

620-221 Real and Complex Analysis. Solutions to the final exam

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**Problem 1.**

- (a) Sketch the image of the right half-plane  $\{z \mid \operatorname{Re} z > 0\}$  under the map  $f(z) = \operatorname{Log} z$ .
- (b) Sketch the image of the horizontal strip  $\{z \mid 5\pi/3 < \operatorname{Im} z < 8\pi/3\}$  under the map  $f(z) = e^z$ .

**Solution:**

- (a) Note that the right half plane can be characterised as  $\{z : -\pi/2 < \operatorname{Arg} z < \pi/2\}$  and  $\operatorname{Im}(\operatorname{Log} z) = \operatorname{Arg}(z)$  so that the image is the horizontal strip:  $-\pi/2 < \operatorname{Im} z < \pi/2$ .
- (b) Note that  $\operatorname{Arg}(e^z) = \operatorname{Im}(z)$  so that the image is the segment  $\{z : 5\pi/3 < \operatorname{Arg} z < 8\pi/3\}$ , that is the open half-plane on the left of the line  $\operatorname{Arg} z = 5\pi/3$ .

**Problem 2.**

- (a) Give a definition of an open and a closed subset of  $\mathbb{C}$ . Give an example which is neither open nor closed.
- (b) Give a definition of a convex and a star-shaped subset of  $\mathbb{C}$ . Give an example of a star-shaped set which is not convex.

**Solution:**

- (a) Standard definitions. An example is the closed unit disc omitting one point. It is not open as no point of the bounding circle lies in a neighbourhood within the disc and it is not closed since the omitted point is a limit point of the set.
- (b) Standard definitions. An example is any standard (kindergarten) star shape

**Problem 3.**

Find all real numbers  $a$  and  $b$  so that  $u(x, y) = ax^2 - by^2$  is the real part of an analytic function  $f : \mathbb{C} \rightarrow \mathbb{C}$ , and determine all such analytic functions  $f$ .

**Solution:**

If  $u$  is the real part of an analytic function, then it has to be harmonic. So

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2a - 2b = 0.$$

Hence  $a = b$ . To determine the harmonic conjugate we use the C-R equations,

$$\frac{\partial u}{\partial x} = 2ax = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -2ay = -\frac{\partial v}{\partial x}.$$

Form the first equation we get

$$v(x, y) = 2axy + \varphi(x)$$

for some function  $\varphi$  depending only on  $x$ . So using the second equation we get,

$$\frac{\partial v}{\partial x} = 2ay + \varphi'(x) = 2ay, \quad \text{that is, } \varphi'(x) = 0,$$

and  $\varphi(x) = c$  for some constant  $c$ . A harmonic conjugate  $v(x, y) = 2axy + c$ , and analytic functions  $f$  having  $u$  as a real part are of the form

$$f(z) = a(x^2 - y^2) - 2axy + c, \text{ for } c \in \mathbb{R}.$$

**Problem 4.**

Show that if  $f = u + iv$  is analytic on a domain  $D$  and  $uv = 1$ , then  $f$  is constant. State theorems that you use to draw your conclusions.

**Solution:**

(1) Differentiating  $uv = 1$  with respect to  $x$  and  $y$  gives,

$$\begin{aligned} u_x v + uv_x &= 0 \\ u_y v + uv_y &= 0. \end{aligned}$$

Using the Cauchy-Riemann equations  $u_x = v_y$  and  $u_y = -v_x$  we get from the above equations,

$$\begin{aligned} u_x v - uv_y &= 0 \\ u_y v + uv_x &= 0. \end{aligned}$$

After multiplying the first equation by  $v$  and the second by  $u$ , and using  $uv = 1$  we get

$$\begin{aligned} u_x v^2 - u_y &= 0 \\ u_y + u^2 u_x &= 0. \end{aligned}$$

This leads to  $u_x[u^2 + v^2] = 0$  and to  $u_x = 0$  (since  $uv = 1$  and so  $u^2 + v^2$  is never zero), and  $u_y = 0$ . By the Cauchy-Riemann equations, also  $v_x = v_y = 0$ . Since  $D$  is a domain (open and *connected*),  $u$  and  $v$  are constants. So  $f = u + iv$  is constant.

**Problem 5.**

- (a) State Liouville's theorem for analytic functions.
- (b) Assume that  $f : \mathbb{C} \rightarrow \mathbb{C}$  is analytic and satisfies  $f(z+i) = f(z+1) = f(z)$  for all  $z \in \mathbb{C}$ . Prove that  $f$  is constant.

**Solution:**

(a) **Liouville's theorem:** If  $f : \mathbb{C} \rightarrow \mathbb{C}$  is entire (analytic in the whole plane) and bounded, then  $f$  is constant.

(b) Any function  $f$  satisfying the assumption satisfies  $f(z) = f(z_1)$  where  $z_1$  lies in the unit square with vertices  $0, 1, 1 + i, i$ . But the (closed) unit square is closed and bounded and so  $f$  achieves a maximum there. Thus  $f$  is bounded on the whole plane. So applying Liouville's theorem,  $f$  is constant.

