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Integral Equations and Operator Theory

The σg-Drazin inverse and the generalized Mbekhta decomposition

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Abstract. In this paper we define and study an extension of the g-Drazin for elements of a Banach algebra and for bounded linear operators based on an isolated spectral set rather than on an isolated spectral point. We investigate salient properties of the new inverse and its continuity, and illustrate its usefulness with an application to differential equations. Generalized Mbekhta subspaces are introduced and the corresponding extended Mbekhta decomposition gives a characterization of circularly isolated spectral sets.

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1. Isolated spectral sets

The g-Drazin inverse in a unital Banach algebra $\mathcal{A}$ is a useful generalization of the ordinary inverse in the case when $a \in \mathcal{A}$ is quasipolar, that is when 0 is not an accumulation point of the spectrum of a (see [8]). The g-Drazin inverse finds many applications, in particular to singular differential equations in the case of operator algebras.

The special form of a is a limitation on this versatile concept. In many situation a similar generalized inverse can be defined even in the case when the element is not quasipolar. The first mention of this generalized concept appeared in [9]; it was subsequently used by Tran in [16] applied to differential equations.

In this paper we define and study such a generalization of the g-Drazin of an element of Banach algebra based on isolated spectral sets rather than isolated spectral points. For any $a \in \mathcal{A}$, $\mathsf{Sp}(a)$, $r(a)$ and $\mathsf{Res}(a)$ denote the spectrum, spectral radius and resolvent set of a relative to the algebra $\mathcal{A}$, respectively. We write $R(\lambda; a) = (\lambda - a)^{-1}$ for the resolvent of a. We write $\mathsf{Sp}(\mathcal{B}; a)$ for the spectrum of $a \in \mathcal{B}$ with respect to a subalgebra $\mathcal{B}$ of $\mathcal{A}$; we attach a similar meaning to $\mathsf{Res}(\mathcal{B}; a)$. The group of invertibles in $\mathcal{A}$ will be denoted by $\mathcal{A}^{\mathsf{inv}}$.

First we recall the following well known concept.

Definition 1.1. A (possibly empty) subset σ of $\mathsf{Sp}(a)$ is called an *isolated spectral set* of a if σ and $\mathsf{Sp}(a) \setminus \sigma$ are both closed. When σ is an isolated spectral set, $\mathsf{Sp}(a) \setminus \sigma$ is also an isolated spectral set called the isolated spectral set *complementary* to σ.

Isolated spectral sets were introduced and studied by Dunford and Schwartz in [3], where they are called spectral sets. We prefer the term isolated spectral sets to distinguish them from spectral sets in the sense of von Neumann. Equivalently, σ is an isolated spectral subset of a if there exist disjoint open sets $\Delta \supset \sigma$ and $\Omega \supset \mathsf{Sp}(a) \setminus \sigma$. This is especially useful in applications of the holomorphic calculus to the element a. The element $p = f(a)$, where f is the characteristic function of Δ in $\Delta \cup \Omega$, is called the *spectral idempotent* of a corresponding to σ. If $\sigma = \emptyset$, then $p = 0$; if $\sigma = \mathsf{Sp}(a)$, then $p = 1$. When dealing with a set σ in the complex plane, we use the symbol $|\sigma|$ for the set $\{|\lambda| : \lambda \in \sigma\}$, and σ^n for the set $\{\lambda^n : \lambda \in \sigma\}$.

If p is an idempotent in $\mathcal{A}$, then $p\mathcal{A}p$ is a closed subalgebra of $\mathcal{A}$ with the unit p. If $ap = pa$, then

$$\mathsf{Sp}(p\mathcal{A}p; ap) \cup \{0\} = \mathsf{Sp}(ap), \quad (p \neq 1) \tag{1.1}$$

(see, for instance, [8, Lemma 2.3]). In this paper we will consistently use the notation

$$p' = 1 - p$$

for the complementary idempotent to p. It is well known that

$$\mathsf{Sp}(a) = \mathsf{Sp}(p\mathcal{A}p; ap) \cup \mathsf{Sp}(p'\mathcal{A}p'; ap'). \tag{1.2}$$

Indeed, if u and v are the inverses of $\lambda p - ap$ and $\lambda p' - ap'$ in the algebras $p\mathcal{A}p$ and $p'\mathcal{A}p'$, respectively, then $u + v$ is the inverse of $\lambda - a$ in $\mathcal{A}$. Conversely, if w is the inverse of $\lambda - a$ in $\mathcal{A}$, then w commutes with p and p', and wp, wp' are the inverses of $\lambda p - ap$, $\lambda p' - ap'$ in the appropriate algebras. This proves the equation

$$\mathsf{Res}(a) = \mathsf{Res}(p\mathcal{A}p; ap) \cap \mathsf{Res}(p'\mathcal{A}p'; ap'), \tag{1.3}$$

which is equivalent to (1.2), and shows that

$$R(\lambda; a) = R_p(\lambda; ap) + R_{p'}(\lambda; ap'), \tag{1.4}$$

where R_p and $R_{p'}$ are the resolvents in $p\mathcal{A}p$ and $p'\mathcal{A}p'$, respectively.

We give a characterization of isolated spectral sets of $a \in \mathcal{A}$, and a characterization of spectral idempotents which does not require functional calculus (though the proof does).

Theorem 1.2. *Let $a \in \mathcal{A}$. A set $\sigma \subset \mathbb{C}$ is an isolated spectral set of a if and only if there exists an idempotent $p \in \mathcal{A}$ commuting with a such that*

$$\mathsf{Sp}(p\mathcal{A}p; ap) \cap \mathsf{Sp}(p'\mathcal{A}p'; ap') = \emptyset \quad \textit{and} \quad \mathsf{Sp}(p\mathcal{A}p; ap) = \sigma. \tag{1.5}$$

In this case p is the spectral idempotent of a corresponding to σ, and p' the spectral idempotent corresponding to $\tau = \mathsf{Sp}(a) \setminus \sigma$.

Proof. Let σ be an isolated spectral set of a with the spectral idempotent p and the complementary isolated spectral set τ. We show that $\mathsf{Sp}(p\mathcal{A}p; ap) \subset \sigma$: Let $\xi \notin \sigma$. Choose disjoint open sets Δ and Ω such that $\sigma \subset \Delta$, $\xi \notin \Delta$ and $\tau \subset \Omega$. Define h on $\Delta \cup \Omega$ by setting $h(\lambda) = (\xi - \lambda)^{-1}$ if $\lambda \in \Delta$, and $h(\lambda) = 0$ if $\lambda \in \Omega$. Let f be the characteristic function of Δ in $\Delta \cup \Omega$; then $p = f(a)$. By the holomorphic calculus, $(\xi p - ap)h(a) = f(a) = p$. Note that $h(a) \in p\mathcal{A}p$ as $h(a)p = h(a)$. Since everything in sight commutes, $\xi p - ap$ is invertible in $p\mathcal{A}p$. Thus $\mathsf{Sp}(p\mathcal{A}p; a) \subset \sigma$. Since p' is the spectral idempotent for τ, the preceding argument yields $\mathsf{Sp}(p'\mathcal{A}p'; ap') \subset \tau$. From (1.2) and $\sigma \cap \tau = \emptyset$,

$$\sigma = \mathsf{Sp}(a) \cap \sigma = \mathsf{Sp}(p\mathcal{A}p; ap) \cap \sigma = \mathsf{Sp}(p\mathcal{A}p; ap).$$

Similarly, $\tau = \mathsf{Sp}(p'\mathcal{A}p'; ap')$. This proves (1.5).

