Department of Mathematics and Statistics
University of Melbourne
620-221: Real and Complex Analysis
Mid-Semester Test, 23 April 2008.
Solutions.

(1) Find the modulus and argument of

$$\frac{(2i)^3}{(1+i)^5}.$$

Solution: The modulus is $|2i|^3/|1+i|^5 = 2^3/\sqrt{2}^5 = \sqrt{2}$. The argument is $3\arg(2i) - 5\arg(1+i) = 3\pi/2 - 5\pi/4 = \pi/4$.

(2) Sketch, on the complex plane, the following set of complex numbers:

$$\{z \in \mathbb{C} : |z-i| < 1, |z-1| < 1\}.$$

Write down a complex number which is in the set.
Solution: This is the intersection of two circles. A complex number in the set is $(1+i)/2$.

(3) Is the set $\{z : |z| > 1, \mathrm{Re}(z) \leq 0\}$ an open subset of the complex plane? Give careful reasons for your answer.
Solution: No. Consider the number $2i$ (for example). It is in the set as $|2i| > 1$ and $\mathrm{Re}(2i) = 0$. But any disc centered at $2i$ must contain a point strictly in the right half-plane. Thus no disc centered at $2i$ can lie entirely in the set.

(4) Show, *without using the Heine-Borel theorem* (that is, directly from the definition of compact), that the real line is not a compact set.
Solution: $\mathbb{R} = \cup_n(-n, n)$ which shows that the real line is the union of an infinite collection of open sets without being the union of any finite number of them.

(5) For what real values of a and b is the function f given by

$$f(x+iy) = (ax^2 + y^2) + i(bxy)$$

an entire function of $z = x + iy$?
Solution: The Cauchy-Riemann equations must be satisfied for all x, y. That is $2ax = bx$ and $2y = -by$ for all x and y. Thus $b = -2$ and $a = -1$.

(6) If the real part of an entire function of $z = x + iy$ is given by $x^2 - y^2 + 2y$, what can the imaginary part be?
Solution: Suppose v is the imaginary part. We must have $v_y = 2x$ and so $v = 2xy + g(x)$ for some function g of x alone. But also $v_x = -(-2y+2)$ and so $g(x) = -2x + c$ for c a constant. Thus the imaginary part is $2xy - 2x$.

(7) For what z does the power series

$$\sum_{n=0}^{\infty} \frac{z^n}{2^{n^2}} = 1 + \frac{z}{2} + \frac{z^2}{16} + \frac{z^3}{2^9} + \dots$$

converge?
Solution: We use the ratio test. We have

$$a_n/a_{n+1} = 2^{-n^2}/2^{-(n+1)^2} = 2^{(n+1)^2-n^2} = 2^{2n+1}.$$

Thus the sequence a_n/a_{n+1} is unbounded and so the power series converges for all z

(8) Expand the function $f(z) = \frac{z}{1+3z}$ as a power series centered at $z = 0$. What is the radius of convergence of this power series?
Solution: We write

$$\frac{z}{1+3z} = z \times \frac{1}{1-(-3z)} = z\sum_{n=0}^{\infty}(-3z)^n = z\sum_{n=0}^{\infty}(-3)^n z^n = \sum_{n=1}^{\infty}(-3)^{n-1}z^n.$$

The expansion of the geometric series is valid for $|3z| < 1$ and so the radius of convergence is $1/3$.

(9) Suppose that f is represented by a power series, centered at 0 and with radius of convergence 2. Suppose also that $f(1/n) = 1$ for $n = 1, 2, 3, 4 \dots$. Write down $f(i)$ and give a brief explanation for your answer.
Solution: The series $\{1/n\}$ has a limit at 0 and so, as f agrees with the constant function 1 at all points of the sequence then it agrees with 1 at all points of the disc of convergence. So $f(i) = 1$.

(10) Find the real and imaginary parts of $\cos(i)$.
Solution: We have $\cos(i) = (\exp(i^2) + \exp(-i^2))/2)$ and so the value is $1/2(e + e^{-1}) = \cosh(1)$.