Department of Mathematics and Statistics
620-221: Real and Complex Analysis, 2007
Exercises 1: Complex Numbers

(1) Write the numbers in the form $x + iy$.

(a) $\dfrac{1-i}{1+i}$ (c) $\dfrac{(2+i)^2}{(6i-(1-2i))^2}$

(b) $\dfrac{(8+2i)-(1-i)}{(2+i)^2}$ (d) $(2+i)(-1-i)(3-2i)$

(2) Write each of the following in polar form:

(a) $(1-i)(-\sqrt{3}+i)$ (b) $\dfrac{-1+i\sqrt{3}}{2+2i}$

(3) Identify and sketch the set of points satisfying:

(a) $|z-(1+i)| = 1$ (f) $0 < \operatorname{Im} z < \pi$

(b) $1 < |2z-6| < 2$ (g) $-\pi < \operatorname{Re} z < \pi$

(c) $|z-1|^2 + |z+1|^2 < 8$ (h) $|\operatorname{Re} z| < |z|$

(d) $|z-1| + |z+1| \leq 2$ (i) $\operatorname{Re}(iz+2) > 0$

(e) $|z-1| < |z|$ (j) $|z-i|^2 + |z+i|^2 < 2$

(4) Fix $\rho > 0$, $\rho \neq 1$, and fix $z_0, z_1 \in \mathbb{C}$. Show that the set of z satisfying $|z-z_0| = \rho|z-z_1|$ is a circle. Sketch it for $\rho = 1/2$ and $\rho = 2$, with $z_0 = 0$ and $z_1 = 1$. What happens when $\rho = 1$?

(5) Sketch the following sets:

(a) $|\operatorname{Arg} z| < \pi/4$ (c) $|z| = \operatorname{Arg} z$

(b) $0 < \operatorname{Arg}(z-1-i) < \pi/3$ (d) $\log|z| = -2\operatorname{Arg} z$

(6) Find all values of the following:

(a) $(1+i)^8$ (c) $8^{1/3}$

(b) $\sqrt{i}$ (d) $\sqrt{1-i}$

(7) Suppose P is a polynomial with real coefficients. If $P(z) = 0$, show that $P(\overline{z}) = 0$. Deduce that, if P has odd degree, then P has a real root.

(8) For $n \geq 1$, show that

$$1 + z + \cdots + z^n = (1 - z^{n+1})/(1-z) \text{ if } z \neq 1.$$

(9) Fix $n \geq 1$. If the nth roots of 1 are $\omega_0, \dots, \omega_{n-1}$, show that they satisfy:

(a) $(z-\omega_0)(z-\omega_1)\cdots(z-\omega_{n-1}) = z^n - 1$

(b) $\omega_0 + \omega_1 + \cdots + \omega_{n-1} = 0$

(c) $\omega_0\omega_1\cdots\omega_{n-1} = (-1)^{n-1}$

(d) (Harder) $\sum_{j=0}^{n-1} \omega_j^k = \begin{cases} 0, & 1 \leq k \leq n-1, \\ n, & k = n. \end{cases}$

(10) Fix $R > 1$ and $n \geq 1$, $m \geq 0$. Show that

$$\left| \frac{z^m}{z^n + 1} \right| \leq \frac{R^m}{R^n - 1}, \quad |z| = R.$$

When does equality hold?

(11) Prove Lemma **??**.

(12) Show that a disc on the sphere bounded by a circle centered at the North Pole maps, under stereographic projection, to the *exterio*r of a circle on the plane.

(13) (Harder) If z is a point on the unit circle (that is, $|z| = 1$), show that

$$\operatorname{Arg}\left(\frac{z-1}{z+1}\right) = \begin{cases} \pi/2 & \text{if } \operatorname{Im} z > 0 \\ -\pi/2 & \text{if } \operatorname{Im} z < 0. \end{cases}$$

(14) (Harder) Let $a, b, c, d \in \mathbb{C}$ with $ad \neq bc$. Define a function $\phi : \mathbb{C} \to \mathbb{C}$ by

$$\phi(z) = \frac{az + b}{cz + d}.$$

What is the image of a line or a circle under ϕ ? (Hint: it may be useful to use the result of Question 4 here).