Conversely assume that p is an idempotent satisfying (1.5). Then we have $\mathsf{Sp}(p\mathcal{A}p; ap) = \sigma$, and $\tau = \mathsf{Sp}(p'\mathcal{A}p'; ap')$ are compact disjoint sets satisfying $\mathsf{Sp}(a) = \sigma \cup \tau$ in view of (1.2). Hence σ, τ are isolated spectral sets of a. To prove that p is the spectral idempotent of a corresponding to σ, choose disjoint open sets $\Delta \supset \sigma$ and $\Omega \supset \tau$. If f is the characteristic function of Δ in $\Delta \cup \Omega$, then in view of (1.4),

$$f(a) = f_p(ap) + f_{p'}(ap') = f_p(ap) = 1_{p\mathcal{A}p} = p,$$

where f_p and $f_{p'}$ denote the application of f in $p\mathcal{A}p$ and $p'\mathcal{A}p'$, respectively. □

Let us comment on two special cases of the preceding theorem. If $p = 0$, then $p\mathcal{A}p = \{0\}$, $1 = 0$, and every element in $p\mathcal{A}p$ is invertible with the inverse 0. Then $p' = 1$, and equation (1.5) is fulfilled with $\sigma = \emptyset$ and $\mathsf{Sp}(p'\mathcal{A}p'; ap') = \mathsf{Sp}(a)$. If $p = 1$, then (1.5) holds with $\sigma = \mathsf{Sp}(a)$ and $\mathsf{Sp}(p'\mathcal{A}p'; ap') = \mathsf{Sp}(\{0\}; 0) = \emptyset$.

The case $\sigma = \{0\}$ is described in the following corollary. Recall that $a \in \mathcal{A}$ is quasipolar if 0 is not an accumulation point of the spectrum of a.

Corollary 1.3. *An element $a \in \mathcal{A}$ is quasipolar if and only if there exists an idempotent $p \in \mathcal{A}$ commuting with a such that*

$$\mathsf{Sp}(ap) = \{0\} \quad \textit{and} \quad ap' + \xi p \in \mathcal{A}^{\mathsf{inv}} \textit{ for some } \xi \neq 0. \tag{1.6}$$

Proof. An element $a \in \mathcal{A}$ is invertible if and only if (1.6) holds with $p = 0$. By Theorem 1.2, $\sigma = \{0\}$ is an isolated spectral set of a if and only if there exists an idempotent p ($\neq 0$) commuting with a such that

$$\mathsf{Sp}(p\mathcal{A}p; ap) = \{0\} \quad \text{and} \quad 0 \notin \mathsf{Sp}(p'\mathcal{A}p'; ap').$$

The second condition says that ap' is invertible in $p'\mathcal{A}p'$. It is not difficult to show that this happens if and only if $ap' + \xi p$ is invertible in $\mathcal{A}$ for some $\xi \neq 0$. □

When we observe that $ap' + \xi p \in \mathcal{A}^{\mathsf{inv}}$ if and only if $a + \xi p \in \mathcal{A}^{\mathsf{inv}}$ for some $\xi \neq 0$ (under the assumption $\mathsf{Sp}(ap) = \{0\}$), we recover [7, Theorem 4.2].

The characterization of isolated spectral sets given in Theorem 1.2 goes outside the Banach algebra $\mathcal{A}$, as it depends on the spectra of elements in the algebras $p\mathcal{A}p$ and $p'\mathcal{A}p'$. The following result characterizes isolated spectral sets in terms of the Banach algebra $\mathcal{A}$ alone (see also [8, Theorem 3.2]).

Theorem 1.4. *Let $a \in \mathcal{A}$. A set $\sigma \subset \mathbb{C}$ is an isolated spectral set of a if and only if there exists an idempotent $p \in \mathcal{A}$ commuting with a such that the following conditions are satisfied*:

(i) $$\sigma = \begin{cases} \mathsf{Sp}(ap) & \text{if } ap + p' \notin \mathcal{A}^{\mathsf{inv}}, \\ \mathsf{Sp}(ap) \setminus \{0\} & \text{if } ap + p' \in \mathcal{A}^{\mathsf{inv}}; \end{cases}$$

(ii) *there exists $\xi \in \mathbb{C}$ such that $a - \mu - \xi p \in \mathcal{A}^{\mathsf{inv}}$ for all $\mu \in \sigma$.*

Proof. First we check that the formula in condition (i) describes the spectrum of ap in the algebra $p\mathcal{A}p$. This follows from (1.1) and the observation that $ap \in (p\mathcal{A}p)^{\mathsf{inv}}$ if and only if $ap + p' \in \mathcal{A}^{\mathsf{inv}}$.

Suppose first that σ is an isolated spectral set of a with the spectral idempotent p and the complementary spectral set τ. Then p commutes with a, and (i) holds by Theorem 1.2. Let $r = \sup\{|\lambda| : \lambda \in \sigma\}$, let ξ be a complex number satisfying $|\xi| > 2r$, and let $\mu \in \sigma$. Choose disjoint open sets $\Delta \supset \sigma$ and $\Omega \supset \tau$, and define h by $h(\lambda) = \lambda - \mu - \xi$ if $\lambda \in \Delta$, and $h(\lambda) = \lambda - \mu$ if $\lambda \in \Omega$. Then

$h(a) = a - \mu - \xi p$. We show that $h(\lambda) \neq 0$ for all $\lambda \in \mathsf{Sp}(a)$. Let $\lambda \in \sigma$. Then $|h(\lambda)| \geq |\xi| - |\mu + \lambda| > 0$ as $|\xi| > 2r$. If $\lambda \in \tau$, then $h(\lambda) = \lambda - \mu$ as $\sigma \cap \tau = \emptyset$. Hence $h(a)$ is invertible, which proves (ii).

Conversely, assume that an idempotent p commutes with a and satisfies conditions (i)–(ii). Condition (i) is equivalent to $\mathsf{Sp}(p\mathcal{A}p; ap) = \sigma$. For every $\mu \in \sigma$, $\xi p + \mu - a \in \mathcal{A}^{\mathsf{inv}}$. Then $\mu p' - ap' = (\xi p + \mu - a)p'$ is invertible in $p'\mathcal{A}p'$, which shows that $\mathsf{Sp}(p'\mathcal{A}p'; ap') \cap \sigma = \emptyset$. By Theorem 1.2, σ is an isolated spectral point of a. □

If $0 \in \sigma$, then condition (i) in the preceding theorem simplifies to $\sigma = \mathsf{Sp}(ap)$ as in this case $\mathsf{Sp}(p\mathcal{A}p; ap) = \mathsf{Sp}(ap)$. A variant of this result for closed operators was given in [16, Theorem 2.1].

2. The σg-Drazin inverse

Definition 2.1. We say that $a \in \mathcal{A}$ is *g-Drazin invertible* if there exists an idempotent p commuting with a such that ap is quasinilpotent in $p\mathcal{A}p$ ($\mathsf{Sp}(p\mathcal{A}p; ap) = \{0\}$ or $ap = 0$) and ap' is invertible in $p'\mathcal{A}p'$. The *g-Drazin inverse* a^{D} of such a is the inverse of ap' in $p'\mathcal{A}p'$: $(ap')a^{\mathsf{D}} = p' = a^{\mathsf{D}}(ap')$.

From Corollary 1.3 it follows that a is g-Drazin invertible if and only if a is quasipolar with the spectral idempotent p. A different, but equivalent, definition was given in [7], and the properties of the g-Drazin inverse were studied there in some detail. A special case of the g-Drazin inverse is the *Drazin inverse* arising when $a^k p = 0$ for some integer $k \geq 0$. If $k = 1$, the Drazin inverse b of a is called the *group inverse*. It is characterized by the equations

$$ab = ba, \quad bab = b, \quad aba = a. \tag{2.1}$$

In order to further generalize the g-Drazin inverse we replace $\{0\}$ in the preceding definition by an isolated spectral set. An example of such a construction was given in [9], and for closed linear operators in [16]. In this paper we present a detailed study of the properties of this generalized g-Drazin inverse, and give applications.