**Problem 6.**

Evaluate the integral

$$\int_{\gamma} \frac{e^z}{z(1-z)^3} dz,$$

where  $\gamma$  is

(a) the circle  $|z| = 1/2$

(b) the circle  $|z - 1| = 1/2$

with the counterclockwise orientation.

**Solution:**

The function  $\frac{e^z}{z(1-z)^3}$  is analytic on  $\mathbb{C} \setminus \{0, 1\}$ . (a) So  $\frac{e^z}{(1-z)^3}$  is analytic on  $\overline{D}(0, 1/2)$  and by the (generalised) Cauchy integral formula,

$$\int_{|z|=1/2} \frac{e^z}{z(1-z)^3} dz = 2\pi i \frac{e^z}{(1-z)^3} \Big|_{z=0} = 2\pi i.$$

(b) The function  $\frac{e^z}{z}$  is analytic on  $\overline{D}(1, 1/2)$ , and by the Cauchy integral formula,

$$\int_{|z|=1/2} \frac{e^z}{z(1-z)^3} dz = - \int_{|z|=1/2} \frac{e^z}{z(z-1)^3} dz = - \frac{2\pi i}{2!} \frac{d^2}{dz^2} \frac{e^z}{z} \Big|_{z=1} = -\pi e i.$$

**Problem 7.**

Prove that if an infinite series  $\sum_{k=0}^{\infty} a_k$  of complex numbers converges absolutely, then  $\sum_{k=0}^{\infty} a_k$  converges. Give an example showing that the converse does not hold.

**Solution:**

Let  $s_n = \sum_{k=0}^n s_k$  and  $S_k = \sum_{k=0}^n |a_k|$ . It is enough to show that  $\{s_k\}$  is Cauchy. Since  $\sum_{k=0}^{\infty} a_k$  converges absolutely,  $S_k$  converges. This means that  $\{S_k\}$  is Cauchy. Given  $\varepsilon > 0$  there exists  $N$  such that

$$|S_m - S_n| \leq \varepsilon, \quad \text{for all } n, m \geq N.$$

For  $n \geq m \geq N$ ,

$$|s_n - s_m| = \left| \sum_{k=m+1}^n a_k \right| \leq \sum_{k=m+1}^n |a_k| = S_n - S_m < \varepsilon.$$

So  $\{s_n\}$  is Cauchy and the result follows since a sequence converges if and only if it is Cauchy.

**Problem 8.**

Find the disk of convergence for the following power series:

$$(a) \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} (z+i)^n \qquad (b) \sum_{n=0}^{\infty} (n+2^n)z^n$$

**Solution:**

(a) Applying the ratio test:

$$\frac{(2n)!(n+1)!}{(2(n+1))!(n!)^2} = \frac{(n+1)(n+1)}{(2n+1)(2n+2)} = \frac{(n+1)}{2(2n+1)} \rightarrow 1/4.$$

So the disk of convergence of the series has centre  $-i$  and radius  $1/4$ .

(b) Applying the Cauchy-Hadamard test we look for a limit of  $\sqrt[n]{n+2^n}$ . Since  $2^n \leq n+2^n = 2^n + 2^n = 2 \cdot 2^n$ ,  $n \geq 2$ , we get  $2 \leq \sqrt[n]{n+2^n} < 2\sqrt[n]{2}$ . By the sandwich theorem,  $\lim_{n \rightarrow \infty} \sqrt[n]{n+2^n} = 2$ , so that the disk of convergence has centre 0 and radius  $1/2$ .

**Problem 9.**

Let  $\{f_n\}$  be a sequence of complex valued functions  $f_n : D \subset \mathbb{C} \rightarrow \mathbb{C}$ , and let  $f : D \rightarrow \mathbb{C}$ .

- (a) Give a definition of the uniform convergence of  $\{f_n\}$  to  $f$  on  $D$ .  
 (b) Prove: Let  $\gamma$  be a piecewise smooth curve in the complex plane. If  $f_n$  is continuous on  $\gamma$  for every  $n$ , and  $\{f_n\}$  converges uniformly to  $f$  on  $\gamma$ , then  $\int_{\gamma} f_n(z) dz$  converges to  $\int_{\gamma} f(z) dz$ .

**Solution:**

(a) A sequence  $\{f_n\}$  converges uniformly to  $f$  on  $D$  if for every  $\varepsilon > 0$  there exists  $N$  such that

$$|f_n(x) - f(x)| < \varepsilon, \quad \text{for every } x \in D \text{ and every } n \geq N.$$

(b) Denote by  $l(\gamma)$  the length of  $\gamma$  ( $< \infty$ ). Let  $\varepsilon > 0$ . Since  $f_n \rightarrow f$  uniformly on  $\gamma$ , there exists  $N$  such that

$$|f_n(z) - f(z)| < \varepsilon/l(\gamma),$$

for all  $z \in \gamma$  and all  $n \geq N$ . Then, in view of the standard estimate,

$$\begin{aligned} \left| \int_{\gamma} f_n(z) dz - \int_{\gamma} f(z) dz \right| &= \left| \int_{\gamma} [f_n(z) - f(z)] dz \right| \\ &\leq \int_{\gamma} |f_n(z) - f(z)| |dz| < \frac{\varepsilon}{l(\gamma)} \cdot l(\gamma) = \varepsilon \end{aligned}$$

for all  $n \geq N$ .

