Department of Mathematics and Statistics
620-221: Real and Complex Analysis 2008
Assignment 2

Answers to this assignment should be returned by 9.00am on Wednesday 21 May.

1. Find the first four terms (to the coefficient of z^3) of the Taylor series for

$$\frac{1}{1+\operatorname{Log}(\cos(z))}$$

about $z=0$. Show, using Taylor's Theorem, that the radius of convergence of this series is $\arccos(1/e)$.

Solution: We shall use the formula $a_n = (1/n!)f^n(z_0)$ where a_n is the coefficient of $(z-z_0)^n$ in the power series for f centered at z_0.

In this case,

$$\begin{aligned}
f(z) &= \frac{1}{1+\operatorname{Log}(\cos(z))} \\
f'(z) &= \frac{\tan z}{(1+\operatorname{Log}(\cos(z)))^2} \\
f''(z) &= \frac{2\tan^2 z}{(1+\operatorname{Log}(\cos(z)))^3} + \frac{\sec^2 z}{(1+\operatorname{Log}(\cos(z)))^2} \\
f'''(z) &= \frac{2\sec^2 z\tan z}{(1+\operatorname{Log}(\cos(z)))^2} + \frac{6\sec^2 z\tan z}{(1+\operatorname{Log}(\cos(z)))^3} + \frac{6\tan^3 z}{(1+\operatorname{Log}(\cos(z)))^4}
\end{aligned}$$

Thus $a_0 = f(0) = 1; a_1 = f'(0) = 0; a_2 = (1/2)f''(0) = 1/2; a_3 = f'''(0) = 0$. Hence the beginning of the series is

$$\frac{1}{1+\operatorname{Log}(\cos(z))} = 1 + \frac{1}{2}z^2 + \ldots.$$

Taylor's Theorem tells us that the radius of the power series is the distance to the nearest singularity. The singularities occur when $\operatorname{Log}(\cos(z)) = -1$ or when $\cos z$ is real and non-positive. The former occurs when $\cos z = \exp(-1)$ and the latter when z is outside the range $[-\pi/2, \pi/2]$. Thus the closest singularity must be $\arccos(1/e)$.

2. Suppose that a function f is analytic on a set that contains the closure of a domain D. Suppose also that f is non-constant on D but has constant modulus on the boundary of D (that is, those points that lie in the closure of D but not in D itself). Show that there must be a point z in D so that $f(z) = 0$.

   Solution: Let $\overline{D}$ denote the closure of D and $\delta(D)$ the boundary. The maximum modulus principle implies that the maximum modulus of f for all points in $\overline{D}$ must be taken in $\delta(D)$. If f had no zero within D then we can apply the same argument to $1/f$ to show that the minimum modulus of f must be taken in $\delta(D)$. (Note that f must also be non-zero on $\delta(D)$ otherwise its maximum modulus is zero and so it is constant on $\overline{D}$.) As $|f|$ is constant on $\delta(D)$, this implies that the maximum and minimum moduli are equal. That is, f has constant modulus on D (even on $\overline{D}$). But then, by (a minor adaptation of) Exercise 10b) of the notes, f is constant on D. Thus, as f is non-constant, it must have a zero within D.

3. The aim of this question is to show that the definition

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

   yields a function which is defined and analytic when the real part of z is greater than 1. You may assume that this series converges when z is real and greater than 1.

   (a) Show that, if n is a positive natural number then n^z (with its usual definition) defines an entire function which is never zero. Deduce that $1/n^z$ is also an entire function.

   Solution: The usual definition of n^z is $n^z = \exp(z \operatorname{Log}(n))$. Since $c = \operatorname{Log} n$ is a clearly defined constant, this is an entire function of z.

   (b) Deduce that the series for $\zeta(z)$ converges absolutely when $\operatorname{Re}(z) > 1$.

   Solution: From

$$\left|\frac{1}{n^z}\right| = \frac{1}{|n^z|} = \frac{1}{|\exp(z \operatorname{Log} n)|} = \frac{1}{\exp(\operatorname{Re}(z) \operatorname{Log} n)} = \frac{1}{n^{\operatorname{Re}(z)}},$$

   we see that the series of absolute values of the terms in $\zeta(z)$ is the series for $\zeta(\operatorname{Re}(z))$...which we are told converges when $\operatorname{Re}(z) > 1$. Thus the series for $\zeta(z)$ converges for $\operatorname{Re}(z) > 1$.

   (c) Suppose that $\delta > 0$ and let H_δ be the open right half-plane given by $\{z : \operatorname{Re}(z) > 1 + \delta\}$. By comparing the series for $\zeta(z)$ with the series for $\zeta(1 + \delta)$, show that the series for $\zeta(z)$ converges uniformly.

Solution: We use the Weierstrass M-test. We have that, whenever $z \in H_\delta$, $\mathrm{Re}(z) > 1 + \delta$ and so

$$\left|\frac{1}{n^z}\right| = \frac{1}{n^{\mathrm{Re}(z)}} < \frac{1}{n^{1+\delta}}.$$

Thus, taking the M_n of the Weierstrass M-test to be $\frac{1}{n^{1+\delta}}$ and noting that the series for $\zeta(1+\delta)$ converges, we have the required result.

(d) Explain why, if $\zeta_k(z) = \sum_{n=1}^{k} \frac{1}{n^z}$, then $\int_\Gamma \zeta_k(z)\,dz = 0$ for any closed contour Γ in H_δ.

Solution: $\zeta_k(z)$ is a finite sum of functions which are analytic in H_δ and so is itself analytic in H_δ. Thus, by Cauchy's Theorem, the integrals are zero.

(e) Deduce that $\int_\Gamma \zeta(z)\,dz = 0$ for any closed contour Γ in H_δ.

Solution: By Proposition 6.7,

$$\int_\Gamma \zeta(z)\,dz = \int_\Gamma \lim_{k\to\infty} \zeta_k(z)\,dz = \lim_{k\to\infty} \int_\Gamma \zeta_k(z)\,dz = \lim_{k\to\infty} 0 = 0$$

where we have used the uniform convergence to apply Proposition 6.7.

(f) Deduce that ζ is an analytic function on H_δ.

Solution: From Theorem 6.5, ζ is the derivative of an analytic function F and so, by Corollary 7.4, is itself analytic.

(g) Deduce that ζ is analytic on the half-plane $\{z : \mathrm{Re}(z) > 1\}$.

Solution: Any point in this half plane is in some H_δ and so ζ is analytic at that point. So ζ is analytic on the whole half-plane.