Definition 2.2. Let σ be an isolated spectral set of $a \in \mathcal{A}$ such that $0 \in \mathsf{Res}(a) \cup \sigma$, and let p be the spectral idempotent of σ. The inverse of ap' in the Banach algebra $p'\mathcal{A}p'$ is called the *σg-Drazin inverse* of a, written $a^{\mathsf{D},\sigma}$. The spectral idempotent of a corresponding to σ will be denoted by $a^{\pi,\sigma}$. We say that $a \in \mathcal{A}$ is *σg-Drazin invertible* if a possesses an isolated spectral set σ such that $0 \in \mathsf{Res}(a) \cup \sigma$.

We must check that $ap' \in (p'\mathcal{A}p')^{\mathsf{inv}}$. For this we observe that the spectrum $\mathsf{Sp}(p'\mathcal{A}p'; ap')$ coincides with τ, the complementary isolated spectral set of σ, and that $0 \notin \tau$ by the assumption $0 \in \mathsf{Res}(a) \cup \sigma$. We observe that $a^{\mathsf{D},\{0\}} = a^{\mathsf{D}}$ if $a \notin \mathcal{A}^{\mathsf{inv}}$, and $a^{\mathsf{D},\emptyset} = a^{\mathsf{D}} = a^{-1}$ if $a \in \mathcal{A}^{\mathsf{inv}}$. Further, $a^{\mathsf{D},\mathsf{Sp}(a)} = 0$. We observe that $\mathsf{Sp}(p\mathcal{A}p; ap) = \sigma$ for any σg-Drazin invertible element a.

The following theorem gives an explicit formula for the σg-Drazin inverse of a.

Theorem 2.3. *Let $a \in \mathcal{A}$ be σg-Drazin invertible with the spectral idempotent p. Then for any $\xi \notin \sigma \cup \{0\}$ and any $\eta \neq 0$,*

$$a^{\mathsf{D},\sigma} = (a - \xi p)^{-1} p' = (ap' - \eta p)^{-1} p'. \tag{2.2}$$

Proof. Let $b = a^{\mathsf{D},\sigma}$. By the definition of the σg-Drazin inverse, $ab = (ap')b = p'$, and $bp = 0$. Recall that $\mathsf{Sp}(ap) = \mathsf{Sp}(p\mathcal{A}p; ap) \cup \{0\} = \sigma \cup \{0\}$. If $\xi \notin \sigma \cup \{0\}$, then

$$(a - \xi p)(\xi b - p) = \xi ab - ap - \xi^2 bp + \xi p = \xi - ap \in \mathcal{A}^{\mathsf{inv}}.$$

Thus $a - \xi p \in \mathcal{A}^{\mathsf{inv}}$, and the first equality in (2.2) follows from $(a - \xi p)b = p'$. The second equality is obtained from the first observing that $ap' \in (p'\mathcal{A}p')^{\mathsf{inv}}$ if and only if $ap' + \eta p \in \mathcal{A}^{\mathsf{inv}}$ for some $\eta \neq 0$. □

If a is σg-Drazin invertible, we have

$$1 - aa^{\mathsf{D},\sigma} = a^{\pi,\sigma}. \tag{2.3}$$

Indeed, for $p = a^{\pi,\sigma}$,

$$1 - a(a - \xi p)^{-1}p' = 1 - (a - \xi p)^{-1}(a - \xi p)p' = 1 - p' = p.$$

The following characterization of the σg-Drazin inverse is reminiscent of the classical definition of the g-Drazin inverse.

Theorem 2.4. *An element $a \in \mathcal{A}$ is σg-Drazin invertible for some set σ if and only if there exists $b \in \mathcal{A}$ such that*

$$ab = ba, \quad bab = b, \quad \mathsf{Sp}(a - aba) \cap \mathsf{Sp}(aba) = \{0\}. \tag{2.4}$$

In this case $b = a^{\mathsf{D},\sigma}$ with $\sigma = \mathsf{Sp}(p\mathcal{A}p; ap)$, where $p = 1 - ab$.

Proof. Suppose that an element b satisfying (2.4) exists. Let $p = 1 - ab$ and $p' = ab$. According to the first two equations in (2.4), p and p' are idempotents commuting with a and b. The third equation in (2.4) is equivalent to $\mathsf{Sp}(ap) \cap \mathsf{Sp}(ap') = \{0\}$. We observe that $(ap')b = (aab)b = a(bab) = ab = p'$, that is, ap' is invertible in $p'\mathcal{A}p'$ with the inverse $b = bp'$. Then $0 \notin \mathsf{Sp}(p'\mathcal{A}p'; ap')$, and

$$\big(\mathsf{Sp}(p\mathcal{A}p; ap) \cup \{0\}\big) \cap \big(\mathsf{Sp}(p'\mathcal{A}p'; ap') \cup \{0\}\big) = \mathsf{Sp}(ap) \cap \mathsf{Sp}(ap') = \{0\},$$

that is, $\mathsf{Sp}(p\mathcal{A}p; ap) \cap \mathsf{Sp}(p'\mathcal{A}p'; ap') = \emptyset$.

Set $\sigma = \mathsf{Sp}(p\mathcal{A}p; ap)$. By Theorem 1.2, σ is an isolated spectral set of a with the spectral idempotent p. By definition, $a^{\mathsf{D},\sigma} = b$, the inverse of ap' in $p'\mathcal{A}p'$. We note that if $0 \notin \sigma$, then $0 \notin \mathsf{Sp}(a) = \sigma \cup \mathsf{Sp}(p'\mathcal{A}p'; ap')$; hence $0 \in \mathsf{Res}(a) \cup \sigma$.

Conversely, if σ is an isolated spectral set of a with the spectral idempotent p such that $0 \in \mathsf{Res}(0) \cup \sigma$, then $b = a^{\mathsf{D},\sigma}$ satisfies (2.4): Indeed, $ab = p' = ba$ (as $b = bp'$ is the inverse of ap' in $p'\mathcal{A}p'$). From this we get $ab = ba$ and $bab = b$. Further, $\mathsf{Sp}(p\mathcal{A}p; ap) \cap \mathsf{Sp}(p'\mathcal{A}p'; ap') = \emptyset$, which combined with (1.1) implies

$$\mathsf{Sp}(a - aba) \cap \mathsf{Sp}(aba) = \mathsf{Sp}(ap) \cap \mathsf{Sp}(ap') = \{0\}$$

as required. □

From the preceding theorem and its proof we glean the following result:

Corollary 2.5. *An element $a \in \mathcal{A}$ is σg-Drazin invertible for some set σ if and only if there exists an idempotent $p \in \mathcal{A}$ commuting with a such that*

$$ap' \in (p'\mathcal{A}p')^{\mathsf{inv}} \quad \textit{and} \quad \mathsf{Sp}(ap) \cap \mathsf{Sp}(ap') = \{0\}.$$

In this case $a^{\mathsf{D},\sigma} = (ap' + p)^{-1}p'$ and $\sigma = \mathsf{Sp}(p\mathcal{A}p; ap)$.

