GQB = PA$ in view of (2.4). Then $I - G = QB$, and $Q(I - G) = Q(QB) = I - G$. Combining this with the last equation in (2.4) we obtain

$$(I - G)Q = Q \quad \text{and} \quad Q(I - G) = I - G. \tag{2.5}$$

Equations (2.4) and (2.5) imply that $P = G^\perp$ and $Q = (I - G)^\perp$. $\square$

We now recover a result of Vidav [16, Theorem 1] originally proved for linear operators on a Hilbert space.

Corollary 2.2. *Let* P, $Q \in \mathfrak{A}$ *be projections. There exists an idempotent* $G \in \mathfrak{A}$ *satisfying* (2.1) *if and only if*

$$\|PQ\| < 1 \quad \text{and} \quad \mathfrak{A} = P\mathfrak{A} + Q\mathfrak{A}. \tag{2.6}$$

Proof. If P and Q are projections, we show that

$$I - PQP \in \mathfrak{A}^{-1} \iff \|PQ\|^2 = \|PQP\| < 1.$$

First we note that PQP is positive and that $\|PQ\|^2 = \|PQ(PQ)^*\| = \|PQP\|$. The equivalence then follows from the inclusion $\sigma(PQP) \subset [0, \|PQP\|]$ and from the fact that $\|PQP\| \in \sigma(PQP)$. The preceding theorem does the rest. □

Remark 2.3. Vidav [16] showed that if $\mathfrak{A} = \mathcal{B}(H)$, the projections $P = G^\perp$ and $Q = (I - G)^\perp$ satisfy

$$\|PQ\| \le \frac{\|G\|}{\sqrt{1 + \|G\|^2}} < 1. \tag{2.7}$$

The proof hinges on spatial arguments. Using Gelfand–Naimark theorem, we note that this estimate is also valid in C^*-algebras. It would be interesting to find a purely algebraic proof of (2.7).

3 Applications to Moore–Penrose inverse

Recall that a *Moore–Penrose inverse* of $A \in \mathfrak{A}$ is an element $B \in \mathfrak{A}$ such that

$$BAB = B, \quad ABA = A, \quad (AB)^* = AB, \quad (BA)^* = BA. \tag{3.1}$$

A Moore–Penrose inverse of A, if it exists, is unique (see Penrose [13]), and will be denoted by $A^\dagger$. For a discussion of the Moore–Penrose inverse in the setting of C^*-algebras see [4, 5].

Range projections can be used as a universal tool in the study of the Moore–Penrose inverse in C^*-algebras, providing a particularly straight path to the inverse and yielding very simple proofs of various facts. For the purpose of such exposition we assume only the above definition of the Moore–Penrose inverse and the fact that any element of $\mathfrak{A}$

can have at most one Moore–Penrose inverse. Recall that $A \in \mathfrak{A}$ is *regular* if there exists $B \in \mathfrak{A}$ such that $ABA = A$. The following theorem is one of the basic results of the theory (see [4, 10]). A proof based on range projections is simpler and more transparent than most proofs found in the literature.

THEOREM 3.1. *Let $A \in \mathfrak{A}$. Then $A^\dagger$ exists if and only if A is regular. If $B \in \mathfrak{A}$ is such that $ABA = A$, then*

$$A^\dagger = ((BA)^*)^\perp B (AB)^\perp. \tag{3.2}$$

Proof. The existence of the Moore–Penrose inverse clearly implies regularity.

Conversely, let $ABA = A$; initially we assume that $BAB = B$. Write $P = BA$ and $Q = AB$. Then P and Q are idempotent, and $AP = A$, $QA = A$. By Proposition 1.4,

$$A(P^*)^\perp = A, \qquad Q^\perp A = A.$$

We set $C = (P^*)^\perp B Q^\perp$, and verify that C satisfies conditions (3.1). Note that $Q^\perp$ and $(P^*)^\perp$ are self-adjoint:

$$\begin{aligned} AC &= A(P^*)^\perp B Q^\perp = QQ^\perp = Q^\perp, \\ CA &= (P^*)^\perp B Q^\perp A = (P^*)^\perp P = (P^*)^\perp, \\ ACA &= (AC)A = Q^\perp A = A, \\ CAC &= (CA)C = (P^*)^\perp (P^*)^\perp B Q^\perp = C. \end{aligned}$$

If $BAB = B$ is not satisfied, write $\widetilde{B} = BAB$. Then $A\widetilde{B}A = A$, $\widetilde{B}A\widetilde{B} = \widetilde{B}$, $A\widetilde{B} = AB$ and $\widetilde{B}A = BA$. The preceding result applies to $\widetilde{B}$ in place of B, and the conclusion and the formula for $A^\dagger$ remain the same. □

