Department of Mathematics and Statistics
620-221: Real and Complex Analysis, 2007
Exercises 3: Differentiable functions

(1) Find the derivatives of the following functions:

(a) $z^2 - 1$ (d) $\dfrac{1}{1-z}$ (g) $\dfrac{az+b}{cz+d}$

(b) $z^n - 1$ (e) $\dfrac{1}{(z^2+3)}$ (f) $\dfrac{z^2+3}{(4z^3+5)^2}$

(c) $(z^2-1)^n$ (h) $(z^2+5)^5(z^4+26z)^{12}$

(2) Show that

$$1 + 2z + 3z^2 + \cdots + nz^{n-1} = \frac{1-z^n}{(1-z)^2} - \frac{nz^n}{1-z}.$$

(3) Use the definition of the complex derivative to find the derivative of:

(a) $z^3 + 2z$ (b) $\dfrac{1}{z}$, $z \neq 0$ (c) $\dfrac{1}{z^2}$, $z \neq 0$

(4) Show from the definition that the functions $f(z) = \operatorname{Re} z$ and $g(z) = \operatorname{Im} z$ are not complex differentiable at any point.

(5) Show that if f is analytic on D, then $g(z)$ defined by $g(z) = \overline{f(\overline{z})}$ is analytic on the reflected domain $D^* = \{\overline{z} \mid z \in D\}$, and that $g'(z) = \overline{f'(\overline{z})}$.

(6) Let f and g be complex differentiable at z_0. Prove:

(a) (sum rule) $(f+g)'(z_0) = f'(z_0) + g'(z_0)$

(b) (product rule) $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$

(7) Verify the Cauchy-Riemann equations for the functions $u(x,y), v(x,y)$ given by

(a) $u(x,y) = x^3 - 3xy^2, v(x,y) = 3x^2y - y^3$,

(b) $u(x,y) = \sin x \cosh y, v(x,y) = \cos x \sinh y$,

(c) $u(x,y) = \dfrac{x}{x^2+y^2}$, $v(x,y) = \dfrac{-y}{x^2+y^2}$ for $x^2 + y^2 \neq 0$.

In ecah case, identify u and v as the real and imaginary parts of a familiar complex function.

(8) Where are the following functions complex differentiable? Where are these functions analytic?

(a) $f(z) = x^4y^5 + ixy^3$

(b) $f(z) = \frac{3}{2}x^2 - xy + ixy^2$

(c) $f(z) = y^2 \sin x + iy$

(d) $f(z) = 1 - y^2 + i(2xy - y^2)$

(9) Let $f(z) = (x^3 + axy^2) + i(bx^2y + cy^3 + 1)$, $a, b, c \in \mathbb{R}$. Find all values of a, b and c so that f is analytic in $\mathbb{C}$.

(10) Let $f = u + iv : D \subset \mathbb{C} \to \mathbb{C}$ be analytic on a disc D. Prove:

(a) If $\overline{f}$ is analytic, then f is constant.

(b) If $|f|$ is constant, then f is constant.

(11) Show that, if $z = x + iy$, then $f(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$ is differentiable on the coordinate axes but is nowhere analytic.

(12) Show that the following functions are harmonic, and find harmonic conjugates:

(a) $u(z) = xy + 3x^2y - y^3$

(b) $u(z) = \sinh x \sin y$

(c) $u(z) = e^{x^2-y^2} \cos(2xy)$

(d) $u(z) = \frac{x}{x^2 + y^2}$