We can give a representation of $a^{\mathsf{D},\sigma}$ in terms of the holomorphic calculus: If σ is an isolated spetral set of a such that $0 \in \mathsf{Res}(a) \cup \sigma$, choose disjoint open sets $\Delta \supset \sigma$ and $\Omega \supset \tau$, $0 \notin \Omega$, and define

$$h(\lambda) = \begin{cases} 0 & \text{if } \lambda \in \Delta, \\ \lambda^{-1} & \text{if } \lambda \in \Omega. \end{cases} \tag{2.5}$$

We show that $a^{\mathsf{D},\sigma} = h(a)$. If f is the characteristic function of Ω in $\Delta \cup \Omega$, then $\lambda f(\lambda) h(\lambda) = f(\lambda)$, and

$$ap'h(a) = f(a) = p'.$$

Since $h(a) \in p'\mathcal{A}p'$ and commutes with ap', it is the inverse of ap' in $p'\mathcal{A}p'$. Thus we have the following

Proposition 2.6. *If $a \in \mathcal{A}$ is σg-Drazin invertible, then*

$$a^{\mathsf{D},\sigma} = h(a), \tag{2.6}$$

where h is defined by (2.5)*.*

If the last condition in (2.4) is replaced by $\mathsf{Sp}(a - aba) = \{0\}$, we obtain the usual g-Drazin inverse a^{D} of a. In this case it makes sense to define the *Drazin index* of a,

$$\operatorname{ind}(a) = \inf\{k \in \mathbb{N} \cup \{0\} : a^{k+1}a^{\mathsf{D}} = a^k\}.$$

It follows that $\operatorname{ind}(a) = 0$ if and only if $a \in \mathcal{A}^{\mathsf{inv}}$, $\operatorname{ind}(a)$ is finite if and only if $a - aba$ is nilpotent, and $\operatorname{ind}(a) = \infty$ if and only if $a - aba$ is quasinilpotent but not nilpotent. Further, $0 < \operatorname{ind}(a) = k < \infty$ if and only if 0 is a pole of order k of the resolvent $R(\lambda; a)$. However, such a definition of the index of a relative to the σg-Drazin inverse with a general isolated spectral set σ is no longer possible.

By (2.6), $a^{k+1}a^{\mathsf{D},\sigma} = g(a)$, where g is equal to 0 on σ and to λ^k on τ. If $\sigma \neq \{0\}$, $g(a) \neq a^k$.

Next we look at some properties of the σg-Drazin inverse.

Theorem 2.7. *Let $a \in \mathcal{A}$ be σg-Drazin invertible. Then:*

(i) *If $\sigma^n \cap \tau^n = \emptyset$, where τ is the complementary spectral set of σ, then $(a^n)^{\mathsf{D},\sigma^n}$ exists, and $(a^n)^{\mathsf{D},\sigma^n} = (a^{\mathsf{D},\sigma})^n$. In this case $(a^n)^{\pi,\sigma^n} = a^{\pi,\sigma}$.*

(ii) *$(a^{\mathsf{D},\sigma})^{\mathsf{D}}$ exists, and $(a^{\mathsf{D},\sigma})^{\mathsf{D}} = a^2 a^{\mathsf{D},\sigma}$.*

(iii) *$((a^{\mathsf{D},\sigma})^{\mathsf{D}})^{\mathsf{D}} = a^{\mathsf{D},\sigma}$.*

(iv) *$a^{\mathsf{D},\sigma}(a^{\mathsf{D},\sigma})^{\mathsf{D}} = aa^{\mathsf{D},\sigma} = 1 - a^{\pi,\sigma}$.*

Proof. (i) Write $b = a^{\mathsf{D},\sigma}$, $p = a^{\pi,\sigma}$. Then a^n commutes with p and p', $(ap)^n = a^n p$, $(ap')^n = a^n p'$, while

$$\mathsf{Sp}(p\mathcal{A}p; a^n p) = \sigma^n \quad \text{and} \quad \mathsf{Sp}(p'\mathcal{A}p'; a^n p') = \tau^n.$$

Since σ^n and τ^n are disjoint, they are isolated spectral sets of a^n by Theorem 1.2. Further, $0 \in \mathsf{Res}(a^n) \cup \sigma^n$. By the definition of $a^{\mathsf{D},\sigma}$, $b(ap') = p'$. Hence

$$b^n(a^n p') = (b^n a^n)p' = (ba)^n p' = (ba)p' = b(ap') = p',$$

that is, b^n is the inverse of $a^n p'$ in $p'\mathcal{A}p'$, and $(a^n)^{\mathsf{D},\sigma^n} = b^n$. We have also shown that $(a^n)^{\pi,\sigma^n} = p$.

(ii) Write $b = a^{\mathsf{D},\sigma}$, $p = a^{\pi,\sigma}$. Then b commutes with p and p', $bp = 0$ is quasinilpotent in $p\mathcal{A}p$ and $bp' = b$ is invertible in $p'\mathcal{A}p'$ with the inverse ap'. By Definition 2.1, b is g-Drazin invertible with $b^{\mathsf{D}} = ap' = a^2 b$.

(iii) and (iv) follow from the preceding result. □

Property (i) of the preceding theorem can be generalized as follows:

Corollary 2.8. *If $a \in \mathcal{A}$ is σg-Drazin invertible and f is an analytic function defined on some open neighbourhood of $\mathsf{Sp}(a)$ such that $f(\sigma) \cap f(\tau) = \emptyset$, where τ*

is the complementary isolated spectral set of σ, then $f(a)$ is ρg-Drazin invertible with $\rho = f(\sigma)$.

By Definition 2.2, the σg-Drazin inverse of a is the inverse of ap' in the Banach algebra $p'\mathcal{A}p'$. It is interesting to see that this inverse is equal to the g-Drazin inverse of ap' in $\mathcal{A}$.

Lemma 2.9. *Let $a \in \mathcal{A}$ be σg-Drazin invertible with the spectral idempotent p. Then ap' is group invertible with*

$$a^{\mathsf{D},\sigma} = (ap')^{\mathsf{D}} \quad \textit{and} \quad a^{\pi,\sigma} = (ap')^{\pi}. \tag{2.7}$$

Proof. We see that $(ap')p = 0$ is quasinilpotent in $p\mathcal{A}p$, and ap' is invertible in $p'\mathcal{A}p'$ with the inverse $a^{\mathsf{D},\sigma}$. Hence $a^{\mathsf{D},\sigma} = (ap')^{\mathsf{D}}$. Further, $1 - a^{\pi,\sigma} = aa^{\mathsf{D},\sigma} = (ap')(ap')^{\mathsf{D}} = 1 - (ap')^{\pi}$. □

The following decomposition of a σg-Drazin invertible element generalizes the important core-quasinilpotent decomposition of a g-Drazin invertible element.

Theorem 2.10. *An element $a \in \mathcal{A}$ is σg-Drazin invertible if and only if $a = x + y$, where $xy = 0 = yx$, x is group invertible, and $\mathsf{Sp}(x) \cap \mathsf{Sp}(y) = \{0\}$. Moreover, such a decomposition is unique.*

Proof. If $a^{\mathsf{D},\sigma}$ exists, we can set $x = ap'$, $y = ap$, where $p = a^{\pi,\sigma}$. Then $xy = 0 = yx$. By the preceding lemma, x is g-Drazin invertible with $xp = (ap')p = 0$, that is, x is group invertible. Further, $\mathsf{Sp}(x) \cap \mathsf{Sp}(y) = \{0\}$ as in the proof of Theorem 2.4.

For the converse assume that the decomposition $a = x + y$ with the given properties exists. Let $p = x^{\pi}$. Then $yp = y$ (as $yp' = yxx^{\mathsf{D}} = 0$), and $xp = 0$ (x is group invertible). Let $b = x^{\mathsf{D}}$. Then $ab = ba$, $bab = b$, $a - aba = ap = y$, and $aba = ap' = x$; hence $\mathsf{Sp}(a - aba) \cap \mathsf{Sp}(aba) = \mathsf{Sp}(y) \cap \mathsf{Sp}(x) = \{0\}$. By Theorem 2.4, $b = a^{\mathsf{D},\sigma}$, where $\sigma = \mathsf{Sp}(p\mathcal{A}p; ap)$.

Next, we prove the uniqueness of such decomposition. Suppose that a has decompositions $a = x + y$ and $a = v + w$, satisfying the conditions of the theorem. Then $x = (x^{\mathsf{D}})^{\mathsf{D}} = (a^{\mathsf{D},\sigma})^{\mathsf{D}} = a^2 a^{\mathsf{D},\sigma} = (v^2 + w^2)v^{\mathsf{D}} = v$ since v is group invertible and $wv^{\mathsf{D}} = wv(v^{\mathsf{D}})^2 = 0$. Consequently $y = w$. This completes the proof. □

We shall call the decomposition $a = x + y$ from the preceding theorem the *core decomposition* of a, and x the *core* of a.

