Department of Mathematics and Statistics
620-221: Real and Complex Analysis 2008
Assignment 2

Answers to this assignment should be returned by 9.00am on Wednesday 21 May.

1. Find the first four terms (to the coefficient of z^3) of the Taylor series for

$$\frac{1}{1+\operatorname{Log}(\cos(z))}$$

about $z = 0$. Show, using Taylor's Theorem, that the radius of convergence of this series is $\arccos(1/e)$.

2. Suppose that a function f is analytic on a set that contains the closure of a domain D. Suppose also that f is non-constant on D but has constant modulus on the boundary of D (that is, those points that lie in the closure of D but not in D itself). Show that there must be a point z in D so that $f(z) = 0$.

3. The aim of this question is to show that the definition

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

yields a function which is defined and analytic when the real part of z is greater than 1. You may assume that this series converges when z is real and greater than 1.

   (a) Show that, if n is a positive natural number then n^z (with its usual definition) defines an entire function which is never zero. Deduce that $1/n^z$ is also an entire function.

   (b) Deduce that the series for $\zeta(z)$ converges absolutely when $\operatorname{Re}(z) > 1$.

   (c) Suppose that $\delta > 0$ and let H_δ be the open right half-plane given by $\{z : \operatorname{Re}(z) > 1 + \delta\}$. By comparing the series for $\zeta(z)$ with the series for $\zeta(1+\delta)$, show that the series for $\zeta(z)$ converges uniformly.

   (d) Explain why, if $\zeta_k(z) = \sum_{n=1}^{k} \frac{1}{n^z}$, then $\int_\Gamma \zeta_k(z)\,dz = 0$ for any closed contour γ in H_δ.

   (e) Deduce that $\int_\Gamma \zeta(z)\,dz = 0$ for any closed contour γ in H_δ.

   (f) Deduce that ζ is an analytic function on H_δ.

   (g) Deduce that ζ is analytic on the half-plane $\{z : \operatorname{Re}(z) > 1\}$.