Department of Mathematics and Statistics
620-221: Real and Complex Analysis, 2007
Exercises 8: Poles, Laurent Series and Residues

(1) Let $f(z) = \dfrac{1}{z(z-1)(z-2)}$. Give the Laurent expansions of f centred at 0 and valid for z in the following annuli:
(a) $0 < |z| < 1$ (b) $1 < |z| < 2$ (c) $2 < |z|$.

(2) Find the Laurent expansion of $\dfrac{1}{(1+z^2)(2+z^2)}$ valid for
(a) $1 < |z| < \sqrt{2}$ (b) $\sqrt{2} < |z|$

(3) Find all possible Laurent expansions in a punctured disc centered at 0 of the following functions:
(a) $\dfrac{1}{z^2 - z}$ (b) $\dfrac{z-1}{z+1}$ (c) $\dfrac{1}{(z^2-1)(z^2-4)}$.

(4) Find a Laurent series valid for $z \neq 0$ for the function $\cos(1/z)$.

(5) Find a Laurent series in the punctured disc $0 < |z-4| < 4$ for the function $\dfrac{z+1}{z(z-4)^3}$.

(6) Find the isolated singularities of the following functions, and determine whether they are removable, essential, or poles. Determine the order of any pole.
(a) $\dfrac{z}{(z^2-1)^2}$ (d) $\tan z = \dfrac{\sin z}{\cos z}$ (g) $\mathrm{Log}\big(1 - 1/z\big)$
(b) $\dfrac{ze^z}{z^2-1}$ (e) $z^2 \sin(1/z)$ (h) $\dfrac{\mathrm{Log}\, z}{(z-1)^3}$
(c) $\sin \dfrac{1}{z-1}$ (f) $\dfrac{\cos z}{z^2 - (\pi/2)^2}$ (i) $e^{1/(z^2+1)}$

(7) Evaluate the following residues:

(a) $\mathrm{Res}\left[\dfrac{1}{z^2+4}, 2i\right]$ (d) $\mathrm{Res}\left[\dfrac{\sin z}{z^2}, 0\right]$ (g) $\mathrm{Res}\left[\dfrac{z}{\mathrm{Log}\, z}, 1\right]$
(b) $\mathrm{Res}\left[\dfrac{1}{z^2+4}, -2i\right]$ (e) $\mathrm{Res}\left[\dfrac{\cos z}{z^2}, 0\right]$ (h) $\mathrm{Res}\left[\dfrac{e^z}{z^5}, 0\right]$

(c) $\text{Res}\left[\frac{1}{z^5-1}, 1\right]$ (f) $\text{Res}\left[\cot z, 0\right]$ (i) $\text{Res}\left[\frac{z^n+1}{z^n-1}, e^{2i\pi k/n}\right]$

(8) Evaluate the following integrals, using the residue theorem.

(a) $\displaystyle\int_{|z|=1} \frac{\sin z}{z^2} dz$ (c) $\displaystyle\int_{|z|=1} \frac{e^z\, dz}{z^2(z^2-4)}$ (e) $\displaystyle\int_{|z-1|=1} \frac{1}{z^8-1} dz$

(b) $\displaystyle\int_{|z|=2} \frac{z}{\cos z} dz$ (d) $\displaystyle\int_{|z|=1} \frac{z^4}{\sin z} dz$ (f) $\displaystyle\int_{|z-1/2|=3/2} \frac{\tan z}{z} dz$

(9) Evaluate

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+4}$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^3} dx$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} dx$

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+a^2)} dx$

(e) $\displaystyle\int_{0}^{2\pi} \frac{\sin^2\theta}{5+3\cos\theta} d\theta$

(10) (Harder) Integrate the function given by $f(z) = \sec \pi z/z^3$ about a suitable contour to find the sum of the series $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}$.

(11) (Hard and rather long; but rewarding)
Consider the integral

$$\int_{\partial P_r} \frac{e^{\pi i (z-1/2)^2}}{1-e^{-2\pi i z}} dz,$$

where P_r is the parallelogram with vertices $\pm(1/2) \pm (1+i)r$.

(a) Use the residue theorem to show that the integral is equal to $(1+i)/\sqrt{2}$

(b) By parameterizing the sides of the parallelogram, show that the integral tends to

$$(1+i)\int_{-\infty}^{\infty} e^{-2\pi t^2} dt, \quad \text{as } r \to \infty.$$

(c) Use (a) and (b) to show that

$$\int_{-\infty}^{\infty} e^{-s^2} ds = \sqrt{\pi}.$$