Corollary 2.11. *The core decomposition $a = x+y$ of a σg-Drazin invertible element $a \in \mathcal{A}$ has the following properties:*

$$a^{\mathsf{D},\sigma} = x^{\mathsf{D}}, \quad a^{\pi,\sigma} = x^{\pi}, \quad \mathsf{Sp}(x) = \tau \cup \{0\}, \quad \mathsf{Sp}(y) = \sigma \cup \{0\},$$

where τ is the complementary isolated spectral set of σ.

3. Representation of the σg-Drazin inverse

In this section we consider sequences of σg-Drazin invertible elements a_n convergent to a σg-Drazin invertible element a.

Theorem 3.1. *Let $a \in \mathcal{A}$ be σg-Drazin invertible with $a^{\pi,\sigma} = p$. Then*

$$a^{\mathsf{D},\sigma} = \lim_{\lambda \to 0} (ap' - \lambda)^{-1} p'. \tag{3.1}$$

Proof. The element ap' is quasipolar, and some punctured neighbourhood $0 < |\lambda| < r$ lies in the resolvent set of ap'. From the Laurent expansion for the resolvent of a quasipolar element,

$$(ap' - \lambda)^{-1} p' = \sum_{n=0}^{\infty} \lambda^n ((ap')^{\mathsf{D}})^{n+1}, \quad 0 < |\lambda| < r.$$

Hence the limit in (3.1) is equal to $(ap')^{\mathsf{D}} = a^{\mathsf{D},\sigma}$. □

For $\sigma = \{0\}$ we recover [7, Theorem 6.1].

Next we give an integral representation of the σg-Drazin inverse.

Theorem 3.2. *Let $a \in \mathcal{A}$ be σg-Drazin invertible with the spectral projection p and with σ satisfying $(\mathsf{Re}\,\sigma) \setminus \{0\} < 0$. Then*

$$a^{\mathsf{D},\sigma} = -\int_0^\infty \exp(ta)p' \,\mathsf{d}t. \tag{3.2}$$

Proof. Let $a = x + y$ be the core decomposition of a. Then $\mathsf{Sp}(x) \setminus \{0\} = \sigma \setminus \{0\}$ lies in the open left half plane and x is group invertible. By [7, Theorem 6.3],

$$x^{\mathsf{D}} = -\int_0^\infty \exp(tx)p' \,\mathsf{d}t.$$

Since $x = ap'$, we have

$$\exp(tx)p' = \exp(tap')p' = \exp(ta)p'.$$

The conclusion of the theorem follows from the equation $a^{\mathsf{D},\sigma} = x^{\mathsf{D}}$. □

For $\sigma = \{0\}$ we recover [2, Theorem 2.2].

4. Continuity of the σg-Drazin inverse

The continuity of the g-Drazin inverse was studied in detail in [10]. Our first result proves the continuity of the σg-Drazin inverse under an additional condition on the convergence of (a_n). We use the following notation: We write $\mathsf{Sp}(a_n) = \sigma \cup \tau_n$ with the disjoint union of isolated spectral sets of a_n; similarly $\mathsf{Sp}(a) = \sigma \cup \tau$.

Theorem 4.1. *Let a_n and a be σg-Drazin invertible elements of $\mathcal{A}$ with the spectral idempotents p_n and p, respectively. Let $a_n \to a$ and $a_n p_n \to ap$. Then the following conditions are equivalent:*

(i) $a_n^{\mathsf{D},\sigma} \to a^{\mathsf{D},\sigma}$,

(ii) $\sup_n \|a_n^{\mathsf{D},\sigma}\| < \infty$,

(iii) $\sup_n r(a_n^{\mathsf{D},\sigma}) < \infty$,

(iv) $p_n \to p$,

(v) $a_n a_n^{\mathsf{D},\sigma} \to a a^{\mathsf{D},\sigma}$.

Proof. Let x_n and x be the core of a_n and a, respectively. Recall that $x_n = a_n - a_n p_n$ and $x = a - ap$. Hence $a_n \to a$ and $a_n p_n \to ap$ imply that $x_n \to x$. We recall that ${x_n}^{\mathsf{D}} = {a_n}^{\mathsf{D},\sigma}$, ${x_n}^{\pi} = {a_n}^{\pi,\sigma} = p_n$, $x^{\mathsf{D}} = a^{\mathsf{D},\sigma}$ and $x^{\pi} = a^{\pi,\sigma} = p$. In addition, $x_n {x_n}^{\mathsf{D}} = a_n a_n^{\mathsf{D}} = 1 - p_n$ and $x x^{\mathsf{D}} = a a^{\mathsf{D},\sigma} = 1 - p$. Since x_n and x are Drazin invertible, the result follows from [10, Theorem 2.4]. □

Remark 4.2. We observe that the conclusions of the preceding theorem remain in force if we merely assume that $x_n \to x$, where x_n and x is the core of a_n and a.

In the case of a mere convergence $a_n \to a$ we have the following rather modest result.

Proposition 4.3. *Let a_n and a be σg-Drazin invertible elements of $\mathcal{A}$, and let p_n and p be the spectral idempotents of a_n and a, respectively. If $a_n \to a$, the following conditions are equivalent*:

(i) $a_n^{\mathsf{D},\sigma} \to a^{\mathsf{D},\sigma}$,

(ii) $a_n a_n^{\mathsf{D},\sigma} \to a a^{\mathsf{D},\sigma}$,

(iii) $p_n \to p$.

Proof. Condition (i) implies (ii) in view of the continuity of the product in $\mathcal{A}$. Further, $a_n a_n^{\mathsf{D},\sigma} = 1 - p_n$ and $a a^{\mathsf{D},\sigma} = 1 - p$; hence (iii) follows from (ii).

Suppose that (iii) holds. According to Theorem 2.3, for any $\xi \notin \sigma \cup \{0\}$,

$$a_n^{\mathsf{D},\sigma} = (a_n - \xi p_n)^{-1} p_n' \to (a - \xi p)^{-1} p' = a^{\mathsf{D},\sigma}.$$

This proves (i), and the proof of the theorem is complete. □

Let us consider the situation when $a_n \to a$ with all the elements σg-Drazin invertible. Clearly, condition (iii) of Theorem 4.1 is necessary for $a_n^{\mathsf{D},\sigma} \to a^{\mathsf{D},\sigma}$.

However, it need not be sufficient. Suppose that (iii) holds with $s = \sup_n r(a_n^{\mathsf{D},\sigma}) > 0$. If $\lambda \in \tau_n$ and h_n is a holomorphic function such that $a_n^{\mathsf{D},\sigma} = h_n(a_n)$, then

$$|\lambda^{-1}| = |h_n(\lambda)| \le r(h_n(a_n)) = r(a_n^{\mathsf{D},\sigma}) \le s.$$

This means that all the sets τ_n lie in the annulus $|\lambda| \ge s^{-1}$. From the spectral mapping theorem we deduce that

$$r_n := \inf |\tau_n| = \frac{1}{r(a_n^{\mathsf{D},\sigma})} \quad \text{if } \tau_n \ne \emptyset.$$

Following [9], we say that a set $\sigma \subset \mathsf{Sp}(a)$ is *circularly isolated* about μ in $\mathsf{Sp}(a)$ if there is a cirle $|\lambda - \mu| = r$ whose interior contains σ and whose exterior contains $\tau = \mathsf{Sp}(a) \setminus \sigma$. Suppose that σ is circularly isolated about 0 in $\mathsf{Sp}(a_n)$ for all n, and write $r = \sup |\sigma|$. Then each τ_n and σ are isolated by a circle $|\lambda| = \rho_n$, where $r_n > \rho_n > r$. However, we may have $r_n \to s^{-1}$, in which case there is no single circle $|\lambda| = \rho$ isolating all the τ_n from σ, and in general we cannot conclude that $p_n = f_n(a_n) \to f(a) = p$. If $\sigma = \{0\}$, then such an isolation obviously exists, and condition (iii) of Theorem 4.1 is seen to be sufficient for $p_n \to p$; thus we recover [8, Theorem 2.4].

