

THE WEIGHTED g -DRAZIN INVERSE FOR OPERATORS

ALEGRA DAJIĆ and J. J. KOLIHA

(Received 10 August 2004)

Communicated by A. Pryde

Abstract

The paper introduces and studies the weighted g -Drazin inverse for bounded linear operators between Banach spaces, extending the concept of the weighted Drazin inverse of Rakočević and Wei (*Linear Algebra Appl.* **350** (2002), 25–39) and of Cline and Greville (*Linear Algebra Appl.* **29** (1980), 53–62). We use the Mbehta decomposition to study the structure of an operator possessing the weighted g -Drazin inverse, give an operator matrix representation for the inverse, and study its continuity. An open problem of Rakočević and Wei is solved.

2000 *Mathematics subject classification*: primary 47A60, 46H30, 47A05, 47A10.

Keywords and phrases: Banach algebra without unit, g -Drazin inverse, bounded linear operator, weighted g -Drazin inverse, the Mbekhta decomposition, ascent and descent.

1. Introduction

In recent papers [13, 14], Rakočević and Wei defined and investigated the weighted Drazin inverse for bounded linear operators between Banach and Hilbert spaces, extending the concept of a weighted Drazin inverse for rectangular matrices introduced by Cline and Greville [6]. The weighted Drazin inverse for operators was previously introduced and studied by Qiao in [12], and further investigated by Wang in [16, 17]. The main purpose of this paper is to introduce and study the weighted g -Drazin inverse for bounded linear operators between Banach spaces X and Y , thus further extending the above mentioned works.

Let us write $\mathcal{B}(X, Y)$ for the set of all bounded linear operators between X and Y , and let W be a nonzero operator in $\mathcal{B}(Y, X)$. The W -weighted

g -Drazin inverse (the Wg -Drazin inverse for short) can be studied in the framework of Banach algebras when we introduce on the space $\mathcal{B}(X, Y)$ the W -product $A \star B = AWB$, and the W -norm $\|A\|_W = \|A\| \|W\|$. This elegant approach which turns $\mathcal{B}(X, Y)$ into a Banach algebra was suggested to the authors of [13, 14] by an anonymous referee. Unless W is invertible (and this would require the spaces X and Y to be isomorphic and homeomorphic), the resulting algebra is without unit.

In our work we remove the restriction of finite polarity of the operator WA (and AW) adopted by Rakočević and Wei [13]. In addition, we solve an open problem posed in [13], and complete and extend the results of Buoni and Faires [3] on the ascent and descent of AB and BA .

In Section 2 we gather relevant results on the g -Drazin inverse in Banach algebras without unit in order to study the Wg -Drazin inverse within the space $\mathcal{B}(X, Y)$, without having to adjoin a unit. Section 3 introduces and studies the weighted g -Drazin inverse between two different Banach spaces. In Section 4 we explore some properties of the weighted g -Drazin inverse, including the core decomposition and an integral representation for the weighted inverse. The ascent and descent for WA and AW is studied in Section 5, and a solution to an open problem posed by Rakočević and Wei in [13] is given there. In Section 6 we compare the Mbekhta decomposition for the operators WA and AW and recover and sharpen a result of Yukhno [19] on rectangular matrices. In the remaining sections we give an operator matrix representation for the Wg -Drazin inverse, compare it with the Moore–Penrose inverse in Hilbert spaces, and give necessary and sufficient conditions for its continuity.

2. The g -Drazin inverse in Banach algebras without unit

Let \mathcal{A} be a Banach algebra. We write $\mathcal{A}^{\text{qnil}}$ for the set of all quasinilpotent elements in \mathcal{A} , that is, elements a satisfying $\lim_{n \rightarrow \infty} \|a^n\|^{1/n} = 0$; the set of all nilpotent elements is denoted by \mathcal{A}^{nil} . If \mathcal{A} is unital, we denote by \mathcal{A}^{inv} the group of all invertible elements in \mathcal{A} . An element $a \in \mathcal{A}$ is *quasipolar* if 0 is not an accumulation point of the spectrum of a . In an algebra without unit, this is equivalent to 0 being an isolated spectral point of a . The set of all quasipolar elements of \mathcal{A} will be denoted by $\mathcal{A}^{\text{qpol}}$. An element $a \in \mathcal{A}$ is *polar* if it is quasipolar and 0 is at most a pole of the resolvent of a . The set of all polar elements is denoted by \mathcal{A}^{pol} .

The following holds [7, Theorems 4.2 and 5.1]:

LEMMA 2.1. *Let \mathcal{A} be a unital Banach algebra. Then $a \in \mathcal{A}$ is quasipolar (polar) in \mathcal{A} if and only there exists $p \in \mathcal{A}$ such that*

$$(2.1) \quad p^2 = p, \quad ap = pa \in \mathcal{A}^{\text{qnil}} \quad (ap = pa \in \mathcal{A}^{\text{nil}}), \quad a + p \in \mathcal{A}^{\text{inv}}.$$

The resolvent $R(\lambda; a) = (\lambda 1 - a)^{-1}$ has a Laurent expansion in some punctured neighbourhood $0 < |\lambda| < r$ of 0 given by

$$(2.2) \quad R(\lambda; a) = \sum_{n=0}^{\infty} \lambda^{-n-1} a^n p - \sum_{n=1}^{\infty} \lambda^{n-1} b^n,$$

where $b = (a + p)^{-1}(1 - p)$.

The element p is uniquely determined by the conditions of the theorem; it is called the *spectral idempotent* of a , and it double commutes with a . The element $q = 1 - p$ is the *support idempotent* of a . The support idempotent of a quasipolar element exists in an algebra without a unit, but not the spectral idempotent. The element $b = (a + p)^{-1}(1 - p)$ defines the *g -Drazin inverse* of a in the case of a unital algebra; b also double commutes with a . We write a^π and a^σ for the spectral idempotent and the support idempotent of a quasipolar element a , respectively.

From now on we assume that \mathcal{A} is a complex Banach algebra without unit.

The *unitization* of \mathcal{A} is the unital Banach algebra $\mathcal{A}_1 = \mathcal{A} \oplus \mathbb{C}$ containing \mathcal{A} as a two sided ideal of codimension 1 [2, p. 15]. Given $a \in \mathcal{A}$, we define the *spectrum* $\text{Sp}(a)$ of a in \mathcal{A} as the spectrum of a considered as an element of the unital Banach algebra \mathcal{A}_1 , that is, the set of all $\lambda \in \mathbb{C}$ such that $\lambda 1 - a \notin \mathcal{A}_1^{\text{inv}}$. Observe that 0 is always in the spectrum of any element of a Banach algebra without unit.

PROPOSITION 2.2. *Let \mathcal{A} be a Banach algebra without unit. Then $a \in \mathcal{A}^{\text{qp ol}}$ ($a \in \mathcal{A}^{\text{po l}}$) if and only if there exists $b \in \mathcal{A}$ such that*

$$(2.3) \quad ab = ba, \quad bab = b, \quad a - aba \in \mathcal{A}^{\text{qnil}} \quad (a - aba \in \mathcal{A}^{\text{nil}}).$$

The element b , if it exists, is unique.

PROOF. We embed \mathcal{A} into its unitization \mathcal{A}_1 .

