Lattice Paths and the Constant Term

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The Paths

The lattice: $\mathbb{Z}^2$
The paths: $v_0 e_1 v_1 \ldots e_t v_t$ where each

- vertex $v_i = (x_i, y_i) \in \mathbb{Z}^2$, with $0 \le y_i \le L$, (for strip height L) and each

- edge $e_i = v_i - v_{i-1}$ is one of the allowed steps

$$e_i = \begin{cases} (1,1) & \text{an } \textit{up} \text{ step} \\ (1,0) & \text{an } \textit{across} \text{ step} \\ (1,-1) & \text{a } \textit{down} \text{ step.} \end{cases}$$

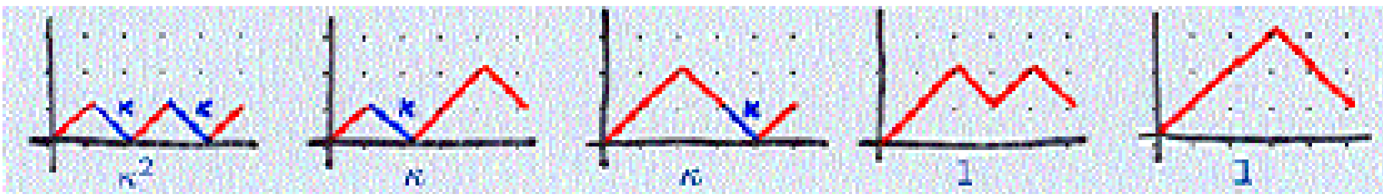
$$w(e_i) = \begin{cases} 1 & e_i \text{ is an } \textit{up} \text{ step} \\ b_{y_i} & e_i \text{ is an } \textit{across} \text{ step} \\ \lambda_{y_i} & e_i \text{ is a } \textit{down} \text{ step,} \end{cases}$$

We want

$$Z_t(y', y; \text{weights}; L) = \text{CT[Laurent Series]}$$

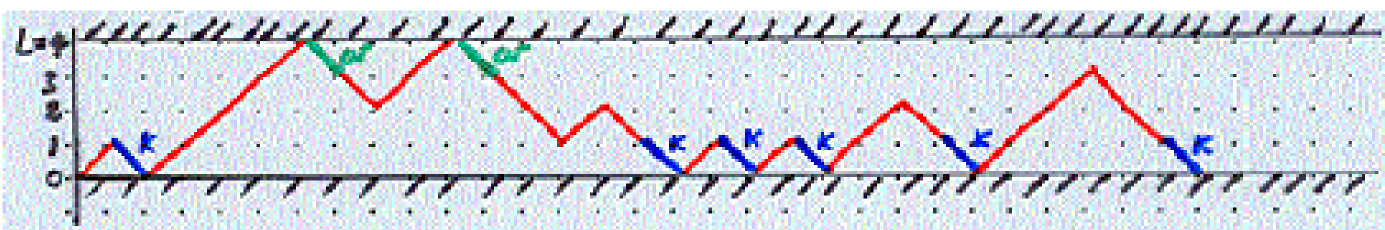
Examples of sets of Paths

Ballot paths of length 5:



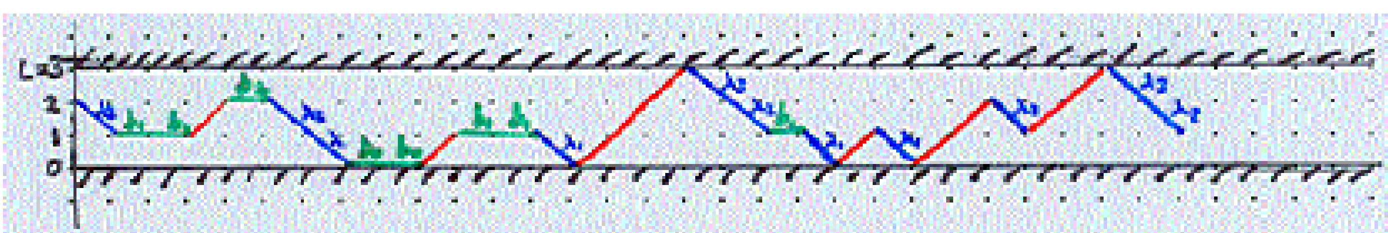
$$Z_5^{Ballot}(0,1;\kappa;\infty) = \kappa^2 + 2\kappa + 2$$