The analysis carried out in the preceding paragraph motivates the following result.

Theorem 4.4. *Let a_n and a be σg-Drazin invertible elements of $\mathcal{A}$, and let there be a Jordan curve Γ such that σ belongs to the interior of Γ, and each τ_n and τ to the exterior of Γ. Let p_n and p be the spectral idempotents of a_n and a, respectively. If $a_n \to a$, then*

$$a_n^{\mathsf{D},\sigma} \to a^{\mathsf{D},\sigma}. \tag{4.1}$$

Proof. Let $\Delta = \operatorname{int}\Gamma$ and $\Omega = \operatorname{ext}\Gamma$. Let h be the function defined on $\Delta \cup \Omega$ by setting $h(\lambda) = 0$ if $\lambda \in \Delta$, and $h(\lambda) = \lambda^{-1}$ if $\lambda \in \Omega$. (Recall that the hypothesis $0 \in \mathsf{Res}(a) \cup \sigma$ guarantees that $0 \notin \tau$, and that we can remove 0 from Ω is necessary.)

Since $\tau \subset \Omega$, $\tau_n \subset \Omega$ for all n, and $\sigma \subset \Delta$, a single holomorphic function h can be used to define

$$a^{\mathsf{D},\sigma} = h(a), \qquad a_n^{\mathsf{D},\sigma} = h(a_n),\ n \in \mathbb{N}.$$

Since $a_n \to a$, we conclude that $h(a_n) \to h(a)$ by [13, Theorem 3.3.7]. Hence (4.1) holds. □

Corollary 4.5. *Let a_n and a be σg-Drazin invertible elements of $\mathcal{A}$ such that $a_n \to a$ and*

$$\max\{r(a^{\mathsf{D},\sigma}), \sup_n r(a_n^{\mathsf{D},\sigma})\} < \frac{1}{\sup|\sigma|}.$$

Then (4.1) *holds.*

Proof. We observe that $\min\{\inf|\tau|, \inf_n \inf|\tau_n|\} = 1/\sup_n r(a_n^{\mathsf{D},\sigma}) > \sup|\sigma|$. Then there exists a circle $|\lambda| = \rho$ whose interior contains σ and whose exterior contains all the τ_n and τ. Then the preceding theorem applies. □

If $\sigma = \{0\}$ and $\mathcal{A} = \mathbb{C}^{n\times n}$ is the Banach algebra of all $n\times n$ complex matrices, the preceding theorem leads to the following result:

Corollary 4.6. *Let A_n and A be matrices in $\mathbb{C}^{n\times n}$. Then $A_n^{\mathsf{D}} \to A^{\mathsf{D}}$ if and only if there exists a constant $\alpha > 0$ such that the nonzero eigenvalues of A_n satisfy $|\lambda| \geq \alpha$ for all n.*

We close the section with the following theorem.

Theorem 4.7. *Let $a \in \mathcal{A}$ be σg-Drazin invertible, let there be a Jordan curve Γ such that σ belongs to the interior of Γ, and τ into the exterior of Γ, and let $a_n \to a$. Then there exists N such that for all $n \geq N$ each a_n is $\sigma_n g$-Drazin invertible for some σ_n, and*

$$a_n^{\mathsf{D},\sigma_n} \to a^{\mathsf{D},\sigma}. \tag{4.2}$$

Proof. By a theorem of Kato [5, Theorem IV.3.16], there exists N such that for all $n \geq N$ the spectrum $\mathsf{Sp}(a_n)$ is separated into two isolated spectral sets $\sigma_n \subset \operatorname{int}\Gamma$ and $\tau_n \subset \operatorname{ext}\Gamma$. Note that $0 \in \mathsf{Res}(a_n) \cup \sigma_n$. Hence each a_n with $n \geq N$ is $\sigma_n g$-Drazin invertible, and (4.2) holds by an argument analogous to the one given in the proof of Theorem 4.4. □

5. The σg-Drazin inverse for operators

If $\mathcal{A}$ is the Banach algebra $\mathcal{B}(X)$ of all bounded linear operators on a complex Banach space X, we can characterize the σg-Drazin inverse of $A \in \mathcal{B}(X)$ in terms of the direct sum of operators.

Theorem 5.1. *An operator $A \in \mathcal{B}(X)$ is σg-Drazin invertible for some set σ if and only if*

$$A = A_1 \oplus A_2, \quad \mathsf{Sp}(A_1) \cap \mathsf{Sp}(A_2) = \emptyset, \quad A_2 \text{ is invertible.} \tag{5.1}$$

In this case $\mathsf{Sp}(A_1) = \sigma$ and $A^{\mathsf{D},\sigma} = 0 \oplus A_2^{-1}$.

Proof. By Theorem 1.2, A is σg-Drazin invertible in $\mathcal{B}(X)$ if and only there exists an idempotent operator $P \in \mathcal{B}(X)$ commuting with A such that AP' is invertible in the algebra $P'\mathcal{B}(X)P'$, and the spectra of AP and AP' in the algebras $P\mathcal{B}(X)P$ and $P'\mathcal{B}(X)P'$ are disjoint. (Here, in accordance with our notation, $P' = I - P$.)

Let $X_1 = R(P)$ and $X_2 = N(P) = R(P')$. Then X is the topological direct sum $X = X_1 \oplus X_2$, and A is decomposed as $A = A_1 \oplus A_2$ relative to this sum. It is known that $\mathsf{Sp}(A_1)$ is the spectrum of AP in the algebra $P\mathcal{B}(X)P$, and $\mathsf{Sp}(A_2)$ is the spectrum of AP' in the algebra $P'\mathcal{B}(X)P'$. Hence AP' is invertible in $P'\mathcal{B}(X)P'$ if and only A_2 is invertible in $\mathcal{B}(X_2)$.

By the definition of the σg-Drazin inverse, $A^{\mathsf{D},\sigma}$ is the inverse of AP' in the algebra $P'\mathcal{B}(X)P'$. Observe that $P = I \oplus 0$ and $P' = 0 \oplus I$. Let $B = 0 \oplus A_2^{-1}$. Then $B = BP'$, that is $B \in P'\mathcal{B}(X)P'$, and

$$(AP')B = (A_1 \oplus A_2)(0 \oplus I)(0 \oplus A_2^{-1}) = (0 \oplus A_2)(0 \oplus A_2^{-1}) = 0 \oplus I = P'.$$

This completes the proof. □

Let us recall that in the preceding theorem the condition $0 \in \sigma \cup \mathsf{Res}(A)$ is assumed when A is σg-Drazin invertible, and can be deduced when (5.1) is assumed. In addition, for any $\xi \notin \sigma \cup \{0\}$ and any $\eta \neq 0$,

$$A^{\mathsf{D},\sigma} = (A - \xi P)^{-1}P' = (AP' - \eta P)^{-1}P'. \tag{5.2}$$

If X is a finite dimensional, then every linear operator T on X has the σg-Drazin inverse for every admissible subset σ of its spectrum as every such set is an isolated spectral set of T.