If a is quasipolar in \mathcal{A} , then it is also quasipolar in \mathcal{A}_1 . Let p be the spectral idempotent of a in \mathcal{A}_1 , and $b = (a + p)^{-1}(1 - p)$ the Drazin inverse of a in \mathcal{A}_1 . Since $1 - p$ is in \mathcal{A} , so is b (\mathcal{A} is an ideal). The equations (2.3) are then easily verified.

Conversely, let equations (2.3) hold. Then $p = 1 - ab$ is the spectral idempotent of a in \mathcal{A}_1 [7, Theorem 4.2], and a is quasipolar, both in \mathcal{A}_1 and \mathcal{A} . From

$$(a + p)b = (a + 1 - ab)b = ab + b - bab = ab = 1 - p$$

and the invertibility of $a + p$ in \mathcal{A}_1 we get $b = (a + p)^{-1}(1 - p)$ in \mathcal{A}_1 (and in \mathcal{A}). This proves the uniqueness of b satisfying (2.3). \square

DEFINITION 2.3. Let \mathcal{A} be a Banach algebra without unit and let $a \in \mathcal{A}^{\text{qpol}}$. We define the g -Drazin inverse a^{D} of a to be the unique element b satisfying (2.3). The *Drazin index* of a quasipolar element a is defined by

$$i(a) = \inf\{k \in \mathbb{N} : (a - a^2 a^{\text{D}})^k = 0\}$$

($\inf \emptyset = \infty$). The g -Drazin inverse of a polar element is called the *Drazin inverse*.

We observe that $a \in \mathcal{A}$ is polar if and only if it is quasipolar and has a finite Drazin index.

As in the unital case, any g -Drazin invertible element a of \mathcal{A} has the ‘core’ decomposition.

PROPOSITION 2.4. *Let \mathcal{A} be a Banach algebra without unit. Then $a \in \mathcal{A}^{\text{qpol}}$ if and only if $a = c + u$, where c is simply polar, u quasinilpotent, and $cu = 0 = uc$. Such a decomposition is unique. In addition,*

$$(2.4) \quad a^{\text{D}} = c^{\text{D}}, \quad a^\sigma = c^\sigma, \quad \text{Sp}(c) = \text{Sp}(a).$$

We can show that $ua^\sigma = 0$, and that the element c , called the *core* of a , satisfies

$$c = aa^\sigma = (a^{\text{D}})^{\text{D}} = a^2 a^{\text{D}}.$$

PROPOSITION 2.5. *Let \mathcal{A} be a Banach algebra without unit and let $a \in \mathcal{A}^{\text{qpol}}$. Then $a^{\text{D}} = a$ if and only if $a^3 = a$.*

PROOF. Suppose that $a^3 = a$ and let $a = c + u$ be the core decomposition of a . We observe that $a^3 = c^3 + u^3$ is the core decomposition for $a^3 = a$. From the uniqueness, $c^3 = c$ and $u^3 = u$. Since $u^3 = u \in \mathcal{A}^{\text{qnil}}$, we conclude that $u = 0$:

$$\lim_{n \rightarrow \infty} \|u\|^{1/3^n} = \lim_{n \rightarrow \infty} \|u^{3^n}\|^{1/3^n} = r(u) = 0.$$

Thus $a = c = aa^\sigma$ is simply polar, and

$$a^{\text{D}} = (a^{\text{D}})^2 a = (a^{\text{D}})^2 a^3 = (a^{\text{D}} a^2)(a^{\text{D}} a) = aa^\sigma = a.$$

Conversely, if $a^{\text{D}} = a$, then $a = (a^{\text{D}})^2 a = a^3$. □

As an example of further properties of the g -Drazin inverse in Banach algebras without unit we prove the following result, which for matrices reduces to Theorem 7.8.4 of Campbell and Meyer [4].

PROPOSITION 2.6. *Let \mathcal{A} be a Banach algebra without unit, and let $a, b \in \mathcal{A}$ be such that $(ba)^2 \in \mathcal{A}^{\text{qpol}}$. Then both ab and ba are g -Drazin invertible, and*

$$(2.5) \quad (ab)^{\text{D}} = a((ba)^2)^{\text{D}} b.$$

PROOF. If $(ba)^2 \in \mathcal{A}^{\text{qpol}}$, then also $(ab)^2$, ab and ba are quasipolar, and $w = ((ba)^2)^{\text{D}} = ((ba)^{\text{D}})^2$ commutes with ba . Set $c = a((ba)^2)^{\text{D}}b = awb$. It is not difficult to show that $(ab)c = c(ab)$ and $(ab)c^2 = c$. The element $ab - (ab)^2c = (a - a(ba)^2w)b$ is quasinilpotent if and only if $x = b(a - a(ba)^2w) = ba - (ba)^3w$ is quasinilpotent. Imbedding \mathcal{A} into its unitization \mathcal{A}_1 , we recall that $p = 1 - (ba)^2w$ is idempotent; hence

$$x = (ba)p \in \mathcal{A}^{\text{qnil}} \iff x^2 = (ba)^2p \in \mathcal{A}^{\text{qnil}} \iff (ba)^2 - (ba)^4w \in \mathcal{A}^{\text{qnil}}.$$

This completes the proof. \square

3. The weighted g -Drazin inverse for operators

Throughout this section we assume that X, Y are nonzero complex Banach spaces and W is a fixed nonzero operator in $\mathcal{B}(Y, X)$, the set of all bounded linear operators on Y to X . First we turn $\mathcal{B}(X, Y)$ into a Banach algebra $\mathcal{B}_W(X, Y)$ (in general without a unit) by introducing a multiplication of elements of $\mathcal{B}(X, Y)$ facilitated by the operator W , and imposing a suitable norm on $\mathcal{B}(X, Y)$.

LEMMA 3.1. *Let $\mathcal{B}_W(X, Y)$ be the space $\mathcal{B}(X, Y)$ equipped with the multiplication*

$$(3.1) \quad A \star B = AWB,$$

and norm $\|A\|_W = \|A\|\|W\|$. Then $\mathcal{B}_W(X, Y)$ becomes a complex Banach algebra; $\mathcal{B}_W(X, Y)$ has a unit if and only if W is invertible, in which case W^{-1} is that unit.

PROOF. The verification of most Banach algebra axioms is straightforward. The positive definiteness of the norm is ensured by the fact that $W \neq 0$. We check the submultiplicativity of the norm: If $A, B \in \mathcal{B}(X, Y)$, then

$$(3.2) \quad \|A \star B\|_W = \|AWB\|\|W\| \leq \|A\|\|W\|\|B\|\|W\| = \|A\|_W\|B\|_W.$$

If W is invertible, then $W^{-1} \in \mathcal{B}(X, Y)$ is the unit in $\mathcal{B}_W(X, Y)$. Conversely, assume that $P \in \mathcal{B}(X, Y)$ is the unit for $\mathcal{B}_W(X, Y)$. Then

$$(3.3) \quad AWP = A = PWA \quad \text{for all } A \in \mathcal{B}(X, Y).$$

For each $y \in Y$ and $f \in X^*$ define $f \otimes y : X \rightarrow Y$ by $(f \otimes y)x = f(x)y$ for all $x \in X$; then $f \otimes y \in \mathcal{B}(X, Y)$. From $PW(f \otimes y)x = (f \otimes y)x$ we get $f(x)PWy = f(x)y$. Selecting x and f so that $f(x) \neq 0$, we obtain $PWy = y$ for any $y \in Y$. From $(f \otimes y)WPx = (f \otimes y)x$ we get $f(WPx)y = f(x)y$. Selecting $y \neq 0$, we get $f(WPx) = f(x)$ for all $f \in X^*$, which implies $WPx = x$ for any $x \in X$. Then W is invertible; setting $A = W^{-1}$ in (3.3), we get $W^{-1} = PWW^{-1} = P$. \square

We observe that if $\mathcal{B}_W(X, Y)$ has the unit W^{-1} , the spaces X and Y are isomorphic and homeomorphic; in particular, X and Y are of the same dimension. Moreover, the norm of the unit in $\mathcal{B}_W(X, Y)$ is equal to $\|W^{-1}\|_W = \|W^{-1}\| \|W\| = \kappa(W)$, known as the condition number of W .