PROPOSITION 3.2. *If $P \in \mathfrak{A}$ is idempotent, then P is regular, $P^\dagger$ exists, and*

$$P^\dagger = (P^*)^\perp P^\perp, \qquad P^\perp = PP^\dagger.$$

Proof. The formula for $P^\dagger$ follows from the preceding theorem. Then

$$PP^\dagger = P(P^*)^\perp P^\perp = ((P^*)^\perp P^*)^* P^\perp = (P^*)^* P^\perp = PP^\perp = P^\perp. \qquad \square$$

We generalize a result of Kato [6, Theorem I.6.35] to C^*-algebras, giving an alternative proof.

Proposition 3.3. *Let P, $Q \in \mathfrak{A}$ be idempotents. Then*

$$\|Q^\perp - P^\perp\| \le \|Q - P\|. \tag{3.3}$$

Proof. By the Kato–Moriya inequality [7],

$$\|QQ^\dagger - PP^\dagger\| \le \max\{\|Q^\dagger\|, \|P^\dagger\|\}\|Q - P\|.$$

If $S \in \mathfrak{A}$ is an idempotent, then by Proposition 3.2, $S^\perp = SS^\dagger$ and $\|S^\dagger\| = \|(S^*)^\perp S^\perp\| \le \|(S^*)^\perp\|\|S^\perp\| \le 1$. This completes the proof. □

Proposition 3.4. *Let $A \in \mathfrak{A}$. Then $A^\dagger$ exists if and only if there exist an idempotent $P \in \mathfrak{A}$ and $B \in \mathfrak{A}$ such that $AP = 0$ and $BA = I - P$. The Moore–Penrose inverse of A is given by*

$$A^\dagger = (I - P^\perp)B(AB)^\perp. \tag{3.4}$$

Proof. Suppose that P and B with the specified properties exist. From $AP = 0$ it follows that $AP^\perp = 0$, and $ABA = A(I - P) = A$. By the preceding theorem $A^\dagger$ exists, and (3.2) holds. Observe that $P = I - BA$ and $I - P^\perp = (I - P^*)^\perp = ((BA)^*)^\perp$ in view of Proposition 1.4.

Conversely, suppose that $A^\dagger$ exists. Then A is regular by Theorem 3.1, and $ABA = A$ for some $B \in \mathfrak{A}$. The result follows when we set $P = I - BA$. □

If we specialize the preceding theorem to the C^*-algebra $\mathfrak{A} = \mathcal{B}(H)$ of all bounded linear operators on a Hilbert space H, we recover [2, Lemma 2]; note that the hypothesis that the range of A is closed made in [2] is redundant, as the existence of $A^\dagger$ follows from other assumptions.

If A is self-adjoint, we have a simple condition for the existence of the Moore–Penrose inverse [10].

Proposition 3.5. *If $A \in \mathfrak{A}$ is self-adjoint, then $A^\dagger$ exists if and only if A is simply polar. In this case $A^\dagger$ is self-adjoint, commutes with A, and $A^\pi = I - A^\dagger A$.*

Proof. Suppose that A is self-adjoint and simply polar. The spectral idempotent $P = A^\pi$ of A is self-adjoint, and $P^\perp = P$. Set $B = (A+P)^{-1}(I-P) = (I-P)(A+P)^{-1}$. Then $AP = 0$ and $BA = I - P$. Hence $A^\dagger$ exists by the preceding corollary, and

$$A^\dagger = (A + A^\pi)^{-1}(I - A^\pi) = (I - A^\pi)(A + A^\pi)^{-1}. \tag{3.5}$$

From (3.5) it follows that $A^\dagger$ is self-adjoint and commutes with A.

Conversely, suppose that A is self-adjoint and that $A^\dagger$ exists. From (3.1) and the uniqueness of $A^\dagger$ we deduce that $A^\dagger$ is also self-adjoint and that A and $A^\dagger$ commute. Set $P = I - AA^\dagger$. Then we can verify that P is idempotent, commutes with A and satisfies $AP = 0$. According to Lemma 1.1, P is the spectral idempotent of A when we show that $A + P \in \mathfrak{A}^{-1}$. But this follows from

$$(A + P)(A^\dagger + P) = I. \qquad \square$$

The preceding result suggests a simple proof of the following known characterization [10] of the existence of $A^\dagger$.