Example 5.2. We give an example of a σ-Drazin inverse of a matrix. Let

$$A = \begin{bmatrix} 2 & -2 & \frac{1}{2} & -\frac{3}{2} & 0 & \frac{1}{2} & \frac{3}{2} \\ 1 & 5 & -1 & 2 & 2 & -2 & -5 \\ 1 & 3 & 0 & 1 & 2 & -1 & -4 \\ 1 & -2 & \frac{1}{2} & -\frac{1}{2} & -1 & \frac{1}{2} & \frac{5}{2} \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} & 2 & -\frac{1}{2} & -\frac{3}{2} \\ 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}.$$

The eigenvalues of A are 0, 2 and 1 with algebraic multiplicities 1, 2 and 4, respectively. Then $\sigma = \{0, 2\}$ is an isolated spectral set of A with $0 \in \sigma$, and $A^{\mathsf{D},\{0,2\}}$ exists. Let $P = A^{\pi,\{0,2\}}$ be the spectral projection relative to $\sigma = \{0, 2\}$. We have $A^{\pi,\{0,2\}} + A^{\pi,1} = A^{\pi,0} + A^{\pi,2} + A^{\pi,1} = I$, that is, $P' = I - P = A^{\pi,1}$:

$$P' = A^{\pi,1} = \begin{bmatrix} 1 & 3 & -1 & 1 & 1 & -1 & -3 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 2 & -\frac{1}{2} & \frac{3}{2} & 1 & -\frac{1}{2} & -\frac{5}{2} \\ 0 & 3 & -\frac{1}{2} & \frac{3}{2} & 2 & -\frac{3}{2} & -\frac{7}{2} \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 1 & 1 & -1 & -2 \end{bmatrix}.$$

We can calculate $A^{\mathsf{D},\{0,2\}}$ as the ordinary Drazin inverse of AP', that is,

$$A^{\mathsf{D},\{0,2\}} = (AP')^{\mathsf{D}} = \begin{bmatrix} 0 & 2 & -\frac{1}{2} & \frac{3}{2} & 1 & -\frac{1}{2} & -\frac{5}{2} \\ 0 & -2 & \frac{3}{2} & -\frac{3}{2} & -2 & \frac{1}{2} & \frac{7}{2} \\ 0 & -2 & \frac{3}{2} & -\frac{3}{2} & -2 & \frac{1}{2} & \frac{7}{2} \\ -1 & 2 & -\frac{1}{2} & \frac{5}{2} & 2 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} \\ -1 & 1 & 0 & 2 & 1 & 0 & -2 \\ 0 & -1 & 1 & -1 & -1 & 0 & 2 \end{bmatrix}.$$

6. The generalized Mbekhta decomposition

Mbekhta's subspaces $H_0(T)$ and $K(T)$ originally defined in [12] for closed linear operators in Hilbert spaces recently received new attention from several authors in [1, 4, 15]. For brevity we shall write $r(T;x) = \limsup_{n\to\infty} \|T^n x\|^{1/n}$, and $\mathcal{S}(T;x)$ for the set of all sequences (x_n) in X such that $Tx_{n+1} = x_n$ for all $n \geq 1$ and $Tx_1 = x$. Mbekhta [12] defined

$$H_0(T) = \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\}, \tag{6.1}$$

$$K(T) = \{x \in X : (\exists\, (x_n) \in \mathcal{S}(T;x))\, (\exists c > 0) \text{ with } \|x_n\| \leq c^n \|x\|\}. \tag{6.2}$$

Following [9], for any $T \in \mathcal{B}(X)$ we define the *generalized quasinilpotent part* $H_r(T)$ and the *generalized analytic core* $K_r(T)$ of T as follows:

Definition 6.1. Let $T \in \mathcal{B}(X)$ and $r > 0$. We define

$$H_r(T) = \{x \in X : r(T;x) < r\}, \tag{6.3}$$

$$K_r(T) = \{x \in X : \exists\, (x_n) \in \mathcal{S}(T;x) \text{ with } \limsup_{n\to\infty} \|x_n\|^{1/n} < r^{-1}\}. \tag{6.4}$$

It is not difficult to show that for any $r > 0$, $H_r(T)$ and $K_r(T)$ are (not necessarily closed) hyperinvariant subspaces of X, and that

$$T^{-1}H_r(T) = H_r(T), \qquad TK_r(T) = K_r(T); \tag{6.5}$$

(they are invariant under any operator commuting with T). These spaces are linked with Mbekhta's subspaces as follows.

Lemma 6.2. *For any $T \in \mathcal{B}(X)$ we have*

$$H_0(T) = \bigcap_{r>0} H_r(T), \qquad K(T) = \bigcup_{r>0} K_r(T). \tag{6.6}$$

Proof. The equality is easily seen for $H_0(T)$.

Let $x \in \bigcup_{r>0} K_r(T)$. Then $x \in K_r(T)$ for some $r > 0$, and there exists $(x_n) \in \mathcal{S}(T;x)$ such that $\limsup_n \|x_n\|^{1/n} < r^{-1}$. We may assume that $x \neq 0$. There exists n_0 such that $\|x_n\|^{1/n} \le r^{-1}\|x\|^{1/n}$ for all $n \ge n_0$. Setting

$$c = \max\{r^{-1}, \max\{(\|x_k\|/\|x\|)^{1/k} : k = 1, \dots, n_0\}\}$$

we get $\|x_n\| \le c^n \|x\|$ for all n. This proves that $x \in K(T)$.

Conversely, if $x \in K(T)$, then there is a sequence $(x_n) \in \mathcal{S}(T;x)$ and a positive constant c such that $\|x_n\| \le c^n\|x\|$ for all n. This implies $\limsup_n \|x_n\|^{1/n} \le c$, and $x \in K_r(T)$ for any r satisfying $0 < r < c^{-1}$. □

From the lemma and the definition of $H_r(T)$ and $K_r(T)$ we see that

$$H_r(T) \searrow H_0(T) \text{ and } K_r(T) \nearrow K(T) \text{ as } r \to 0+\,. \tag{6.7}$$

For completeness we may set $K_0(T) = K(T)$, $H_\infty(T) = X$ and $K_\infty(T) = \{0\}$. We also observe that

$$K_r(T) \subset K(T) \subset R(T^n) \text{ and } N(T^n) \subset H_0(T) \subset H_r(T) \tag{6.8}$$

for all n and all $r \in [0, \infty]$. The proof of Lemma 6.2 yields an alternative definition of $K(T)$:

$$K(T) = \{x \in X : \exists\,(x_n) \in \mathcal{S}(T;x) \text{ with } \limsup_{n\to\infty} \|x_n\|^{1/n} < \infty\}. \tag{6.9}$$

The main result involving the generalized quasinilpotent part and the generalized analytic core of an operator T is the following theorem first obtained in [9, Theorem 5.6]. We give a proof based on the foregoing results.

Theorem 6.3. *Let $T \in \mathcal{B}(X)$ and $0 < r < \infty$. Then T has a spectral set σ circularly isolated by the circle $|\lambda| = r$ if and only if X is the topological direct sum*

$$X = H_r(T) \oplus K_r(T). \tag{6.10}$$

Proof. It will be convenient to write D_r for the open disc $|\lambda| < r$, C_r for the circle $|\lambda| = r$ and A_r for the annulus $|\lambda| > r$.

Suppose first that there exists a spectral set σ of T circularly isolated by C_r. We show that

$$R(P) = H_r(T), \qquad N(P) = K_r(T), \tag{6.11}$$

where P is the spectral projection corresponding to σ. Let $S = T^{\mathsf{D},\sigma}$ be the σg–Drazin inverse of T. Then $r(TP) < r$, and $r(S) < r^{-1}$ by the spectral mapping theorem. The inclusion $R(P) \subset H_r(T)$ can be easily verified. For the converse inclusion assume that $x \in H_r(T)$. Then $P' = (P')^n = (ST)^n = S^nT^n$. Further, $\limsup_n \|P'x\|^{1/n} \le r(S)r(T;x) < r^{-1}r = 1$, which implies $P'x = 0$. Hence $x \in R(P)$.

To prove $N(P) \subset K_r(T)$, for a given $x \in N(P)$ construct a $\mathcal{S}(T;x)$ sequence $x_n = S^nx$, and check that $\limsup_n \|x_n\|^{1/n} < r^{-1}$. For the reverse inclusion show that any $x \in K_r(T)$ satisfies $\limsup_n \|Px\|^{1/n} \le r(TP) \limsup_n \|x_n\|^{1/n} < 1$, that is, $Px = 0$.