For any $n \in \mathbb{N}$ we write $A^{*n} = A \star \cdots \star A$ (n factors). Observe that

$$(3.4) \quad A^{*n} = (AW)^{n-1}A = A(WA)^{n-1}.$$

We write $r_W(\cdot)$ for the spectral radius of elements of $\mathcal{B}_W(X, Y)$. We show that

$$(3.5) \quad r_W(A) = r(AW) = r(WA),$$

where $r(\cdot)$ is the spectral radius in $\mathcal{B}(Y)$ or $\mathcal{B}(X)$. Indeed,

$$\begin{aligned} r(AW) &= \lim_{n \rightarrow \infty} \|(AW)^n\|^{1/n} \leq \lim_{n \rightarrow \infty} (\|(AW)^{n-1}A\| \|W\|)^{1/n} \\ &= \lim_{n \rightarrow \infty} \|A^{*n}\|_W^{1/n} = r_W(A). \end{aligned}$$

Conversely,

$$\begin{aligned} r_W(A) &= \lim_{n \rightarrow \infty} \|A^{*n}\|_W^{1/n} = \lim_{n \rightarrow \infty} \|A^{*n}\|^{1/n} \|W\|^{1/n} \\ &= \lim_{n \rightarrow \infty} \|(AW)^{n-1}A\|^{1/n} \|W\|^{1/n} \\ &\leq \lim_{n \rightarrow \infty} \|(AW)^{n-1}\|^{1/n} \lim_{n \rightarrow \infty} (\|A\| \|W\|)^{1/n} \\ &= \lim_{n \rightarrow \infty} \|(AW)^{n-1}\|^{1/n} = r(AW) \end{aligned}$$

as $\lim_{n \rightarrow \infty} \|(AW)^{n-1}\|^{1/n} = \lim_{n \rightarrow \infty} \|(AW)^n\|^{1/n}$. The second equality in (3.5) follows by symmetry.

DEFINITION 3.2. Let W be a fixed nonzero operator in $\mathcal{B}(Y, X)$. An operator $A \in \mathcal{B}(X, Y)$ is called *Wg-Drazin invertible* if A is quasipolar in the Banach algebra $\mathcal{B}_W(X, Y)$. The *Wg-Drazin inverse* $A^{\mathbf{D}, W}$ of A (or *W-weighted g-Drazin inverse*) is then defined as the g -Drazin inverse B of A in the Banach algebra $\mathcal{B}_W(X, Y)$; $i_W(A)$ is the Drazin index of A in $\mathcal{B}_W(X, Y)$. A polar element of $\mathcal{B}_W(X, Y)$ is called *W-Drazin invertible*, with the *W-Drazin inverse* $A^{\mathbf{D}, W} = B$.

The Wg -Drazin inverse is unique if it exists (Proposition 2.2), and is characterized by the following theorem.

THEOREM 3.3. *Let W be a fixed nonzero operator in $\mathcal{B}(Y, X)$. Then $A \in \mathcal{B}(X, Y)$ is Wg -Drazin invertible with the Wg -Drazin inverse $A^{\mathbf{D}, W} = B \in \mathcal{B}(X, Y)$ if and only if one of the following equivalent conditions holds:*

- (i) AW is quasipolar in $\mathcal{B}(Y)$ with $(AW)^{\mathbf{D}} = BW$.

(ii) WA is quasipolar in $\mathcal{B}(X)$ with $(WA)^D = WB$.

(iii) There exists $B \in \mathcal{B}(X, Y)$ satisfying

$$(AW)B = (BW)A, \quad (BW)^2A = B, \quad (AW)^2BW - AW \in \mathcal{B}(Y)^{\text{qnil}}.$$

(iv) There exists $B \in \mathcal{B}(X, Y)$ satisfying

$$A(WB) = B(WA), \quad A(WB)^2 = B, \quad WB(WA)^2 - WA \in \mathcal{B}(X)^{\text{qnil}}.$$

The Wg -Drazin inverse $A^{D,W}$ of A then satisfies

$$(3.6) \quad A^{D,W} = ((AW)^D)^2A = A((WA)^D)^2.$$

PROOF. Suppose that A has the Wg -Drazin inverse B .

The conditions

$$A \star B = B \star A, \quad B \star A \star B = B, \quad A \star B \star A - A \in \mathcal{B}_W(X, Y)^{\text{qnil}},$$

translate to

$$(3.7) \quad AWB = BWA, \quad (BW)^2A = B, \quad T = (AW)^2B - A \in \mathcal{B}_W(X, Y)^{\text{qnil}}.$$

Let $C = BW$. Then $(AW)C = C(AW)$ and $C^2(AW) = C$ by (3.7). Finally, by (3.5), $r(TW) = r_W(T) = 0$. Hence $(AW)^2C - AW = TW$ is quasinilpotent in $\mathcal{B}(Y)$, and (i) is proved.

Condition (ii) follows from a symmetrical argument. Conditions (i) and (iii) (respectively (ii) and (iv)) are equivalent by the characterization of the g -Drazin inverse given in Proposition 2.2.

Conversely, suppose that $AW \in \mathcal{B}(Y)$ has the g -Drazin inverse C . Let $B = C^2A$. The equations $(AW)C = C(AW)$ and $C^2(AW) = C$ imply

$$\begin{aligned} A \star B &= AWC^2A = C^2AWA = B \star A, \text{ and} \\ B \star A \star B &= (C^2AW)(AWC^2)A = C^2A = B. \end{aligned}$$

Write $A \star B \star A - A = (AWC^2)AWA - A = CAWA - A = S$. Since $SW = C(AW)^2 - AW$ is quasinilpotent in $\mathcal{B}(Y)$, $r_W(S) = r(SW) = 0$, and S is quasinilpotent in $\mathcal{B}_W(X, Y)$. This proves that condition (i) implies that A is Wg -Drazin invertible with $A^{D,W} = C^2A$. The rest follows from Proposition 2.2 by symmetry. \square

From Equation 3.6 we find an expression for the support idempotent $A^{\sigma,W}$ of A in $\mathcal{B}_W(X, Y)$: $A^{\sigma,W} = A \star A^{D,W} = AW((AW)^D)^2A = (AW)^DA$. By symmetry,

$$(3.8) \quad A^{\sigma,W} = (AW)^DA = A(WA)^D.$$

PROPOSITION 3.4. *If $A \in \mathcal{B}(X, Y)$ is Wg -Drazin invertible, then the Drazin indices $i_W(A)$, $i(WA)$ and $i(AW)$ are all finite or all infinite, and satisfy the inequalities*

$$(3.9) \quad \max \{i(AW), i(WA)\} \leq i_W(A) \leq \min \{i(AW), i(WA)\} + 1.$$

PROOF. Let $A^{D,W} = B$ be the Wg -Drazin inverse of A and let $T = (AW)^2B - A$. If $i_W(A) = k < \infty$, then $T^{*k} = 0$; consequently $(TW)^k = (TW)^{k-1}TW = T^{*k}W = 0$; hence $i(AW) \leq i_W(A)$.