THEOREM 3.6. *An element $A \in \mathfrak{A}$ has the Moore–Penrose inverse if and only if A^*A (respectively AA^*) is simply polar. If this is the case,*

$$A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger. \tag{3.6}$$

Proof. Assume that A^*A is simply polar. By the preceding proposition, $C = A^*A$ has the Moore–Penrose inverse $C^\dagger$, a self-adjoint element. We verify that $B = C^\dagger A^*$ is the Moore–Penrose inverse of A:

$$\begin{aligned} AB &= AC^\dagger A^* \quad \text{is self-adjoint,} \\ BA &= C^\dagger C \qquad \text{is self-adjoint,} \\ BAB &= C^\dagger CC^\dagger A^* = C^\dagger A^* = B, \\ ABA &= AC^\dagger C = A; \end{aligned}$$

the last equality follows when we set $U = A(I - C^\dagger C)$ and calculate

$$U^*U = (I - C^\dagger C)A^*A(I - C^\dagger C) = (I - C^\dagger C)(C - CC^\dagger C) = 0.$$

Conversely, assume that $A^\dagger$ exists. A direct verification shows that $A^\dagger(A^\dagger)^*$ is the Moore–Penrose inverse of the self-adjoint element A^*A. Hence A^*A is simply polar by Proposition 3.5.

The statement about AA^* is obtained by writing $AA^* = (A^*)^*A^*$ and applying the preceding result to A^* in place of A. $\square$

We observe that (3.5) and (3.6) yield an alternative formula for $A^\dagger$:

$$A^\dagger = (A^*A + (A^*A)^\pi)^{-1}A^* = A^*(AA^* + (AA^*)^\pi)^{-1}. \tag{3.7}$$

Equation (3.7) helps us to get new necessary and sufficient conditions for the continuity of the Moore–Penrose inverse—see conditions (ii) and (iii) of the following theorem.

THEOREM 3.7. *Let $A, A_n \in \mathfrak{A}$, $n = 1, 2 \ldots,$ be elements which possess the Moore–Penrose inverse. Then the following conditions are equivalent*:

(i) $A_n \to A$ *and* $\|{A_n}^\dagger\|$ *are bounded*;

(ii) $A_n \to A$ *and* $(A_n A_n^*)^\pi \to (AA^*)^\pi$;

(iii) $A_n \to A$ *and* $(A_n^* A_n)^\pi \to (A^*A)^\pi$;

(iv) $A_n \to A$ *and* ${A_n}^\dagger A_n \to A^\dagger A$;

(v) $A_n \to A$ *and* $A_n {A_n}^\dagger \to AA^\dagger$;

(vi) ${A_n}^\dagger \to A^\dagger$ *and* $\|A_n\|$ *are bounded*;

Proof. The equivalence of conditions (iv) and (v) to (vi) and the implication (i) $\Longrightarrow$ (vi) can be found in [4, 14]. The equivalence of (ii) and (iii) to (vi) is a consequence of (3.7) and of the equations $(A^*A)^\pi = I - A^\dagger A$ and $(AA^*)^\pi = I - AA^\dagger$. The implication (vi) $\Longrightarrow$ (i) follows from (i) $\Longrightarrow$ (vi) and the identity $(B^\dagger)^\dagger = B$. □

PROPOSITION 3.8. *If $P_n \in \mathfrak{A}$, $n = 1, 2, \ldots,$ are idempotent, then*

$$P_n \to P \implies P_n^\dagger \to P^\dagger.$$

Hence the Moore–Penrose inverse is continuous on the set of all idempotents in $\mathfrak{A}$.

Proof. The limit of idempotents is an idempotent. Recall that $P^\dagger = (P^*)^\perp P^\perp$ for any idempotent $P \in \mathfrak{A}$. The result then follows by Theorem 1.6. □

A more complete discussion of the continuity of the Moore–Penrose inverse in C^*-algebras is given, for instance, in [5, 11, 14].

4 Application to norms of idempotents

The paper [1] contains a wealth of information about idempotent operators in Hilbert spaces, with many of the results algebraic, involving two idempotents, many dealing with operator norms. In this section we propose a unified approach to deriving and proving results involving norms of idempotents. This approach utilizes the Schur decomposition (1.7) for idempotents and the equation

$$\|A\| = r(A^*A)^{1/2} = r(AA^*)^{1/2}, \tag{4.1}$$

where $r(T)$ is the spectral radius of $T \in \mathfrak{A}$. Though our proofs are generally considerably shorter than those in [1], they require a certain level of spectral sophistication. In particular, we need the following result that may be deduced from [15, Theorem 1.6.15].

