Department of Mathematics and Statistics
620-221: Real and Complex Analysis, 2007
Exercises 6: Integration

(1) Evaluate the following integrals:

(a) $\displaystyle\int_0^1 (2t + it^2)\,dt$ (b) $\displaystyle\int_0^1 (1 + 2it)^3\,dt$ (c) $\displaystyle\int_0^2 \frac{t}{(t^2+i)^2}\,dt$

(2) Compute $\int_\gamma \overline{z}\,dz$ where γ is the circle $|z| = 2$ traversed three times clockwise.

(3) Compute $\int_\gamma \overline{z}^2\,dz$ where γ is the perimeter of the square with vertices $z = 0,\ 1,\ 1+i,\ i$ traversed in that order.

(4) Let γ be the boundary of the triangle with vertices $0, 1, 1+i$ with the counter-clockwise orientation. Evaluate the following integrals:

(a) $\int_\gamma \mathrm{Re}\ z\ dz$ (b) $\int_\gamma \mathrm{Im}\ z\ dz$ (c) $\int_\gamma z\ dz$

(5) Let γ be the unit circle $\{z : |z| = 1\}$, with the usual counterclockwise orientation. Evaluate the following integrals:

(a) $\int_\gamma z^m\ dz$ (b) $\int_\gamma \overline{z}^m\ dz$ (c) $\int_\gamma z^m |dz|$

(6) Compute $\int_\gamma (|z-1+i|^2 - z)\,dz$ where γ is the semicircle $z = 1-i+e^{it}$, $0 \leq t \leq \pi$.

(7) Find a definition for the length of a smooth curve in your favourite calculus textbook and reconcile it with the definition given in Definition **??**.

(8) Show that

$$\left|\int_{|z|=1} \frac{dz}{5z^2+3}\ dz\right| \leq \pi, \qquad \left|\int_{|z-1|=1} \frac{e^z}{z+3}\ dz\right| \leq \frac{2\pi e^2}{3}.$$

(9) Prove that

(a) if C is the circle $|z| = 3$ then

$$\left|\int_C \frac{dz}{z^2 - i}\right| \leq \frac{3\pi}{4};$$

(b) if C is the triangle with vertices $z = 0, 3i, -4$ then

$$\left| \int_C (e^z - \overline{z})\, dz \right| \leq 48;$$

(c) if C is the arc of the unit circle in the first quadrant then

$$\left| \int_C \operatorname{Log} z\, dz \right| \leq \frac{\pi^2}{4}.$$

(10) Evaluate the following integrals, for a path γ that travels from $-\pi i/2$ to $\pi i/2$:
(a) $\int_\gamma z^4\, dz$ (b) $\int_\gamma e^z\, dz$ (c) $\int_\gamma \cos z\, dz$ (d) $\int_\gamma \sinh z\, dz$

(11) Using an appropriate anti-derivative, evaluate $\int_\gamma 1/z\, dz$ for a path γ that travels from $-\pi i$ to πi in the right half-plane, and also for a path γ from $-\pi i$ to πi in the left half-plane. For each path give a precise definition of the anti-derivative used to evaluate the integral.

(12) Evaluate the following integrals using the Cauchy integral formula:
(a) $\displaystyle\int_{|z|=2} \frac{z^n}{z-1}\, dz,\ n \in \mathbb{N}$ (c) $\displaystyle\int_{|z|=1} \frac{\sin z}{z}\, dz$
(b) $\displaystyle\int_{|z|=1} \frac{z^n}{z-2}\, dz, n \geq 0$

(13) If $|a|, |b| < 1$ and $a \neq b$, C is the unit circle and f is analytic on the interior of C, evaluate $\displaystyle\int_C \frac{f(z)}{(z-a)(z-b)}\, dz$. What happens in the limit as a approachs b?

(14) Complete the proof of Theorem **??** by filling in the details for the last paragraph.

(15) Show that, if a is real,

$$\int_0^\pi e^{a\cos\theta} \cos(a \sin\theta)\, d\theta = \pi.$$

(Hint: Evaluate $\int_C \dfrac{e^{az}}{z}\, dz$ on the unit circle.)

(16) Evaluate the following integrals using the Cauchy integral formula:
(a) $\displaystyle\int_{|z|=2} \frac{e^z}{(z+1)(z-3)^2}\, dz$ (c) $\displaystyle\int_{|z|=2} \frac{\cos z}{z^3 + 9z}\, dz$
(b) $\displaystyle\int_{|z-1|=1} \frac{\sin z}{(z-1)(z^2+z+1)}\, dz$

(17) Evaluate $\displaystyle\int_C \frac{dz}{z^2+1}$ for C being.

(a) $|z-i|=1$ (b) $|z+i|=1$ (c) $|z|=2$.

(18) Does the function $1/z(1-z^2)$ have an anti-derivative in the region $0<|z|<1$?

(19) (Harder) Derive the formula

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^{2n} t \; dt = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$$

by calculating the integral of $f(z)=\dfrac{1}{z}\left(z+\dfrac{1}{z}\right)^{2n}$ around the circle $|z|=1$ in two different ways. Firstly, expand the function using the binomial theorem and hence calculate the integral; secondly, parametrise the circle to obtain a suitable integral involving $\cos^{2n}\theta$.