A Dyck path in a strip:



$$Z_{30}^{Dyck}(0,0;\kappa,\omega;4) = \cdots + \kappa^6\omega^2 + \cdots$$

A Motzkin-type path in a strip:



$$Z_{31}^{Motz}(2,1;\lambda\text{'s},\ b\text{'s};3) = \cdots + \lambda_1^4\lambda_2^5\lambda_3^2 b_0^2 b_1^5 b_2 + \cdots$$

A history of the Constant Term

- Constant Term Method (or Ansatz?) solves partial difference equations in the half plane
  Richard Brak, John Essam, Aleks Owczarek (1998)

- CT method equivalent to diagonalizing the Transfer Matrix in the half plane *or* a strip

- Rational Generating Functions, and now a Constant Term Theorem $\cdots$

$$\text{“ } Z_t = \mathrm{CT}\Big[(\rho + \rho^{-1})^t \, R(\rho) \, (1 - \rho^2)\Big] \text{ ”}$$

$\uparrow$

a rational function

Generating Functions ...

Paths in a strip of height L admit a transfer matrix, of order $L+1$, of the form

$$T_L = \begin{pmatrix} b_0 & 1 & 0 & \dots \\ \lambda_1 & b_1 & 1 & \\ 0 & \lambda_2 & b_2 & \\ \vdots & & & \ddots \end{pmatrix}$$

Construct path length generating function:

$$\begin{aligned} G(x)_{y',y} &= \sum_{t \geq 0} x^t (T_L^t)_{y',y} \\ &= (I - xT_L)^{-1}|_{y',y} \\ &= \frac{\mathrm{cof}_{y,y'}(I - xT_L)}{\det(I - xT_L)} \end{aligned}$$

We want the coefficient of x^t, so *divide by* x^{t+1}, and then the partition function is a residue.

Partition function is

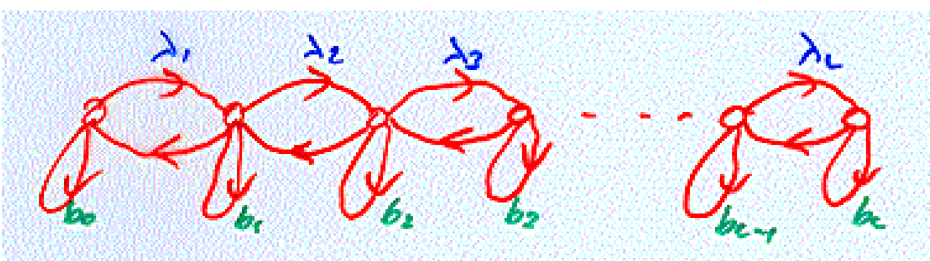
$$Z_t(y', y) = \text{Res}\left[\frac{\text{cof}_{y,y'}(I - xT_L)}{x^{t+1}\det(I - xT_L)}, 0\right]$$

Let the combinatorics do the work...