LEMMA 4.1. *Suppose $A \in \mathfrak{A}$ commutes with an idempotent $Q \in \mathfrak{A}$. Then*

$$\sigma(A) \cup \{0\} = \sigma(AQ) \cup \sigma(A(I-Q)).$$

The following result is the culmination of Propositions 1.3–1.6 in [1].

PROPOSITION 4.2. [1, Proposition 1.7] *Let $P \in \mathfrak{A}$ be idempotent. Then*

$$\|P\| = \|I-P\| = \|P+P^*-I\|. \tag{4.2}$$

Proof. Let $P = Q+N$ be the decomposition (1.7). Then

$$\begin{aligned} PP^* = (Q+N)(Q+N^*) = Q+NN^* &= (I+NN^*)Q, \\ (I-P^*)(I-P) = I-Q+N^*N &= (I+N^*N)(I-Q), \\ (P+P^*-I)^2 &= I+NN^*+N^*N. \end{aligned}$$

We observe that the elements on the right in the preceding display commute with Q, which enables us to use Lemma 4.1:

$$\begin{aligned} \sigma((I-P)^*(I-P)) &= \sigma((I+N^*N)(I-Q)) = \sigma(I+N^*N) \cup \{0\} \\ &= \sigma(I+NN^*) \cup \{0\} = \sigma((I+NN^*)Q) \\ &= \sigma(PP^*); \end{aligned}$$

$$\begin{aligned} \sigma((P+P^*-I)^2) \cup \{0\} &= \sigma((I+NN^*)Q) \cup \sigma((I+N^*N)(I-Q)) \\ &= \sigma(PP^*). \end{aligned}$$

The result follows on application of (4.1). □

PROPOSITION 4.3. [1, Corollary 1.10] *Let $P \in \mathfrak{A}$ be idempotent. Then*

$$\|P^\perp - P\| = \|P^* - P\|. \tag{4.3}$$

Proof. Let again $P = Q + N$ be the decomposition (1.7). Then $P^{\perp} = Q$,

$$U = (Q - P)(Q - P)^* = NN^*,$$
$$V = (P - P^*)(P - P^*)^* = -(N - N^*)^2 = NN^* + N^*N,$$

and

$$\sigma(V) \cup \{0\} = \sigma(NN^* + N^*N) = \sigma(NN^*Q + N^*N(I - Q))$$
$$= \sigma(NN^*) \cup \sigma(N^*N) = \sigma(NN^*) = \sigma(U)$$

by Lemma 4.1. The result again follows by (4.1). □

5 Representations of the range projection

Throughout this section, $P \in \mathfrak{A}$ is an idempotent. Define

$$\mu = \mu(PP^*) = \inf\, \sigma(PP^*) \setminus \{0\}, \tag{5.1}$$
$$\nu = \nu(PP^*) = \sup\, \sigma(PP^*) = \|P\|^2. \tag{5.2}$$

We recall (Proposition 3.5) that 0 is isolated in $\sigma(PP^*)$. Hence $\mu > 0$ and

$$\sigma(PP^*) \subset \{0\} \cup [\mu, \nu]. \tag{5.3}$$

We can then define a function $f : \sigma(PP^*) \to \mathbb{C}$ such that

$$f(0) = 0, \qquad f(\sigma(PP^*) \setminus \{0\}) = 1. \tag{5.4}$$

Observe that $f \in C(\sigma(PP^*))$. In the present section, the letter f is reserved for this particular function.

Before we present our main result, we recall representations of the range projection we encountered previously:

$$P^{\perp} = P(P + P^* - I)^{-1} = (P + P^* - I)^{-1}P^*, \tag{5.5}$$
$$P^{\perp} = PP^*(I - (P - P^*)^2)^{-1} = (I - (P - P^*)^2)^{-1}PP^*, \tag{5.6}$$
$$P^{\perp} = PP^{\dagger}; \tag{5.7}$$

(5.6) follows from (5.5) on remembering that $(P^{\perp})^2 = P^{\perp} = (P^{\perp})^*$.

Proposition 5.1. *The range projection $P^{\perp}$ of P is given by*

$$P^{\perp} = I - (PP^*)^{\pi} = f(PP^*), \tag{5.8}$$

where $(PP^*)^{\pi}$ is the spectral idempotent of PP^* and f is defined by (5.4).

