Department of Mathematics and Statistics
620-221: Real and Complex Analysis, 2007
Exercises 9: Conformal transformations

(1) Find the image under $w = \exp(z)$ of the lines $x =$constant and $y =$constant.

(2) Show that the function $w = \exp(z)$ takes a vertical strip in the z-plane to an annulus in the w-plane and a horizontal strip in the z-plane to a 'wedge' in the w-plane.

(3) (a) Find a conformal transformation between the wedge $\{z : 0 \leq \mathrm{Arg} z \leq \pi/4$ and the horizontal strip $\{z = x + iy : 0 \leq y \leq 1\}$.

(b) Find a conformal transformation between the strip $\{z = x + iy : -2 \leq x \leq 1$ and the unit disk. (Hint: first map the strip to a half-plane.)

(4) Find the fractional linear transformations which send

(a) $1, i, -1$ to $-1, i, 1$ respectively;

(b) $-i, 0, i$ to $0, i, 2i$ respectively;

(c) $-i, i, 2i$ to $\infty, 0, 1/3$ respectively.

Also, for each triple of points, identify the circle or line that they define.

(5) The aim of this exercise is to show that every fractional linear transformation can be written as a product of suitable rotations (about 0), translations, inversions (in the unit circle) ($z \mapsto 1/z$) and magnifications ($z \mapsto rz$ with r real) centered on 0.

(a) Show that each of the above transformations of the plane can be represented as a fractional linear transformation.

(b) Show that the product of an inversion, a translation and an inversion is of the form

$$z \mapsto \frac{z}{\alpha z + 1}.$$

(c) Show that any fractional linear transformation

$$z \mapsto \frac{az + b}{cz + d}$$

with $b \neq 0$ can be written as a product of a transformation as in the previous part, a rotation, a magnification and a translation.

(d) Show that any fractional linear transformation

$$z \mapsto \frac{az+b}{cz}$$

can be written as a product of an inversion, a rotation, a magnification and a translation.

(e) Deduce that all fractional linear transformations can be expressed as a product of translations, of inversion in the unit circle and of rotations and magnifications about 0.