- $\text{cof}_{y,y'}(I - xT_L) = x^L P_{y'}(1/x) P_{L-y}^{(y+1)}(1/x)$

- $\det(I - xT_L) = x^{L+1} P_{L+1}(1/x)$

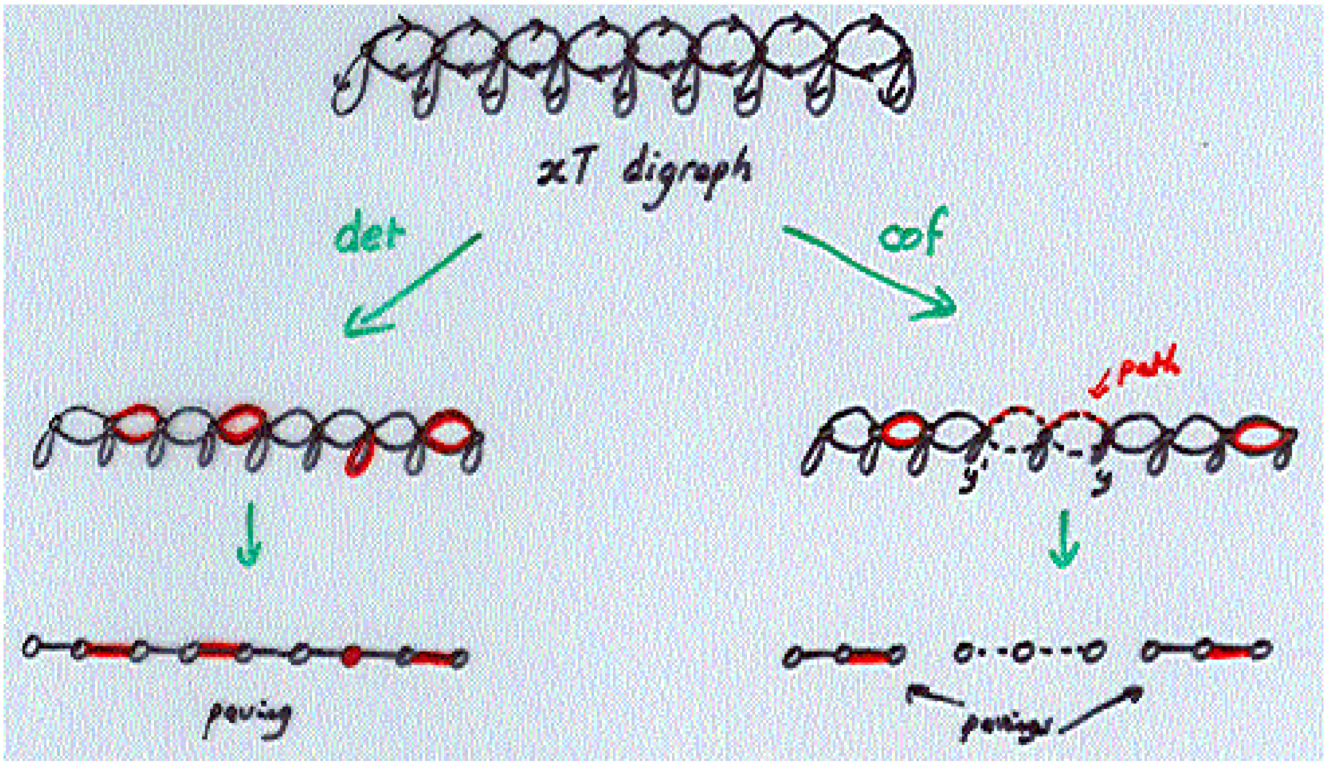
via: xT_L is the adjacency matrix of a digraph.



- The determinant is a weighted sum over non-intersecting cycles.

- For the cofactor, add a path from y' to y.

Cycles $\longleftrightarrow$ Pavings on a line-graph



Weighted pavings are counted by orthogonal polynomials $\{P_k(\mu)\}$ (Viennot, Flajolet 1980) satisfying the recurrence relation

$$P_{k+1}(\mu) = (\mu - b_k)P_k(\mu) - \lambda_k P_{k-1}(\mu),$$

subject to the normalization

$$P_0(\mu) = 1, \qquad P_1(\mu) = \mu - b_0.$$

Note, $P_k^{(j)}(\mu)$'s are the same, except with indexes of b's and λ's increased by j.