Proof. By Proposition 3.2, $P^{\perp} = PP^{\dagger}$, and by Proposition 3.5, $PP^{\dagger} = I - (PP^*)^{\pi}$. The equation $I - (PP^*)^{\pi} = f(PP^*)$ follows from the Gelfand–Naimark calculus for PP^* and from functional representation of a spectral idempotent. □

The following theorem is our main result on representation of the range projection, which generates a number of useful concrete representations.

Theorem 5.2. *Let (f_α) be a net in $C(\sigma(PP^*))$ convergent uniformly on $\sigma(PP^*)$ to the function f defined by* (5.4). *Then*

$$\lim_{\alpha} \|f_\alpha(PP^*) - P^{\perp}\| = 0. \tag{5.9}$$

Proof. Follows from Proposition 5.1 and from the continuity of the Gelfand–Naimark calculus. □

Example 5.3. $P^{\perp} = \lim_{\alpha\to 0}(PP^* + \alpha I)^{-1}PP^*$.

The net $f_\alpha(\lambda) = (\lambda+\alpha)^{-1}\lambda$ converges to f uniformly on the set $\{0\}\cup[\mu,\nu] \supset \sigma(PP^*)$ as $\alpha \to 0$.

Example 5.4. $P^{\perp} = \sum_{k=0}^{\infty}(I - \beta PP^*)^k \beta PP^* = I - \lim_{n\to\infty}(I - \beta PP^*)^{n+1}$, where $0 < \beta < 2\|P\|^{-2}$.

Let $\lambda \in [\mu,\nu]$ and let $0 < \beta < 2/\|P\|^2 = 2/\nu$. Then

$$|\beta\lambda - 1| \le \max\{\beta\nu - 1, 1 - \beta\mu\} < 1.$$

The (discrete) net $f_\alpha(\lambda) = 1 - (1 - \beta\lambda)^{\alpha+1}$ converges to f uniformly on $\{0\} \cup [\mu,\nu] \supset \sigma(PP^*)$ as $\alpha \to \infty$. The fastest convergence is obtained when $\beta\nu - 1 = 1 - \beta\mu$, that is, when $\beta = 2/(\nu+\mu)$. With this choice of β,

$$\begin{aligned}\|P^{\perp} - f_\alpha(PP^*)\| &= \|(f - f_\alpha)(PP^*)\| = \sup\{|1-\beta\lambda|^{\alpha+1} : \lambda \in [\mu,\nu]\} \\ &= |1 - \beta\mu|^{\alpha+1} = \rho^{\alpha+1},\end{aligned}$$

where $\rho = (\nu - \mu)/(\nu + \mu)$.

Specializing this result to bounded linear operators on a Hilbert space H, we recover [1, Proposition 1.1].

EXAMPLE 5.5. $P^{\perp} = \int_0^{\infty} \exp(-tPP^*)PP^*\,dt = I - \lim_{\alpha\to\infty} \exp(-\alpha PP^*)$.

For $\alpha > 0$ set $f_\alpha(\lambda) = \int_0^{\alpha} e^{-t\lambda}\lambda\,d\lambda = 1 - e^{-\lambda\alpha}$. Then $f_\alpha(0) = 0$ for all α, and $f_\alpha(\lambda)$ converges to 1 uniformly on $[\mu, \nu]$ as $\alpha \to \infty$; consequently f_α converges to f uniformly on $\{0\} \cup [\mu, \nu] \supset \sigma(PP^*)$.

EXAMPLE 5.6. Boulmaarouf, Fernandez and Labrousse [1, Proposition 1.11] described a fast convergent recursive algorithm for the calculation of $P^{\perp}$ when P is an idempotent bounded linear operator on a Hilbert space H.

Let $P \in \mathfrak{A}$ be an idempotent, and let $G(\beta) = P + \beta PP^*(I - P)$ for some $\beta > 0$. It is shown in [1] that the value of β that minimizes $\|G^*(\beta) - G(\beta)\|$ is equal to $\beta = \varphi(\|P\|^2)$, where $\varphi(t) = 4/(3t^2+1)$. For any idempotent $Q \in \mathfrak{A}$ set $\Phi(Q) = Q+\varphi(\|Q\|^2)QQ^*(I-Q)$. It is further shown in [1] that the recursively defined sequence

$$Q_{\alpha+1} = \Phi(Q_\alpha), \qquad Q_0 = P,$$

converges to $P^{\perp}$ with

$$\|Q_\alpha - P^{\perp}\| = \frac{2\kappa^{3^n/2}}{1 - \kappa^{3^n}}, \quad \text{where } \kappa = (\|P\| - 1)/(\|P\| + 1).$$

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