**Problem 10.**

Find the Laurent expansion of  $f(z) = \frac{1}{(z-1)(z-2)}$  on the following domains

(a)  $\{|z| < 1\}$

(b)  $\{1 < |z| < 2\}$

(c)  $\{2 < |z|\}$

**Solution:**

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

(a)

$$\frac{1}{z-2} = \frac{-1}{2} \frac{1}{1-(z/2)} = \frac{-1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

for  $|z| < 2$ , and

$$-\frac{1}{z-1} = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

for  $|z| < 1$ . So for  $|z| < 1$ ,

$$\frac{1}{(z-1)(z-2)} = \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^{n+1}}\right) z^n = \sum_{n=0}^{\infty} \frac{2^{n+1} - 1}{2^{n+1}} z^n.$$

(b)

$$\frac{1}{z-2} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

for  $|z| < 2$ , and

$$\frac{1}{z-1} = \frac{1}{z} \frac{1}{1-(1/z)} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^{n+1}$$

for  $|z| > 1$ . So for  $1 < |z| < 2$ ,

$$\frac{1}{(z-1)(z-2)} = \sum_{-\infty}^{\infty} a_n z^n,$$

where

$$a_n = \frac{-1}{2^{n+1}}, \quad n \geq 0, \quad a_n = -1, \quad n < 0.$$

(c)

$$\frac{1}{z-2} = \frac{1}{z} \frac{1}{1-(2/z)} = \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}$$

for  $|z| > 2$ , and for  $|z| > 1$  as above, so

$$\frac{1}{(z-1)(z-2)} = \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} - \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^{n+1} = \sum_{n=0}^{\infty} [2^n - 1] \frac{1}{z^{n+1}}.$$

**Problem 11.**

- (a) Give definitions of a removable singularity, a pole and an essential singularity.
- (b) Let  $z_0 = 0$ . Classify each of the following as having a removable singularity, a pole, or an essential singularity at  $z_0$ . Determine the order of a pole, and the principal part at a pole.

(i)  $z^2 \sin\left(\frac{1}{z}\right)$

(ii)  $\frac{e^{2z} - 1}{z}$

(iii)  $\frac{\cos z - 1}{z^4}$

**Solution:** (a) If  $\sum_{-\infty}^{\infty} a_n(z-z_0)^n$  is the Laurent expansion of  $f$  on  $D(z_0, r) \setminus \{z_0\}$ , then

- (i)  $a_k = 0$  for all  $k < 0$ .  
(ii) For some  $N \geq 1$ ,  $a_k = 0$  for all  $k < -N$  and  $a_{-N} \neq 0$ .  
(iii) Neither (i) nor (ii) holds.

In case (i),  $z_0$  is called a removable singularity, in case (ii)  $z_0$  is a pole of order  $N$  and in case an essential singularity.

- (b)  
(i)

$$z^2 \sin\left(\frac{1}{z}\right) = z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{z^{2n+1}}\right),$$

and so  $z_0 = 0$  is an essential singularity.

- (ii)

$$\frac{e^{2z} - 1}{z} = \frac{1}{z} \sum_{n=1}^{\infty} \frac{2^n z^n}{n!} = 2 + \sum_{n=2}^{\infty} \frac{2^n z^{n-1}}{n!},$$

so that  $z_0 = 0$  is a removable singularity.

- (iii)

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots,$$

and so

$$\frac{\cos z - 1}{z^4} = -\frac{1}{2} \frac{1}{z^2} + \frac{1}{24} - \frac{z^2}{6!} + \cdots.$$

So  $z_0 = 0$  is a pole. The principal part is the sum of negative powers of  $z$ , so

$$P(z) = -\frac{1}{2} \cdot \frac{1}{z^2},$$

and the order of  $z_0 = 0$  is 2.

**Problem 12.**

Evaluate the following integral using the residue theorem

$$\int_{|z|=2} \frac{e^z}{z^2 - 1} dz$$

where the circle  $|z| = 2$  has counterclockwise orientation.

**Solution:**

The function  $\frac{e^z}{z^2 - 1}$  has isolated singularities at  $z_1 = -1$  and  $z_2 = 1$ . Both are simple poles, so with  $\eta = \pm 1$

$$\operatorname{Res}\left[\frac{e^z}{z^2 - 1}, \eta\right] = \left.\frac{e^z}{2z}\right|_{z=\eta} = \frac{e^\eta}{2\eta}.$$

Thus, by the Residue theorem

$$\int_{|z|=2} \frac{e^z}{z^2 - 1} dz = 2\pi i \left[ \frac{e^1}{2} - \frac{e^{-1}}{2} \right] = 2\pi i \sinh 1.$$