Conversely assume that X is the topological direct sum $X = H_r(T) \oplus K_r(T)$. Then $T = T_1 \oplus T_2$ relative to this sum as the subspaces $H_r(T), K_r(T)$ are invariant under T. For each $x \in H_r(T)$, $r(T_1;x) = r(T;x) < r$. According to [6, Corollary 2.1], this implies $r(T_1) < r$, and $\mathsf{Sp}(T_1) \subset D_r$.

The operator T_2 is bijective, and hence invertible, on $K_r(T)$ in view of (6.5). For any $x \in K_r(T)$ we use $T^nx_n = x$ to show that

$$r(T_2^{-1};x) = \limsup_{n\to\infty} \|T_2^{-n}x\|^{1/n} = \limsup_{n\to\infty} \|x_n\|^{1/n} < r^{-1}$$

where $(x_n) \in \mathcal{S}(T;x)$. By [6, Corollary 2.1], $r(T_2^{-1}) < r^{-1}$, so $\mathsf{Sp}(T_2) \subset A_r$. Combining this with $\mathsf{Sp}(T_1) \subset D_r$, we conclude that σ is circularly isolated by C_r. □

It is interesting to observe that the condition that $X = H_r(T) \oplus K_r(T)$ is a topological direct sum in the preceding theorem can be relaxed:

Corollary 6.4. *Let $T \in \mathcal{B}(X)$ and for some $r > 0$ let $X = H_r(T) \oplus K_r(T)$ be an algebraic direct sum with at least one of the component spaces closed. Then T has a spectral set σ circularly isolated in $\mathsf{Sp}(T)$ by the circle $|\lambda| = r$.*

This is proved in [9, Theorem 5.9]. Specializing σ to $\{0\}$, we have the following result [9, Corollary 5.11]:

Corollary 6.5. *An operator $T \in \mathcal{B}(X)$ is quasipolar if and only if for each sufficiently small $r > 0$, $X = H_r(T) \oplus K_r(T)$ with at least one of the component spaces closed.*

The original result of Mbekhta in [12] states that 0 is not an accumulation point of $\mathsf{Sp}(T)$ if and only if X is the topological sum $X = H_0(T) \oplus K(T)$. In [14], Schmoeger improved this result by showing that it is enough to have an algebraic direct sum with $K(T)$ closed to ensure that 0 is not an accumulation spectral point of T. We show that it is enough to have only $H_0(T)$ closed. (See also [9, Corollary 5.11].) First a useful known lemma.

Lemma 6.6. *Let H be a closed subspace of a Banach space X, invariant under $T \in \mathcal{B}(X)$. Define $A : H \to H$ by $Ax = Tx$ for all $x \in H$, and $B : X/H \to X/H$ by $B(x + H) = Tx + H$. If both A and B are invertible, then so is T.*

Proof. We note that A, B are bounded linear operators acting on Banach spaces. Assuming that they are invertible, we show that T is bijective. If $Tx = 0$, then $B(x + H) = H$, and $x \in H$ by the injectivity of B. Hence $Ax = 0$, and $x = 0$.

Let $y \in X$ be arbitrary. By the surjectivity of B, there exists $x \in X$ such that $Tx + H = B(x + H) = y + H$. Hence $y - Tx \in H$, and $T(x + A^{-1}(y - Tx)) = Tx + (y - Tx) = y$. □

Theorem 6.7. *An operator $T \in \mathcal{B}(X)$ is quasipolar if and only if $X = H_0(T) \oplus K(T)$ with at least one of the component spaces closed.*

Proof. The topological direct decomposition $X = H_0(T) \oplus K(T)$ for a quasipolar operator T was proved in [7, 12, 14].

Let $X = H_0(T) \oplus K(T)$ be an algebraic direct sum. Schmoeger [14] proved that T is quasipolar if $K(T)$ is closed. Suppose that $H_0(T)$ is closed. If A is the restriction of T to $H = H_0(T)$, then A is quasinilpotent. Let B be the operator on X/H defined by $B(x + H) = Tx + H$, and let $B(x + H) = Tx + H = H$. Then $Tx \in H$, and $x \in H$ since $T^{-1}H = H$. Hence B is injective. Let $y \in X$. By hypothesis, $y = h + k$ with $h \in H$ and $k \in K(T)$. Since $K(T) = TK(T)$, there exists $x \in K(T)$ such that $Tx = k$. Then $y + H = Tx + H = B(x + H)$, and B is surjective. Thus B is invertible.

There exists $\delta > 0$ such that $A(\lambda) = \lambda I_H - A$ and $B(\lambda) = \lambda I_{X/H} - B$ are both invertible whenever $0 < |\lambda| < \delta$. Applying Lemma 6.6 with $T(\lambda) = \lambda I - T$, $A(\lambda)$ and $B(\lambda)$ in place of T, A, B, we conclude that $T(\lambda)$ is invertible whenever $0 < |\lambda| < \delta$. This proves that 0 is not an accumulation point of $\mathsf{Sp}(T)$. □

Several authors recently applied Mbekhta's subspace theorem to derive certain properties of special classes of operators. Gong and Wang [4] were concerned with compact operators, Bouamama [1] studied Riesz operators, while Schmoeger extended some of the results of [1] and [4] to meromorphic operators. Several of their results can be further generalized when Corollary 6.5 and Theorem 6.7 are taken into account.

7. Application

The concept of the σg-Drazin inverse can be applied to closed linear operators on Banach spaces. Trung Dinh Tran used the special case of the inverse to study the solutions of a certain differential equation in a Banach space. His definition is based on an analogue of our equation (5.2).

Tran considered the differential equation studied by S. G. Krein [11, Chapter 3],

$$\frac{\mathsf{d}^2 x(t)}{\mathsf{d}t^2} = B^2 x(t) + f(t), \quad t \in [0, T], \tag{7.1}$$

where B is the infinitesimal generator of a strongly continuous group of bounded linear operators $T(t)$ on a Banach space X, and f is continuously differentiable on $[0, T]$. Assuming that B has an isolated spectral set σ containing 0 and circularly isolated by $|\lambda| = r$, Tran obtained the following result involving the σg-Drazin inverse of the generator B:

Theorem 7.1. (Tran [16, Theorem 3.1]) *The unique solution of the equation* (7.1) *with $T = r^{-1}$ and the initial conditions*

$$x(0) = u_0 \quad \textit{and} \quad \left.\frac{\mathsf{d}}{\mathsf{d}t}\right|_0 x(t) = v_0$$

is given by

$$\begin{aligned} x(t) = {} & \sum_{j=1}^{\infty} B^{2(j-1)} P F^{(2)}(t) \\ & + \frac{1}{2}(T(t) + T(-t))(I - P)u_0 + \frac{1}{2} B^{\mathsf{D},\sigma}(T(t) - T(-t))(I - P)v_0 \\ & + \int_0^t B^{\mathsf{D},\sigma}(T(t-s) - T(s-t))(I - P)f(s)\, ds, \quad t \in [0, r^{-1}], \end{aligned}$$

provided that $u_0 \in \mathcal{D}(B^2)$, $v_0 \in \mathcal{D}(B)$, and

$$\sum_{j=1}^{\infty} B^{2(j-1)} P F^{(2j)}(0) = P u_0, \quad \sum_{j=1}^{\infty} B^{2(j-1)} P F^{(2j-1)}(0) = P v_0,$$

where $P = B^{\pi,\sigma}$ and $F^{(k)}$ is the kth primitive of f.

In closing we comment that other results on differential equations in Banach spaces involving isolated spectral points of the infinitesimal generator can be extended to the case when the generator has the σg-Drazin inverse for some isolated spectral set σ. Work on such problems is in progress.

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