Let AW have the g -Drazin inverse C and let $S = CAWA - A$. If $i(AW) = k < \infty$, then $(SW)^k = 0$, and $S^{*(k+1)} = (SW)^kS = 0$, that is, $i_W(A) \leq k + 1$. This proves the inequality for $i(AW)$ in (3.9).

It is known that for any $A \in \mathcal{B}(X, Y)$ and $W \in \mathcal{B}(Y, X)$,

$$(3.10) \quad \text{Sp}(AW) \setminus \{0\} = \text{Sp}(WA) \setminus \{0\}.$$

Hence AW is g -Drazin invertible in $\mathcal{B}(Y)$ if and only if WA is g -Drazin invertible in $\mathcal{B}(X)$. The inequality for $i(WA)$ in (3.9) is obtained by symmetry. \square

EXAMPLE 3.5. The inequality $i(AW) \leq i_W(A)$ (respectively $i(WA) \leq i_W(A)$) in (3.9) can be strict. Let

$$W = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix},$$

and let \mathcal{B}_W be the space $\mathcal{M}_{2,3}(\mathbb{C})$ of all complex 2×3 matrices with the multiplication (3.1). By the preceding theorem, every element $A \in \mathcal{M}_{2,3}(\mathbb{C})$ has a g -Drazin inverse of finite Drazin index in \mathcal{B}_W since the matrix AW has the conventional Drazin inverse $(AW)^D$ in $\mathcal{M}_{2,2}(\mathbb{C})$. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $AW = 0 = B$, where B is the g -Drazin inverse of A in \mathcal{B}_W , and $T = (AW)^2B - A = -A$. Since $T \neq 0$ and $T \star T = AWA = 0$, we have $i_W(A) = 2$. On the other hand, $i(AW) = i(0) = 1$. An example of a strict inequality between $i(WA)$ and $i_W(A)$ can be obtained from the present example and the following proposition involving the dual spaces X^* and Y^* of X and Y .

PROPOSITION 3.6. *$A \in \mathcal{B}(X, Y)$ is Wg -Drazin invertible if and only if the adjoint $A^* \in \mathcal{B}(Y^*, X^*)$ of A is W^*g -Drazin invertible. In this case*

$$(3.11) \quad (A^*)^{D,W^*} = (A^{D,W})^*.$$

PROOF. Since $\text{Sp}((AW)^*) = \text{Sp}(AW)$, $(AW)^*$ is quasipolar if and only if AW is quasipolar. Hence $W^*A^* = (AW)^*$ is g -Drazin invertible if and only if AW is. By Theorem 3.3, A^* is W^*g -Drazin invertible if and only if A is Wg -Drazin invertible. Equation (3.11) follows on application of Proposition 2.2. \square

EXAMPLE 3.7. (Rakočević and Wei [13].) If $A \in \mathcal{B}(X, Y)$ is a finite rank operator, then A has a finite index Wg -Drazin inverse for any nonzero $W \in \mathcal{B}(Y, X)$. If $W \in \mathcal{B}(Y, X)$ is a nonzero operator of finite rank, then any $A \in \mathcal{B}(X, Y)$ has a finite index Wg -Drazin inverse.

4. Further properties of the Wg -Drazin inverse

First we briefly explore a duality between $A^{D,W}$ and $W^{D,A}$ provided $A \in \mathcal{B}(X, Y)$ and $W \in \mathcal{B}(Y, X)$. From Theorem 3.3 we see that the weighted g -Drazin inverse $W^{D,A}$ exists if and only if $A^{D,W}$ exists. Equation (3.6) gives rise to the following relations:

$$(4.1) \quad W^{D,A}A = (WA)^D = WA^{D,W},$$

$$(4.2) \quad AW^{D,A} = (AW)^D = A^{D,W}W.$$

We can then express $W^{D,A}$ in terms of $A^{D,W}$ and vice versa:

$$(4.3) \quad W^{D,A} = WA^{D,W}WA^{D,W}W,$$

$$(4.4) \quad A^{D,W} = AW^{D,A}AW^{D,A}A.$$

We observe that in Equation (4.4), the operators $AW^{D,A}$ and $W^{D,A}A$ are simply polar (that is, of index 1 or 0): *e.g.* $AW^{D,A} = AW((AW)^D)^2 = (AW)^D$; the simple polarity of the g -Drazin inverse of AW is well known (see [7]). Specialized to matrices, this proves the necessary part of Theorem 3 in [6].

PROPOSITION 4.1. *Let $A \in \mathcal{B}(X, Y)$ be Wg -Drazin invertible. Then the following are true:*

- (i) $A = A^{D,W}$ if and only if $A = A^{*3} = AWAWA$.
- (ii) $(A^{D,W})^{D,W} = (AW)^\sigma A = A(WA)^\sigma$.
- (iii) $(A^{D,W})^{\sigma,W} = A^{\sigma,W}$.
- (iv) For any $n \in \mathbb{N}$, $(A^{D,W})^{*n} = ((AW)^D)^{n+1}A = A((WA)^D)^{n+1} = (A^{*n})^{D,W}$.

PROOF. (i) Follows from Proposition 2.5 applied to $\mathcal{B}_W(X, Y)$.

(ii) Applying the results of [7] while working in the Banach algebra $\mathcal{B}_W(X, Y)$, we have $(A^{\mathcal{D}, W})^{\mathcal{D}, W} = A \star A^{\sigma, W} = AW(AW)^{\mathcal{D}}A = (AW)^{\sigma}A$.

(iii) In the proof of [7, Theorem 5.2] it is shown that a quasipolar element and its g -Drazin inverse have the same support idempotent.

(iv) Induction on n . □

Part (ii) of the preceding theorem implies that $(A^{\mathcal{D}, W})^{\mathcal{D}, W} = A$ if and only if $(AW)^{\sigma}A = A(A(WA)^{\sigma} = A)$. This is equivalent to A being simply polar in $\mathcal{B}_W(X, Y)$.

From [7, Theorem 5.5] we can deduce the following result.

PROPOSITION 4.2. *Let $A, B \in \mathcal{B}(X, Y)$ be Wg -Drazin invertible. If $AWB = BWA$, then AWB is Wg -Drazin invertible with*

$$(4.5) \quad (AWB)^{\mathcal{D}, W} = A^{\mathcal{D}, W}WB^{\mathcal{D}, W}.$$

We now turn our attention to an analogue of the core decomposition for the weighted g -Drazin inverse.