The partition function in x is:

$$Z_t(y',y) = \text{Res}\left[\frac{P_{y'}(1/x)P_{L-y}^{(y+1)}(1/x)}{x^{t+2}P_{L+1}(1/x)},0\right]$$

$$= \text{CT}\left[\frac{P_{y'}(1/x)P_{L-y}^{(y+1)}(1/x)}{x^{t+1}P_{L+1}(1/x)},0\right]$$

Can we evaluate this CT, ... even for the simplest polynomials?

$$F_{k+1}(\mu) = \mu F_k(\mu) - F_{k-1}(\mu) \qquad (1)$$

Substituting Ansatz $F_k = \rho^k$ into (1) gives a quadratic in ρ with two solutions:

$$\rho_{\pm} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$

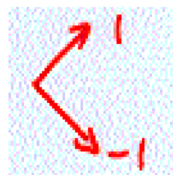
Then $F_k(\mu) = \frac{\rho_+^{k+1} - \rho_-^{k+1}}{\rho_+ - \rho_-}$ implies

$$F_k(\mu) = \frac{(\mu + \sqrt{\mu^2-4})^{k+1} - (\mu - \sqrt{\mu^2-4})^{k+1}}{2^{k+1}\sqrt{\mu^2-4}}$$

A very combinatorial substitution

Context: unweighted ballot paths (no horizontal steps)...

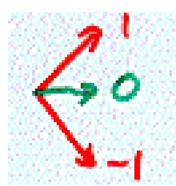
$$1/x = \mu \mapsto \rho + \rho^{-1}$$



$$F_k(\rho + \rho^{-1}) = \frac{\rho^{k+1} - \rho^{-(k+1)}}{\rho - \rho^{-1}}$$

Context: unweighted motzkin-type paths ...

$$1/x = \mu \mapsto \rho + 1 + \rho^{-1}$$



Lemma 1. *Let $f(x)$ be a rational function, and let $x(\rho) := 1/(a\rho + b + c\rho^{-1})$ be a change of variables; $a, c \neq 0$. Then*

$$\mathit{Res}\,[f(x), \{x, 0\}] = \mathit{Res}\left[f(x(\rho))\frac{dx}{d\rho}, \{\rho, 0\}\right].$$

Proof. Several methods. Not trivial. □

gives a very combinatorial answer ...

Theorem 2. *Let $\{P_k(\mu)\}_{k\geq 0}$ be the family of orthogonal polynomials generated by the digraph of a tri-diagonal transfer matrix, T_L, for a lattice path enumeration problem in a strip. Let y', y be the starting and ending heights, respectively, of the paths; and let L be the height of the strip. Then*

$$Z_t(y', y; \textit{weights}; L) =$$

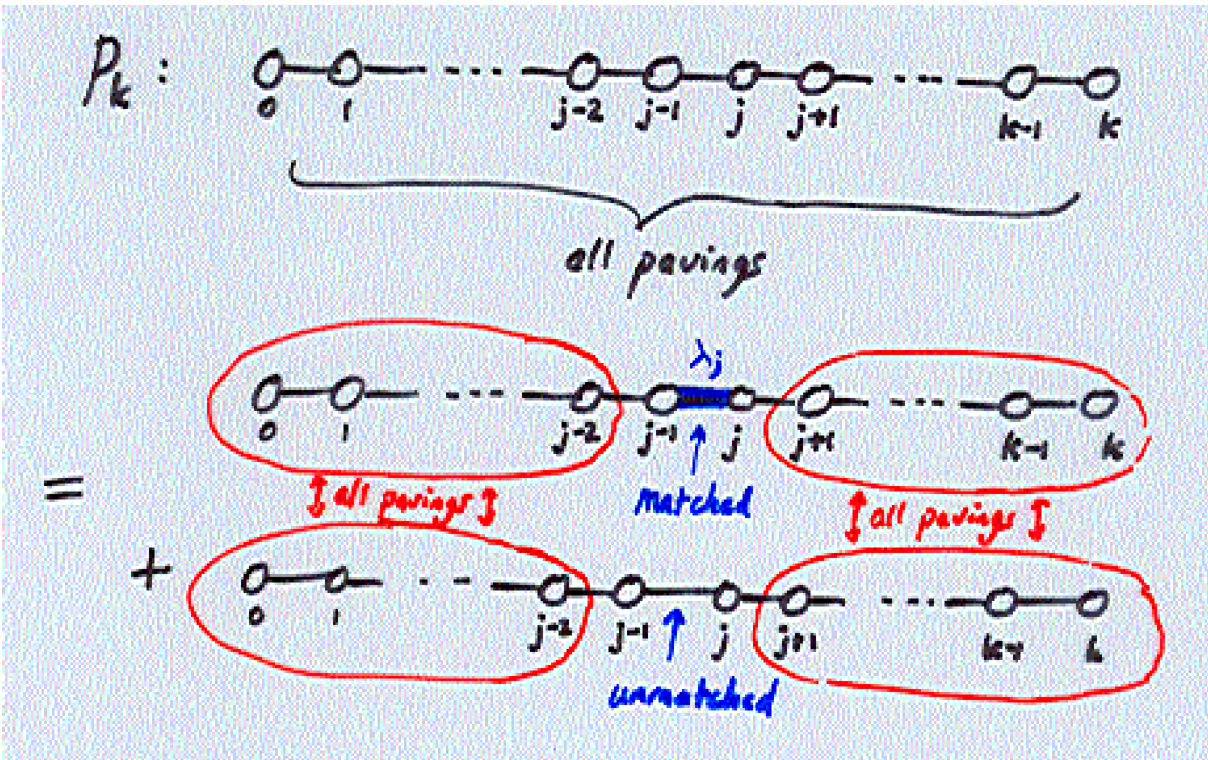
$$CT\left[\frac{(1-\rho^2)(\rho + b + \rho^{-1})^{t-1-(y-y')}\tilde{P}_{y'}(\rho)\tilde{P}^{(y+1)}_{L-y}(\rho)}{-\rho\tilde{P}_{L+1}(\rho)}\right]$$

