

Department of Mathematics and Statistics
620-221: Real and Complex Analysis, 2007

Exercises 2: Convergence and related properties

- (1) Evaluate the limits of the following sequences $\{z_n\}$:
- (a) $z_n = \frac{n}{n+1}$ (c) $z_n = \frac{2n^p + 5n + 1}{n^p + 3n + 1}$, $p > 1$
(b) $z_n = \frac{n}{n^2 + 1}$ (d) $z_n = \frac{in}{n^2 + i}$
- (2) For which values of $z \in \mathbb{C}$ do the following sequences $\{z_n\}$ converge?
- (a) $z_n = n^n z^n$ (c) $z_n = \frac{z^n}{n^n}$
(b) $z_n = \frac{z^n}{n!}$ (d) $z_n = (z + \bar{z})^n$, $z \neq 0$.
- (3) Determine whether the following sequences converge. If they converge, find the limit.
- (a) $\frac{1}{(1+i)^n}$ (b) $\frac{n}{(1+i)^n}$ (c) $\frac{n!}{(1+i)^n}$
- (4) Classify each of the following sets in \mathbb{C} as open, closed, or neither open nor closed.
- (a) $A = \{z \mid |z| \geq 1\}$ (d) $D = \{z \mid -\pi < \text{Im } z \leq \pi\}$
(b) $B = \{z \mid |z| > 1\}$ (e) $E = \mathbb{C} \setminus (0, 1)$
(c) $C = \mathbb{C} \setminus [0, 1]$ (f) $F = \{z \mid |\text{Re } z| + |\text{Im } z| \leq 1\}$
- (5) Describe geometrically each of the following sets. Which are open and which are closed? Find the closure of each set.
- (a) $\{z : |z - 1 + i| \geq |z - 1 - i|\}$ (d) $\{z : |z|^2 > z + \bar{z}\}$
(b) $\{z : |zi| \neq |z - i|\}$ (e) $\{z : \text{Im}((z + i)/2i) \geq 0\}$
(c) $\{z : |z - 1| < 1, |z| = |z - 2|\}$
- (6) Define $x_1 = 0$, and define by induction $x_{n+1} = x_n^2 + \frac{1}{4}$ for $n \geq 1$. Show that the sequence converges and that the limit is $1/2$.
- (7) Define the sequence $\{s_n\}$ by setting

$$s_n = \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} = \sum_{k=1}^n \frac{1}{k!}.$$

Use the Cauchy criterion to show that $\{s_n\}$ converges.

(8) Prove

- (a) If $z_n \rightarrow z$ and $u_n \rightarrow u$, then $z_n + u_n \rightarrow z + u$ and $z_n u_n \rightarrow zu$.
- (b) If $\{z_n\}$ and $\{u_n\}$ are Cauchy sequences of complex numbers, then $\{z_n + u_n\}$ and $\{z_n u_n\}$ are also Cauchy.

(9) Set $z = x + iy$ and let $f(z) = \frac{x^2}{x^2 + y^2} + 2i$. Does $f(z)$ have a limit as $z \rightarrow 0$?

(10) At what points of \mathbb{C} is the function $\text{Arg}(z)$ continuous?

(11) Prove that the union of arbitrarily many open sets is open. What about the union of arbitrarily many closed sets?

(12) Let S be a (path-)connected set of complex numbers and let f be a continuous, complex-valued, function defined on S . Prove that $f(S)$ is (path-)connected.

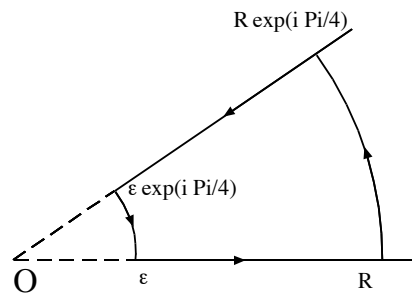
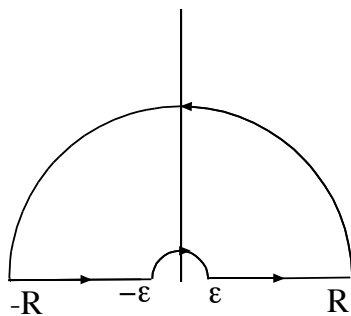
(13) Let S be a compact set of complex numbers and let f be a continuous, complex-valued, function defined on S . Prove that $f(S)$ is compact.

(14) Show that the least upper bound of a subset S of \mathbb{R} is either in S or is a limit point of S .

(15) For each of the following sets and pairs of points, define, if possible, a path in the set joining the two points and a step path in the set joining the two points.

- (a) $|z| < 2$; $\{1 + i, 1 - i\}$
- (b) $1 \leq |z| < 2$; $\{\sqrt{2}, -\sqrt{2}\}$
- (c) $|z| = 1$; $\{-i, i\}$
- (d) $|1 - |z|| > 1/2$; $\{i/3, 49(1 + i)\}$

(16) Give explicit functions $[0, 1] \rightarrow \mathbb{C}$ which give the following closed paths.



(17) Show

- (a) The slit plane $\mathbb{C} \setminus (-\infty, 0]$ is star-shaped.
- (b) The slit plane $\mathbb{C} \setminus [-1, 1]$ is not star-shaped.

(18) Find the boundaries of the following sets:

- (a) $\{z \mid 1 < |z| < 2\}$;
- (b) $\{z \mid 1 \leq |z| \leq 2\}$;
- (c) $\{z \mid z = x + iy \text{ with } x \text{ and } y \text{ rational}\}$;