THEOREM 4.3. *An operator $A \in \mathcal{B}(X, Y)$ is Wg -Drazin invertible if and only if there exist operators $C, U \in \mathcal{B}(X, Y)$ such that*

$$(4.6) \quad A = C + U, \quad CWU = 0, \quad UWC = 0,$$

$$(4.7) \quad (CW)^{\sigma}C = C, \quad UW \in \mathcal{B}(Y)^{\text{qnil}}.$$

Such operators are uniquely determined, and $C = (A^{\mathcal{D}, W})^{\mathcal{D}, W} = (AW)^{\sigma}A$. Further,

$$(4.8) \quad (AW)^{\mathcal{D}} = (CW)^{\mathcal{D}}, \quad (AW)^{\sigma} = (CW)^{\sigma}, \quad \text{Sp}(AW) \cup \{0\} = \text{Sp}(CW).$$

PROOF. We apply Theorem 2.4 to $\mathcal{B}_W(X, Y)$: A is Wg -Drazin invertible if and only if there exist $C, U \in \mathcal{B}(X, Y)$ such that $A = C + U$, $C \star U = CWU = 0$, $U \star C = UWC = 0$, C is simply polar in $\mathcal{B}_W(X, Y)$, and U is quasinilpotent in $\mathcal{B}_W(X, Y)$. The element $C \in \mathcal{B}_W(X, Y)$ is simply polar if and only if $C \star C^{\sigma, W} = C$. From the equation

$$C \star C^{\sigma, W} = CW(CW)^{\mathcal{D}}C = (CW)^{\sigma}C$$

we conclude that the simple polarity of $C \in \mathcal{B}_W(X, Y)$ is equivalent to $(CW)^{\sigma}C = C$. Finally, $r_W(U) = r(UW)$, and UW is quasinilpotent in $\mathcal{B}(X, Y)$ if and only if U is quasinilpotent in $\mathcal{B}_W(X, Y)$. This proves the equivalence of (4.6) and (4.7) to the Wg -Drazin invertibility of A . Explicitly, $C = A \star A^{\sigma, W} = (AW)^{\sigma}A$.

Towards (4.8) we note that in view of Theorem 2.4,

$$(AW)^{\mathbb{D}} = ((AW)^{\mathbb{D}})^2 AW = A^{\mathbb{D},W} W = C^{\mathbb{D},W} W = ((CW)^{\mathbb{D}})^2 CW = (CW)^{\mathbb{D}}.$$

Therefore

$$(4.9) \quad (CW)^{\sigma} = (CW)^{\mathbb{D}} C = (AW)^{\mathbb{D}} C = (AW)^{\mathbb{D}} (AW)^{\sigma} A = (AW)^{\mathbb{D}} A = (AW)^{\sigma}.$$

If $\mathbf{Sp}_W(A)$ denotes the spectrum of A as an element of the Banach algebra $\mathcal{B}_W(X, Y)$ without unit, then it can be shown that

$$\mathbf{Sp}_W(A) = \mathbf{Sp}(AW) \cup \{0\}.$$

Hence $\mathbf{Sp}(AW) \cup \{0\} = \mathbf{Sp}_W(A) = \mathbf{Sp}_W(C) = \mathbf{Sp}(CW)$ (as $0 \in \mathbf{Sp}(CW)$). This completes the proof. \square

The statement of the theorem remains true when (4.7) is replaced by $C(WC)^{\sigma} = C$, $WU \in \mathcal{B}(X)^{\text{anil}}$, and AW, CW in (4.8) are replaced by WA, WC , respectively.

We close the section with an integral representation of the Wg -Drazin inverse. The representation of the g -Drazin inverse given by Castro *et al.* [5, Theorem 2.2] is valid also for Banach algebras without unit. Applying this result to $\mathcal{B}_W(X, Y)$, we get the integral representation

$$(4.10) \quad A^{\mathbb{D},W} = - \int_0^{\infty} \exp(tA) \star A^{\sigma,W} dt$$

provided A is Wg -Drazin invertible and the nonzero spectrum $\mathbf{Sp}_W(A) \setminus \{0\}$ lies in the open left half plane. We express $\exp(tA) \star A^{\sigma,W}$ in terms of the usual multiplication of operators:

$$A^{\star n} \star A^{\sigma,W} = (AW)^{n-1} AW A^{\sigma,W} = (AW)^n A^{\sigma,W}.$$

Hence

$$\exp(tA) \star A^{\sigma,W} = \sum_{n=0}^{\infty} \frac{t^n}{n!} (AW)^n A^{\sigma,W} = \exp(tAW) A^{\sigma,W}.$$

Note that in general $\exp(tAW)$ belongs to the unitization of $\mathcal{B}_W(X, Y)$ but not to $\mathcal{B}_W(X, Y)$, while $\exp(tAW) A^{\sigma,W}$ is in $\mathcal{B}_W(X, Y)$. We summarize our findings:

PROPOSITION 4.4. *Let $A \in \mathcal{B}(X, Y)$ be Wg -Drazin invertible such that $\mathbf{Sp}(WA) \setminus \{0\}$ lies in the left open half plane. Then*

$$(4.11) \quad A^{\mathbb{D},W} = - \int_0^{\infty} \exp(tAW) A^{\sigma,W} dt.$$

If the Drazin index $i(AW)$ is finite and the set $\text{Sp}((AW)^{m+1}) \setminus \{0\}$ lies in the left open half plane for some $m \geq \min\{i(AW), i(WA)\} + 1$, then

$$(4.12) \quad A^{\text{D},W} = - \int_0^\infty \exp(t(AW)^{m+1})(AW)^{m-1}A dt.$$

PROOF. Equation (4.11) follows from our calculations preceding the theorem. For (4.12) we find

$$(A^{\star(m+1)})^{\star n} \star A^{\star m} = A^{\star(m+1)n+m} = (AW)^{(m+1)n}(AW)^{m-1}A,$$

and

$$\exp(tA^{\star(m+1)}) \star A^{\star m} = \exp(t(AW)^{m+1})(AW)^{m-1}A.$$

Equation (4.12) then follows from [5, Theorem 2.4]. \square

As expected from symmetry, there is also a WA version of the preceding theorem. If we specialize Equation (4.12) to matrices, we recover [18, Theorem 1]. The inequality $m \geq \min\{i(AW), i(WA)\} + 1$ in the preceding theorem can be relaxed to $m \geq i_W(A)$.

Using the core decomposition of a Wg -Drazin invertible operator $A \in \mathcal{B}(X, Y)$, we obtain yet another integral representation for $A^{\text{D},W}$:

COROLLARY 4.5. *Let $A \in \mathcal{B}(X, Y)$ be Wg -Drazin invertible such that $\text{Sp}((WA)^2) \setminus \{0\}$ lies in the open left half plane, and let $A = C + U$ be the core decomposition of A . Then*

$$(4.13) \quad A^{\text{D},W} = C^{\text{D},W} = - \int_0^\infty \exp(t(CW)^2)C dt.$$

PROOF. This follows from (4.12) when we note that $i_W(C) = 1$. \square

5. Ascent and descent

We recall that the ascent and descent of an operator $T \in \mathcal{B}(X)$ are defined by

$$\begin{aligned} \text{asc}(T) &= \inf \{k \in \mathbb{N} \cup \{0\} : N(T^{k+1}) = N(T^k)\}, \\ \text{des}(T) &= \inf \{k \in \mathbb{N} \cup \{0\} : R(T^{k+1}) = R(T^k)\} \end{aligned}$$

($\inf \emptyset = \infty$). Rakočević and Wei [13] ask whether the finiteness of $\text{asc}(AW)$ and $\text{des}(WA)$ is sufficient for A to have the W -weighted Drazin inverse. An equivalent question is whether $\text{asc}(AW)$ and $\text{asc}(WA)$ are always both finite or both infinite.