where $\tilde{P}_k(\rho) := P(\rho + b + \rho^{-1})$, for

$$b = \begin{cases} 0 & \text{ballot paths; i.e. diagonal of } T_L \text{ all 0's,} \\ 1 & \text{motzkin paths.} \end{cases}$$

One last thing: how to get P's with weights

More combinatorics:



In particular, for the $\kappa - \omega$ problem,

$$\begin{aligned} P_k &= F_k \\ &-(\kappa - 1)F_{k-2} \\ &-(\omega - 1)F_{L-1}F_{k-L-1} \\ &+(\kappa - 1)(\omega - 1)F_{L-3}F_{k-L-1}. \end{aligned}$$

Results:

since 1971, DiMarzio & Rubin

Finally.. the open problem, general κ and ω.

Theorem

The weight polynomial, $Z_{2r}(\kappa, \omega; L)$ for Dyck paths of length $2r$, in a strip of width L with vertex weights κ on the line $y = 0$ and ω on the line $y = L$ is given by

$$Z_{2r}(\kappa, \omega; L) = CT\left[\left(\rho + \frac{1}{\rho}\right)^{2r}(1-\rho^2)\frac{A\,\rho^L - B\,\rho^{-L}}{AC\,\rho^L - BD\,\rho^{-L}}\right]$$

where

$$A = \rho^2 - \hat{\omega}$$
$$B = 1 - \hat{\omega}\rho^2$$
$$C = \rho^2 - \hat{\kappa}$$
$$D = 1 - \hat{\kappa}\rho^2$$

$\hat{\kappa} = \kappa - 1$, $\hat{\omega} = \omega - 1$.

Corollary

$$Z_{2r}(\kappa, \omega; L) = \sum_{m \geq 1} \sum_{p,q \geq 0} \sum_{s_1,s_2=0}^{m} (-1)^{s_1+s_2} \hat{\kappa}^{s_2+q} \hat{\omega}^{s_1+p}$$

$$\times \binom{m}{s_1} \binom{m+p-1}{p} \left[C_{r;k+1} \binom{m-1}{s_2} \binom{m+q-1}{q} \right.$$

$$\left. - C_{r;k} \binom{m}{s_2} \binom{m+q}{q} \right]$$

where $k = r + s_1 + s_2 - p - q - (L+2)m$, $C_{n;x} = \binom{2n}{n-x} - \binom{2n}{n-x-1}$ *and* $\hat{\kappa} = \kappa - 1$, $\hat{\omega} = \omega - 1$.