

Department of Mathematics and Statistics  
620-221: Real and Complex Analysis, 2007

Exercises 7: Applications of the Cauchy Theorems

- (1) Show that, for all  $z$  with  $|z - 1| < 1$ ,

$$\operatorname{Log} z = (z - 1) - \frac{(z - 1)^2}{2} + \frac{(z - 1)^3}{3} - \dots + (-1)^{n-1} \frac{(z - 1)^n}{n} + \dots$$

- (2) Use the Taylor expansion to show that, for all  $z$ ,

$$\exp(z) = \sum_{n=0}^{\infty} \frac{e(z - 1)^n}{n!}.$$

- (3) Use the Taylor expansion to show that

$$\frac{e^z}{1 + z} = 1 + \frac{1}{2}z^2 - \frac{1}{3}z^3 + \frac{3}{8}z^4 - \frac{11}{30}z^5 + \dots$$

On what domain is the expansion valid?

- (4) Show that, for all  $z$  with  $|z + 1| < 1$ ,

$$\frac{1}{z^2} = \sum_{n=0}^{\infty} (n + 1)(z + 1)^n.$$

Deduce a value for the sum of the series  $\sum_{n=0}^{\infty} (n + 1)/2^n$ .

- (5) Find a power series expansion for  $\sin^2(z)$ .

- (6) Evaluate the following integrals using the generalised Cauchy integral formula:

$$\begin{array}{ll} \text{(d)} \int_{|z|=1} \frac{\cosh z}{z^3} dz & \text{(f)} \int_{|z-(1+i)|=5/4} \frac{\operatorname{Log} z}{(z - 1)^2} dz \\ \text{(e)} \int_{|z|=1} \frac{e^z}{z^n} dz, \quad n \in \mathbb{Z} & \text{(g)} \int_{|z|=1} \frac{dz}{z^2(z^2 - 4)e^z} dz \end{array}$$

- (7) Use power series to show that the only function  $f$  which is analytic at 0 and satisfies

$$f''(z) + f(z) = 0 \quad f(0) = 0 \quad f'(0) = 1$$

is given by  $f(z) = \sin z$  in a neighbourhood of 0.

- (8) Calculate the first four derivatives of the function  $1/\cos z$  and use them to show that

$$\frac{1}{\cos z} = 1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 + \dots$$

On what domain is this expansion valid?

- (9) Let  $\alpha$  be a non-zero complex number. Find a generalisation of the binomial expansion by finding a power series expansion, about 0, for  $(1+z)^\alpha$ . (You may find it useful to define an appropriate ‘binomial coefficient’  $\binom{\alpha}{k}$  for an arbitrary complex number  $\alpha$  and non-negative integer  $k$ .) On what domain is this expansion valid?
- (10) Show that if  $u$  is a harmonic function on  $\mathbb{R} \times \mathbb{R}$  that is bounded above, then  $u$  is constant. *Hint:*  $u$  is the real part of some analytic function  $f = u + iv : \mathbb{C} \rightarrow \mathbb{C}$ . Consider  $e^f$ . Apply Liouville’s theorem.
- (11) Suppose that  $f$  is a function analytic in the closed disc  $\{z : |z| \leq 1\}$ . Show that there exists a natural number  $n$  so that  $f(1/n) \neq 1/(n+1)$ .
- (12) Show that if  $f$  is an entire function, and there is a nonempty disk such that  $f$  does not attain any value in the disk, then  $f$  is constant.
- (13) Find the maximum and minimum values of the modulus of the function  $z^2 - z$  on the disc  $\{z : |z| \leq 1\}$ .
- (14) Suppose that  $f : \mathbb{C} \rightarrow \mathbb{C}$  is entire such that  $f(z)/z^n$  is bounded for  $|z| \geq R$ . Show that  $f$  is a polynomial of degree at most  $n$ . (Hint: Use the Cauchy estimates to show that  $f^{n+1}(z)$  is zero.)
- (15) Suppose that  $f$  is entire and that  $|f'(z)| \leq |z|$  for all  $z$ . Show that  $f(z) = a + bz^2$  with  $|b| \leq 1/2$ .
- (16) If  $f$  is an entire function which satisfies  $f(z+1) = f(z)$  and  $f(z+i) = f(z)$  for all  $z$ , then show that  $f$  is constant.