In this connection it is interesting to recall that Buoni and Faires [3] studied the ascent and descent for the operators $\lambda I - BA$ and $\lambda I - AB$, where $A, B \in \mathcal{B}(X)$, and proved, *inter alia*, that for any $\lambda \neq 0$,

$$(5.1) \quad \text{asc}(AB - \lambda I) = \text{asc}(BA - \lambda I), \quad \text{des}(AB - \lambda I) = \text{des}(BA - \lambda I);$$

however, the case $\lambda = 0$ was left open. Later Barnes [1] proved by different methods that the ascents of $I - RS$ and $I - SR$ are equal for $R \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(Y, X)$. It can be shown that the arguments in [3] concerning descent are valid also when $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, X)$. Thus (5.1) is valid for operators between different spaces. The following theorem, dealing with the ascent and descent in general, completes the results of Buoni and Faires in the case $\lambda = 0$.

THEOREM 5.1. *Let $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, X)$. Then the ascents (descents) of AB and BA are both finite or both infinite, and satisfy the inequalities*

$$(5.2) \quad \begin{aligned} \text{asc}(AB) - 1 &\leq \text{asc}(BA) \leq \text{asc}(AB) + 1, \\ \text{des}(AB) - 1 &\leq \text{des}(BA) \leq \text{des}(AB) + 1. \end{aligned}$$

PROOF. Suppose that $\text{asc}(AB) = p < \infty$. If there existed

$$x \in N((BA)^{p+2}) \setminus N((BA)^{p+1}),$$

we would have

$$(AB)^{p+2}Ax = A(BA)^{p+2}x = 0,$$

and $B(AB)^pAx = (BA)^{p+1}x \neq 0$, that is,

$$(AB)^pAx \neq 0.$$

Then Ax would belong to $N((AB)^{p+2}) \setminus N((AB)^p)$, which is empty by assumption. This contradiction proves $N((BA)^{p+1}) = N((BA)^{p+2})$, which shows that $\text{asc}(BA) \leq p + 1 = \text{asc}(AB) + 1$. A symmetrical argument gives $\text{asc}(AB) \leq \text{asc}(BA) + 1$. This proves the first inequality in (5.2).

Let $\text{des}(AB) = p < \infty$. Suppose

$$(5.3) \quad x \in R((BA)^{p+1}) \setminus R((BA)^{p+2}).$$

Then there exists $x' \in X$ such that

$$x = (BA)^{p+1}x' = B(AB)^pAx' = By,$$

where $y = (AB)^pAx' \in R((AB)^p) = R((AB)^{p+2})$. Hence $y = (AB)^{p+2}y'$ for some $y' \in Y$, and

$$(BA)^{p+2}By' = B(AB)^{p+2}y' = By = x$$

contrary to (5.3). This proves that $R((BA)^{p+1}) = R((BA)^{p+2})$, so that $\text{des}(BA) \leq p + 1$. \square

The inequalities in (5.2) can be strict; this follows from Example 3.5 since for matrices $i(AB) = \text{asc}(AB) = \text{des}(AB)$.

The following theorem gives a solution to the open problem of Rakočević and Wei [13, p. 28].

THEOREM 5.2. *Let $A \in \mathcal{B}(X, Y)$ and $W \in \mathcal{B}(Y, X) \setminus \{0\}$. Then A is W -Drazin invertible if and only if one of the following equivalent conditions hold:*

- (i) AW is polar in $\mathcal{B}(Y)$.
- (ii) WA is polar in $\mathcal{B}(X)$.
- (iii) $\text{asc}(AW)$ and $\text{des}(WA)$ are both finite.
- (iv) $\text{asc}(WA)$ and $\text{des}(AW)$ are both finite.

PROOF. Suppose that A is W -Drazin invertible. By Theorem 3.3, AW is quasipolar, and by (3.9) we have $i(AW) \leq i_W(A)$, which proves that AW is polar. Conversely, if AW is polar, then $i_W(A) \leq i(AW) + 1$, and A is W -Drazin invertible.

(i) \implies (ii). Since AW is quasipolar, so is WA by (3.10). By (3.9) again, $i(WA) \leq i(AW) + 1$, and WA is polar.

(ii) \implies (iii). It is well known that if WA is polar, then $\text{asc}(WA)$ and $\text{des}(WA)$ are both finite. But $\text{asc}(AW) \leq \text{asc}(WA) + 1$ by Theorem 5.1 and (iii) follows.

(iii) \implies (iv). Follows from Theorem 5.1 as $\text{asc}(WA) \leq \text{asc}(AW) + 1$ and $\text{des}(AW) \leq \text{des}(WA) + 1$.

(iv) \implies (i). Since $\text{asc}(AW) \leq \text{asc}(WA) + 1$, both $\text{asc}(AW)$ and $\text{des}(AW)$ are finite; this implies that AW is polar. \square

6. The Mbekhta decomposition for WA and AW

As before, X, Y are Banach spaces and W a nonzero operator in $\mathcal{B}(Y, X)$. In order to obtain an operator matrix representation for the weighted g -Drazin inverse of an operator $A \in \mathcal{B}(X, Y)$, we first recall the Mbekhta decomposition for a quasipolar operator. For any operator $T \in \mathcal{B}(X)$ we define spaces $H_0(T)$ and $K(T)$ as follows:

$$H_0(T) = \{x \in X : \lim_{n \rightarrow \infty} \|T^n x\|^{1/n} = 0\},$$

$$K(T) = \{x \in X : \exists x_n \in X, x_n = Tx_{n+1}, x_0 = x, \sup_{n \in \mathbb{N}} \|x_n\|^{1/n} < \infty\}.$$

Both spaces are hyperinvariant under T , $H_0(T) \supset N(T^n)$, and $K(T) \subset R(T^n)$ for all $n \in \mathbb{N}$. Further,

$$TK(T) = K(T), \quad T^{-1}H_0(T) = H_0(T).$$

PROPOSITION 6.1. (See [8, 11].) *The following conditions on $T \in \mathcal{B}(X)$ are equivalent:*

- (i) T is quasipolar.
- (ii) X is the topological direct sum $X = K(T) \oplus H_0(T)$.
- (iii) $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 quasinilpotent.

Condition (ii) can be weakened to $X = K(T) \oplus H_0(T)$ being only an algebraic sum with at least one of the spaces closed (see [10] and [15]).

THEOREM 6.2. *Let $A \in \mathcal{B}(X, Y)$ and $W \in \mathcal{B}(Y, X)$. If WA is quasipolar, then so is AW ,*

$$(6.1) \quad \begin{aligned} A(K(WA)) &= K(AW), & A^{-1}(H_0(AW)) &= H_0(WA), \\ W(K(AW)) &= K(WA), & W^{-1}(H_0(WA)) &= H_0(AW), \end{aligned}$$

and the spaces $K(WA)$, $K(AW)$ are isomorphic and homeomorphic.

