Department of Mathematics and Statistics
620-221: Real and Complex Analysis 2008
Assignment 1

Solutions to this assignment should be returned by 9.00am on Wednesday 2 April.

(1) Solve $z^2 - (6+5i)z + (2+16i) = 0$.
Solution: We use the standard formula for quadratics. This gives

$$\begin{aligned} z &= \frac{1}{2}\left((6+5i) \pm \sqrt{(6+5i)^2 - 4(2+16i)}\right) \\ &= \frac{1}{2}\left((6+5i) \pm \sqrt{36+60i-25-8-64i}\right) = \frac{1}{2}\left((6+2i) \pm \sqrt{3-4i}\right) \end{aligned}$$

We need to calculate a square root for $3-4i$. This is best done by writing $3-4i = (a+bi)^2$ to obtain $3 = a^2 - b^2$, $-4 = 2ab$. We can solve this either by inspection or by noting that $3 = a^2 - 4/a^2$. This leads to a quadratic in a^2 which has roots $-1, 4$. Since a^2 is positive, we have $a^2 = 4$ and so $a = \pm 2$. Thus, $b = -1$ or $b = +1$ and so $3 - 4i = (\pm(2-i))^2$. Thus the two roots are

$$\frac{1}{2}\left((6+5i) \pm (2-i)\right) = 4+2i \quad \text{and} \quad 2+3i.$$

(2) Describe the projections onto the Riemann sphere of the following subsets of the complex plane. (Geographical descriptions, such as 'North Pole', 'line of latitude' etc, are acceptable as descriptions.)

(a) the real and imaginary axes;
(b) the unit disk—$\{z : |z| \le 1\}$;
(c) the right half plane—$\{z : \mathrm{Re}(z) > 0\}$;
(d) the annulus—$\{z : 1 < |z| < 2\}$;
(e) the exterior of the unit disc—$\{z : |z| > 1\}$.

Solution: Note that, in some cases, the projection described below contains the 'North Pole'. This should be omitted for an accurate description.

(a) Both these projections are great circles; let us say the circle of longitude 0 and 180 for the real axis and the circle of longitude 90 and 270 for the imaginary axis. let us assume that longitude 0 corresponds to the positive real axis and longitude 90 to the positive imaginary axis.

(b) The formula yields that if $|z| = 1$ then the image of z has height $1/2$ and so lie on the equator. Also an interior point of the disc lies above the equator. Thus the image of the disc and its interior is the 'southern hemisphere'.

(c) according to our identification in the first part, this will be the hemisphere between longitude -90 and longitude +90 which includes longitude 0.

(d) We have identified the image of the unit circle. For the circle of radius 2, the formula gives that this is the set of points with z-value $4/5$ and this becomes the circle of latitude with angle $\theta = \arcsin(3/5)$. Thus the image is that region north of the equator with latitude less than $\arcsin(3/5)$.

(e) This must be the complement of the interior of the unit disc and hence is the 'northern hemisphere'.

(3) Show that the function defined on the complex numbers by

$$f(z) = \frac{z}{1 + |z|}$$

is continuous. (You should prove from first principles that the absolute value $z \mapsto |z|$ is a continuous function. You should then quote standard results on continuous functions rather than continue to use first principles.)

What is the image, under f, of the closed right half plane—that is, $\{z : \operatorname{Re}(z) \geq 0\}$. Explain why this shows that the image of a closed set under a continuous mapping need not be closed.

Solution: Firstly we prove the continuity of the absolute value function. Suppose $\epsilon > 0$. If $|z - z_0| < \epsilon$ then $||z| - |z_0|| \leq |z - z_0| < \epsilon$. That is, (taking $\delta = \epsilon$) we have that $|z - z_0| < \delta$ implies that $||z| - |z_0|| < \epsilon$. Thus $\lim_{z \to z_0} |z| = |z_0|$ and so the absolute value function is continuous at z_0.

Let P denote the closed left half-plane. For any point z we have $|f(z)| = |z|/(1 + |z|) < 1$ and, if $z \in P$ then $\operatorname{Re}(f(z)) \geq 0$ and so $f(z) \in P$. We claim that the image of P is the intersection of the open unit circle and P; it is clear from what we have said that the image of P is inside this intersection.

Let w be any point in this intersection; that is, $|w| < 1$ and $\text{Re}(w) \geq 0$. Set $z = w/(1 - |w|)$. Then $|z| = |w|/(1 - |w|)$ and $1 + |z| = 1/(1 - |w|)$ and so

$$f(z) = \frac{z}{1 + |z|} = \frac{\frac{w}{1-|w|}}{\frac{1}{1-|w|}} = w.$$

Thus the image of P is exactly this intersection. Note that P is a closed set but the intersection is not closed. For example, the point $w = 1$ is not in the intersection but it is clearly a limit point of the intersection.