Department of Mathematics and Statistics
620-221: Real and Complex Analysis, 2007
Exercises 4: Power Series

(1) Determine the radius of convergence and the convergence behaviour on the circle of convergence for the series:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{z^k}{k^2}$ (b) $\displaystyle\sum_{k=1}^{\infty} k! z^k$ (c) $\displaystyle\sum_{k=1}^{\infty} k z^k$

(2) Show that $\displaystyle\sum_{k=1}^{\infty} z^{k!}$ converges for all $|z| < 1$ but diverges for all $|z| = 1$.

(3) Find the radius of convergence of the following power series:

(a) $\displaystyle\sum_{k=0}^{\infty} 2^k z^k$ (e) $\displaystyle\sum_{k=1}^{\infty} 2^k z^{k!}$ (i) $\displaystyle\sum_{k=0}^{\infty} \frac{3^k z^k}{4^k + 5^k}$

(b) $\displaystyle\sum_{k=0}^{\infty} \frac{k}{6} z^k$ (f) $\displaystyle\sum_{k=1}^{\infty} (2 + (-1)^n)^n z^n$ (j) $\displaystyle\sum_{k=3}^{\infty} (\log k)^{k/2} z^k$

(c) $\displaystyle\sum_{k=0}^{\infty} (kz)^k$ (g) $\displaystyle\sum_{k=0}^{\infty} (1+i)^k z^k$ (k) $\displaystyle\sum_{k=0}^{\infty} \frac{k^k}{1 + 2^k k^k} z^k$

(d) $\displaystyle\sum_{k=1}^{\infty} k z^{2k}$ (h) $\displaystyle\sum_{k=0}^{\infty} \frac{z^{3k}}{2^k}$ (l) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k z^{2k}}{k^2 + k}$

(4) Determine for which z the following series converge:

(a) $\displaystyle\sum_{k=1}^{\infty} (z-1)^k$ (c) $\displaystyle\sum_{k=0}^{\infty} 2^k (z-2)^{3k}$ (e) $\displaystyle\sum_{k=1}^{\infty} k^k (z-3)^k$

(b) $\displaystyle\sum_{k=10}^{\infty} \frac{(z-i)^k}{k!}$ (d) $\displaystyle\sum_{k=1}^{\infty} \frac{(z+i)^k}{k^2}$ (f) $\displaystyle\sum_{k=3}^{\infty} \frac{2^k}{k^2} (z-2-i)^{2k}$

(5) Find the power series expansion of the following functions about the given point: expanded about the indicated point.

(a) $\displaystyle\frac{1}{1-2z}, \quad z = 0$ (c) $\displaystyle\frac{z^2}{4-z}, \quad z = i$

(b) $\displaystyle\frac{1}{z}, \quad z = 2 + i$ (d) $\displaystyle\frac{z - (1/2)}{1 - (z/2)}, \quad z = 0$

(6) Let C be the circle of convergence for the series $\sum_{k=1}^{\infty} a_n z^n$. The following statements may or may not be true. Decide whether thay are true and either prove them or give a counterexample. (The series $\sum_k z^k/k$ may prove useful.)

(a) If $\sum_{k=1}^{\infty} a_n z^n$ converges at some point z on C then it converges everywhere on C.

(b) If $\sum_{k=1}^{\infty} a_n z^n$ converges absolutely at some point z on C then it converges absolutely everywhere on C.

(c) (Harder) If $\sum_{k=1}^{\infty} a_n z^n$ converges at all points z on C with the possible exception of one point, then it converges everywhere on C. (Hint: try showing that

$$\sum_{k=1}^{n} \frac{z^k}{k} = \sum_{k=1}^{n-1} A_k \left(\frac{1}{k} - \frac{1}{k+1} \right) + \frac{1}{n} A_n$$

where $A_n = \sum_{k=1}^{n} z^k$.)

(7) Show that the least upper bound of the set of limit points of a sequence is itself a limit point of the sequence.

(8) Show that $L = \limsup a_n$ if and only if, whenever $\epsilon > 0$ then only finitely many of the a_n lie above $L + \epsilon$ but infinitely many of the a_n lie above $L - \epsilon$.