PROOF. The result on quasipolarity follows from (3.10). We introduce the following notation:

$$(6.2) \quad X_1 = K(WA), \quad X_2 = H_0(WA), \quad Y_1 = K(AW), \quad Y_2 = H_0(AW);$$

X and Y are then decomposed into the topological direct sums $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$. The operator matrices

$$(6.3) \quad T = \begin{bmatrix} WA & 0 \\ 0 & AW \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}$$

represent commuting operators in $\mathcal{B}(X \oplus Y)$ with T quasipolar. The support projection T^σ of T double commutes with T , that is, the matrix

$$T^\sigma = \begin{bmatrix} (WA)^\sigma & 0 \\ 0 & (AW)^\sigma \end{bmatrix}$$

commutes with the matrix S . This gives $A(WA)^\sigma = (AW)^\sigma A$. Since $(WA)^\sigma$ is the projection of X onto X_1 along X_2 , and $(AW)^\sigma$ is the projection of Y onto Y_1 along Y_2 , we have $A(X_i) \subset Y_i$ ($i = 1, 2$). The inclusions $W(Y_i) \subset X_i$ ($i = 1, 2$) are obtained by symmetry.

Note that $A(X_2) \subset Y_2$ is equivalent to $X_2 \subset A^{-1}(Y_2)$. In order to prove $A^{-1}(Y_2) \subset X_2$, assume that $Ax \in Y_2$. Then $x = k + h$ with $k \in X_1$ and $h \in X_2$, and $Ax = Ak + Ah \in Y_2$ implies that $Ak = 0$. From $k \in N(A) \subset N(WA) \subset X_2$ we obtain $k \in X_1 \cap X_2 = \{0\}$. Hence $x = h \in X_2$.

Let $A_0 : X_1 \rightarrow Y_1$ be the restriction of A . If $x \in X_1$ and $Ax = 0$, then $x = 0$ by the same argument as in the preceding paragraph. Hence A_0 is injective. Suppose that $y \in Y_1$. Since $AWY_1 = Y_1$, there exists $u \in Y_1$ such

that $y = AWu$. But $WY_1 \subset X_1$, and so $Wu \in X_1$. This proves that A_0 is surjective. Therefore A_0 is a bounded linear bijection from X_1 to Y_1 , and (6.1) is proved. \square

In particular, if AW is quasipolar, then the spaces $K(AW)$ and $K(WA)$ have the same dimension being isomorphic.

If A and W are rectangular matrices of orders $m \times n$ and $n \times m$ respectively, we recover the result of Yuhno [19, Theorem]. For this we note that the operator $T : \mathbb{C}^m \rightarrow \mathbb{C}^m$ with the matrix WA is polar, and that $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 nilpotent; T_1 operates on $X_1 = K(T) = R(T^p)$, where p is the index of T . The eigenvalues of T_1 are the nonzero eigenvalues of WA . Let λ be a nonzero eigenvalue of WA , and x_1, \dots, x_k a chain of generalized eigenvectors of WA corresponding to λ , that is,

$$WAx_1 = \lambda x_1 + x_2, \quad \dots, \quad WAx_{k-1} = \lambda x_{k-1} + x_k, \quad WAx_k = \lambda x_k.$$

In view of the decomposition of T as $T = T_1 \oplus T_2$, where T_1 operates on X_1 , we can take $x_i \in X_1$ for all i . If $y_i = Ax_i$ ($i = 1, \dots, k$), then y_1, \dots, y_k is a chain of generalized eigenvectors of AW corresponding to λ (this follows from the bijectivity of the operator $x \mapsto Ax$ restricted from X_1 to Y_1). All chains corresponding to nonzero eigenvalues of WA are matched in this way. This leads to the following structure theorem for WA and AW .

PROPOSITION 6.3. *Let A and W be rectangular matrices of orders $m \times n$ and $n \times m$, respectively. The matrices WA and AW (of orders $n \times n$ and $m \times m$, respectively) have Jordan forms*

$$\begin{bmatrix} U & 0 \\ 0 & N_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U & 0 \\ 0 & N_2 \end{bmatrix},$$

where U is a matrix in Jordan form corresponding to the nonzero eigenvalues of WA (and AW), while N_1 and N_2 are nilpotent matrices in Jordan form, in general of different orders.

Recall that the entries of N_1 and N_2 are zero except for the superdiagonals of the two matrices, which consist of 0s and 1s.

7. An operator matrix representation of the Wg -Drazin inverse

From the Mbekhta decomposition theorem (Proposition 6.1) it follows that an operator $T \in \mathcal{B}(X)$ is quasipolar if and only if it can be expressed as the direct sum $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 quasinilpotent; the g -Drazin inverse of T is given by

$$T^{\text{D}} = T_1^{-1} \oplus 0.$$

Our aim is to derive an analogous formula for the Wg -Drazin inverse using the results of the preceding section.

THEOREM 7.1. *Let $A \in \mathcal{B}(X, Y)$ and $W \in \mathcal{B}(Y, X) \setminus \{0\}$. Then A is Wg -Drazin invertible if and only if there exist topological direct sums $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$ such that*

$$(7.1) \quad A = A_1 \oplus A_2, \quad W = W_1 \oplus W_2,$$

where $A_i \in \mathcal{B}(X_i, Y_i)$, $W_i \in \mathcal{B}(Y_i, X_i)$, with A_1, W_1 invertible, and W_2A_2 and A_2W_2 quasinilpotent in $\mathcal{B}(X_2)$ and $\mathcal{B}(Y_2)$, respectively. The Wg -Drazin inverse of A is given by

$$(7.2) \quad A^{D,W} = (W_1A_1W_1)^{-1} \oplus 0$$

with $(W_1A_1W_1)^{-1} \in \mathcal{B}(X_1, Y_1)$ and $0 \in \mathcal{B}(X_2, Y_2)$.

PROOF. If WA is quasipolar, the decomposition exists with X_i and Y_i given by (6.2). By the preceding theorem, A maps X_1 into Y_1 , and X_2 into Y_2 , that is,

$$(7.3) \quad A = A_1 \oplus A_2,$$

with $A_i \in \mathcal{B}(X_i, Y_i)$, $i = 1, 2$. Similarly, since W maps Y_1 into X_1 and Y_2 into X_2 ,

$$(7.4) \quad W = W_1 \oplus W_2,$$

where $W_i \in \mathcal{B}(Y_i, X_i)$, $i = 1, 2$. Hence

$$(7.5) \quad WA = W_1A_1 \oplus W_2A_2, \quad AW = A_1W_1 \oplus A_2W_2$$

relative to $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$. Since WA and AW are quasipolar, W_1A_1 and A_1W_1 are invertible, and W_2A_2 and A_2W_2 are quasinilpotent. Hence A_1 and W_1 are invertible.

The Wg -Drazin inverse of A is equal to

$$A((WA)^D)^2 = (A_1 \oplus A_2)((W_1A_1)^{-2} \oplus 0) = (W_1A_1W_1)^{-1} \oplus 0.$$

Conversely, if the decompositions with the specified properties exist, then $AW = (A_1W_1) \oplus (A_2W_2)$ is quasipolar as A_1W_1 is invertible and A_2W_2 quasinilpotent. Then A is Wg -Drazin invertible. \square

From the necessary part of the preceding theorem we recover [18, Theorem 2] when we specialize the operators to finite matrices. From Theorem 6.2 applied to finite matrices we deduce that the ranks of $(AW)^m$ and $(WA)^m$ are equal for any $m \geq \max\{\text{ind}(AW), \text{ind}(WA)\}$. (This is used, but not proved, in the derivation of [18, Theorem 2]).

From the commutativity of the operator matrices given in (6.3) and the double commutativity of the g -Drazin inverse we deduce that $(AW)^D A = A(WA)^D$, which leads to a new equality for $A^{D,W}$ derived from (3.6):

$$(7.6) \quad A^{D,W} = (AW)^D A (WA)^D.$$

8. Relation to the Moore–Penrose inverse

We briefly address the relation of the Wg -Drazin inverse to the Moore–Penrose inverse in Hilbert spaces (see [13, page 28]). Let H, K be Hilbert spaces and let $A \in \mathcal{B}(H, K)$. It is well known that

$$\begin{aligned} R(A) \text{ closed} &\iff R(A^*) \text{ closed} \\ \iff A^*A \text{ simply polar} &\iff AA^* \text{ simply polar.} \end{aligned}$$

This means that $A \in \mathcal{B}(H, K)$ is A^*g -Drazin invertible if and only if the range of A is closed. We note that

$$(8.1) \quad (A^{\mathbf{D}, A^*})^* = (A^*)^{\mathbf{D}, A}.$$

We can then prove that the operator

$$A^\dagger = (A^*)^{\sigma, A} = A^* A^{\mathbf{D}, A^*} A^*$$

is the Moore–Penrose inverse characterized by the equations

$$(8.2) \quad A^\dagger AA^\dagger = A^\dagger, \quad AA^\dagger A = A, \quad (A^\dagger A)^* = A^\dagger A, \quad (AA^\dagger)^* = AA^\dagger.$$

We offer a sample of such proof:

$$A^\dagger AA^\dagger = (A^*)^{\sigma, A} A (A^*)^{\sigma, A} = (A^*)^{\sigma, A} \circ (A^*)^{\sigma, A} = (A^*)^{\sigma, A} = A^\dagger,$$

where $T \circ S = T A S$, and

$$AA^\dagger A = AA^* A^{\mathbf{D}, A^*} A^* A = A \star A^{\mathbf{D}, A^*} \star A = A,$$

where $T \star S = T A^* S$. Other equations in (8.2) can be proved similarly.

9. Continuity of the Wg -Drazin inverse

THEOREM 9.1. *Let $A_n \rightarrow A_0$ in $\mathcal{B}(X, Y)$ and $W_n \rightarrow W_0 \neq 0$ in $\mathcal{B}(Y, X)$, where each A_n is $W_n g$ -Drazin invertible, $n = 0, 1, 2, \dots$. Then the following conditions are equivalent:*

- (i) $A_n^{\mathbf{D}, W_n} \rightarrow A_0^{\mathbf{D}, W_0}$.
- (ii) $\sup_n \|A_n^{\mathbf{D}, W_n}\| < \infty$.
- (iii) $(A_n W_n)^{\mathbf{D}} \rightarrow (A_0 W_0)^{\mathbf{D}}$.
- (iv) $A_n^{\sigma, W_n} \rightarrow A_0^{\sigma, W_0}$.

PROOF. We rely on the continuity results for the g -Drazin inverse obtained in [9].

Condition (i) clearly implies (ii). Suppose that (ii) holds. Since

$$(A_n W_n)^D = ((A_n W_n)^D)^2 (A_n W_n) = A_n^{D, W_n} W_n,$$

we have $\sup_n \|(A_n W_n)^D\| < \infty$. By [9, Theorem 2.4], $(A_n W_n)^D \rightarrow (A_0 W_0)^D$.

If (iii) holds, then

$$A_n^{\sigma, W_n} = (A_n W_n)^D A_n \rightarrow (A_0 W_0)^D A_0 = A_0^{\sigma, W_0}.$$

Let (iv) hold. From the equation

$$(A_n W_n)^\sigma = (A_n W_n)^D A_n W_n = A_n^{\sigma, W_n} W_n$$

we deduce that $(A_n W_n)^\sigma \rightarrow (A_0 W_0)^\sigma$. Using [9, Theorem 2.4] again, we obtain $(A_n W_n)^D \rightarrow (A_0 W_0)^D$. Hence

$$A_n^{D, W_n} = ((A_n W_n)^D)^2 A_n \rightarrow ((A_0 W_0)^D)^2 A_0 = A_0^{D, W_0}$$

and the theorem is proved. \square

From the preceding theorem we recover [13, Theorem 5.1] when we specialize the result to a finite index weighted Drazin inverse.

References

- [1] B. A. Barnes, ‘Common operator properties of the linear operators RS and SR ’, *Proc. Amer. Math. Soc.* **126** (1998), 1055–1061.
- [2] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer, Berlin, 1973.
- [3] J. J. Buoni and J. D. Faires, ‘Ascent, descent, nullity and defect of products of operators’, *Indiana Univ. Math. J.* **25** (1976), 703–707.
- [4] S. L. Campbell and C. D. Meyer, *Generalized Inverses of Linear Transformations*, Pitman, London, 1979.
- [5] N. Castro González, J. J. Koliha and Yimin Wei, ‘On integral representation of the Drazin inverse in Banach algebras’, *Proc. Edinburgh Math. Soc.* **45** (2002), 327–331.
- [6] R. E. Cline and T. N. E. Greville, ‘A Drazin inverse for rectangular matrices’, *Linear Algebra Appl.* **29** (1980), 53–62.
- [7] J. J. Koliha, ‘A generalized Drazin inverse’, *Glasgow Math. J.* **38** (1996), 367–381.
- [8] ———, ‘Isolated spectral points’, *Proc. Amer. Math. Soc.* **124** (1996), 3417–3424.
- [9] J. J. Koliha and V. Rakočević, ‘Continuity of the Drazin inverse II’, *Studia Math.* **131** (1998), 167–177.

- [10] J. J. Koliha and Pak Wai Poon, ‘Spectral sets II’, *Rend. Circ. Mat. Palermo (2)* **47** (1998), 293–310.
- [11] M. Mbekhta, ‘Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux’, *Glasgow Math. J.* **29** (1987), 159–175.
- [12] San Zhen Qiao, ‘The weighted Drazin inverse of a linear operator on a Banach space and its approximation’ (Chinese), *Numer. Math. J. Chinese Univ.* **3** (1981), 296–305.
- [13] V. Rakočević and Y. Wei, ‘A weighted Drazin inverse and applications’, *Linear Algebra Appl.* **350** (2002), 25–39.
- [14] ———, ‘The representation and approximation of the W -weighted Drazin inverse of linear operators in Hilbert spaces’, *Appl. Math. Comp.* **141** (2003), 455–470.
- [15] Ch. Schmoeger, ‘On isolated points of the spectrum of a bounded linear operator’, *Proc. Amer. Math. Soc.* **117** (1993), 715–719.
- [16] Guo Rong Wang, ‘Approximation methods for the W -weighted Drazin inverse of linear operators in Banach spaces’ (Chinese), *Numer. Math. J. Chinese Univ.* **10** (1988), 74–81.
- [17] ———, ‘Iterative methods for computing the Drazin inverse and the W -weighted Drazin inverse of linear operators based on functional interpolation’, *Numer. Math. J. Chinese Univ.* **11** (1989), 269–280.
- [18] Yimin Wei, ‘Integral representation of the W -weighted Drazin inverse’, *Appl. Math. Comp.* **144** (2003), 3–10.
- [19] L. F. Yukhno, ‘An eigenvalue property of the product of two rectangular matrices’, *Comp. Maths Math. Phys.* **36** (1996), 555–557.

Department of Mathematics and Statistics
The University of Melbourne
VIC 3010
Australia
e-mail: a.dajic@pgrad.unimelb.edu.au, j.koliha@ms.unimelb.edu.au